On the Number of Graphical Forest Partitions

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Abstract

A **graphical partition** of the even integer n is a partition of n where each part of the partition is the degree of a vertex in a simple graph and the degree sum of the graph is n. In this note, we consider the problem of enumerating a subset of these partitions, known as **graphical forest partitions**, graphical partitions whose parts are the degrees of the vertices of forests (disjoint unions of trees). We shall prove that

$$gf(2k) = p(0) + p(1) + p(2) + \ldots + p(k-1)$$

where $gf(2k)$ is the number of graphical forest partitions of $2k$ and $p(j)$ is the ordinary partition function which counts the number of integer partitions of j.

1 Introduction

A **partition** of a positive integer n is a sequence of positive integers, in no particular order, whose sum is n. For example, $5+3+2+2+1+1+1$ is a partition of 15. Each number in a partition is called a **part** of that partition. The partition function $p(n)$ counts the number of partitions of the integer n.

In this note we will consider only those partitions of n that are graphical sequences (and denote the number of these partitions by $g(n)$). A **graphical sequence** is a sequence whose terms represent the degrees of the vertices in a **simple graph**, a graph that can be drawn without any multiple edges or loops.

Finding a closed formula for $g(n)$ has proven difficult. Indeed, even the asymptotics of $g(n)$ are still unknown. However, several results regarding $g(n)$ are known. For instance, a lower bound for this function has been found. This lower bound is $p(n) - p(n-1)$, which is also the number of partitions of n with all successive ranks negative [4]. Moreover, it is also known [7] that an upper bound for $g(n)$ is $(.25 + o(1))p(n)$. Finally, Pittel [6] has shown that

$$\frac{g(n)}{p(n)} \to 0 \text{ as } n \to \infty.$$

The interested reader may also wish to see [2] and [5] for additional discussion regarding $g(n)$.

Because of the difficulty in finding a closed formula for $g(n)$, we chose to restrict $g(n)$ even further by considering only those graphical partitions of n which correspond to forests, with the hope that a closed form might become apparent. (Here we use the term **forest** to mean a union of trees.) We denote the number of graphical forest partitions of n by $gf(n)$.

The goal of this note is to prove that, for all $k \geq 1$,

$$gf(2k) = p(0) + p(1) + p(2) + \ldots + p(k-1).$$

2 The Results

First off, we let $gf(n,t)$ be the number of graphical forest partitions of n into exactly t parts. Our first goal is to prove the following result.

Theorem 2.1. *For $s > 1$, $gf(2k, k+s) = gf(2k-2, k+s-2)$.*

Proof. If a forest realizes a sequence counted by $gf(2k, k+s)$, at least two of its vertices in different components must have degree 1. Deleting these two and joining their neighbors by an edge gives a forest realization of a sequence counted by $gf(2k-2, k+s-2)$. Conversely, adding two new vertices joined by an edge to a forest realization of a sequence counted in $gf(2k-2, k+s-2)$ creates a new sequence counted by $gf(2k, k+s)$. □

We turn now to our main theorem.

Theorem 2.2. *For all $k \geq 2$, $gf(2k) = p(0) + p(1) + \ldots + p(k-1)$.*

Proof. It is known [8, Problem 2.1.12] that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k+1} > 0$ is the degree sequence of a tree if and only if $\lambda_1 + \lambda_2 + \cdots + \lambda_{k+1} = 2k$. Letting $p(n, t)$ denote the number of partitions of n with exactly t parts, then the number of graphical tree partitions of $2k$ is $p(2k, k+1)$ which equals $p(k-1)$.

Thus, $gf(2k, k+1) = p(k-1)$ and

$$gf(2k) = \sum_{s=1}^{k} gf(2k, k+s).$$

Finally, from Theorem 2.1 above, we know $gf(2k, k+s) = gf(2k-2, k+s-2)$ and the result follows. □

3 On Computing $gf(2k)$

Thanks to the results above, we see that finding the number of graphical forest partitions of $2k$ simply involves finding the values of the ordinary partition function $p(n), 0 \leq n \leq k-1$.

A quick word on asymptotics is worth noting here. It is known [3, Section 3] that

$$p(0) + \cdots + p(n-1) \sim p(n)\frac{\sqrt{6n}}{\pi},$$

which means

$$gf(2k) \sim p(k)\frac{\sqrt{6k}}{\pi}.$$

Moreover, we know [1, p. 70] that

$$p(k) \sim \frac{1}{4k\sqrt{3}} \exp\left[\pi \left(\frac{2k}{3}\right)^{1/2}\right],$$

which implies

$$gf(2k) \sim \frac{\sqrt{2}}{4\pi\sqrt{k}} \exp\left[\pi \left(\frac{2k}{3}\right)^{1/2}\right].$$

Next, we mention two ways to determine exact values of $gf(2k)$. First, we can utilize the generating function for $p(n)$ and Euler's Pentagonal Number Theorem to develop the following recurrence for $p(n)$:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \ldots$$

where the values being subtracted in the arguments on the right–hand side are the pentagonal numbers $\frac{3}{2}m^2 - \frac{1}{2}m$ for integers m. The interested reader should see [1, p. 11].

Alternatively, we can compute the values of $gf(2k)$ by developing a generating function for $gf(2k)$ and expanding it using a computer algebra system. Since the generating function for $p(n)$ is given by

$$\sum_{k=0}^{\infty} p(k)q^k = \prod_{n\geq 1} \frac{1}{1-q^n},$$

we see that the coefficient of q^k in

$$\frac{q}{1-q} \prod_{n\geq 1} \frac{1}{1-q^n}$$

is $p(0) + p(1) + \ldots + p(k-1)$. Thus,

$$\sum_{k=0}^{\infty} gf(2k)q^k = \frac{q}{1-q} \prod_{n\geq 1} \frac{1}{1-q^n}.$$

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