Which Graphs Have Hall Number Two?

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Abstract

We make further progress towards the forbidden-induced-subgraph characterization of the graphs with Hall number ≤ 2. We solve several problems posed in [4] and, in the process, describe all "partial wheel" graphs with Hall number > 2 with every proper induced subgraph having Hall number ≤ 2.

1 Introduction

Throughout, G will denote a finite simple graph and L will denote a list assignment to the vertices of G, i.e., a function from $V(G)$ into the collection of finite subsets of C, an infinite set (of "colors", or symbols). A proper L-coloring of G is a selection $\varphi(v) \in L(v)$ for each $v \in V(G)$ such that if u and v are adjacent in G, then $\varphi(u) \neq \varphi(v)$. [This last can be restated: for each $\sigma \in C, \varphi^{-1}(\sigma) = \{v \in V(G); \varphi(v) = \sigma\}$ is an independent set of vertices in G.]

The study of list-colorings, started by Vizing [11] and independently by Erdös, Rubin, and Taylor [2], departs from the question of when (under what conditions on G and L) is there a proper L-coloring of G? Interest has fastened mainly on the choice number, or list-chromatic number: $c(G)$ is the smallest positive integer among those m such that there is a proper L-coloring of G whenever $|L(v)| \geq m$ for all $v \in V(G)$. It is clear that

*supported by ONR grant no. N00014-97-1-1067

$c(G) \geq \chi(G)$, the chromatic number of G, and it is known that $c(G)$ can be quite a bit larger than $\chi(G)$; for instance, $c(K_{m,m}) \sim \log_2 m$ [9]. Curiosity is drawn to the extremes: how much larger than $\chi(G)$ can $c(G)$ be (for instance, can $c(G)/(\chi(G)\log|V(G)|)$ be arbitrarily large?) and, at the other extreme, for which G is $c(G) = \chi(G)$?

Here is a necessary condition for a proper L-coloring which does not directly refer to the size of the lists $L(v)$, $v \in V(G)$. We say that G and L *satisfy Hall's condition* iff for each subgraph H of G,

$$|V(H)| \leq \sum_{\sigma \in C} \alpha(\sigma, L, H) \tag{1}$$

where $\alpha(\sigma, L, H)$ is the independence number of the subgraph of H induced by $\{u \in V(H); \sigma \in L(u)\}$. Note that if you were trying to properly L-color H, for each $\sigma \in C$ $\alpha(\sigma, L, H)$ is an upper bound on the number of vertices of H that could be colored with σ. This shows that Hall's condition is necessary for the existence of a proper L-coloring of G.

Removing edges from H will leave $|V(H)|$ unchanged while the right hand side of (1) cannot decrease. Therefore, for G and L to satisfy Hall's condition it suffices that (1) hold for induced subgraphs H of G.

If H is a clique then $\alpha(\sigma, L, H) = 1$ iff σ actually appears on the lists on H: otherwise, $\alpha(\sigma, L, H) = 0$. Therefore $\sum_{\sigma \in C} \alpha(\sigma, L, H) = |\bigcup_{z \in V(H)} L(z)|$, in this case. [Hencefoward, we abbreviate $\bigcup_{z \in V(H)} L(z)$ by $L(H)$.] Noting that a proper L-coloring of a clique G is nothing but a selection of distinct representatives from the "system" of sets $L(v)$, $v \in V(G)$, and that an induced subgraph of a clique is a clique, it is straightforward to see that when G is a clique, Hall's condition on G and L is the same as the condition in Philip Hall's famous theorem on systems of distinct representatives [5]; that theorem, restated, says that Hall's condition is sufficient for the existence of a proper L-coloring of G, when G is a clique.

Hall's condition is so called because of this ancestral connection with Hall's theorem. In [6] it is shown that Hall's condition is far from being sufficient for a proper L-coloring; indeed, the graphs G such that, for arbitrary L, the satisfying of Hall's condition with G is sufficient for the existence of a proper L-coloring, are precisely the "locally clique-like" graphs in which every block (maximal 2-connected subgraph) is a clique.

As in [6] and [8] we define the Hall number $h(G)$ of G to be the smallest positive integer among those m such that there is necessarily a proper L-coloring of G whenever

(i) G and L satisfy Hall's condition and

(ii) $|L(v)| \geq m$ for all $v \in V(G)$.

Thus, the result from [6] mentioned above is: $h(G) = 1$ if and only if every block of G is a clique.

It is easy to see (although not completely trivial; see [8] for a proof) that h is monotone with respect to taking induced subgraphs; that is, if H is an induced subgraph of G, then $h(H) \leq h(G)$. This is the only known decent behavior of the parameter h. Removing an edge can cause the value of h to leap up or down by large amounts (see [7] and [10]), and removing a vertex, which always precipitates a drop in the value of h, can precipitate an extraordinarily large such drop [7].

Indeed, although the parameter h is of indisputable interest from the point of view of "systems of distinct representatives" theory, it would be dismissed as contrived and capricious in the area of graph colorings, were it not for its relation to $c(G)$ and $\chi(G)$ and the problem of when $c(G) = \chi(G)$. This relation is detailed in [8]. Let it suffice here to note that $c(G) = \chi(G)$ if and only if $h(G) \leq \chi(G)$; therefore, the problem of characterizing the graphs G such that $h(G) \leq k$, $k = 1, 2, \ldots$, becomes of definite interest in the quest for the solutions of the equation $c(G) = \chi(G)$. In particular, it is immediate that $h(G) \leq 2$ implies that $c(G) = \chi(G)$.

The monotonicity of h with respect to taking induced subgraphs insures that the problem of characterizing G such that $h(G) \leq k$ has a satisfactory solution, for each k; the collection of graphs with Hall number $\leq k$ has a "forbidden-induced-subgraph" characterization: if we define a graph H to be Hall-k^+-critical iff $h(H) > k$ but $h(H - v) \leq k$ for every $v \in V(H)$, then $h(G) \leq k$ if and only if G has no Hall-k^+-critical induced subgraph.

The Hall-1^+-critical graphs are known (see Theorem A). For any $k \geq 3$, it is unlikely that a full classification of the Hall-k^+-critical graphs will ever be achieved. The purpose of this paper is to push on toward the full classification of the Hall-2^+-critical graphs, and thus a characterization of the graphs with Hall number 2. This characterization seems to us to be one of the two main, currently unsolved problems in the area of Hall's condition and the Hall parameters. (For the other one, see [1].) The pursuit of this characterization began in [8] and continued in [4], where two particular graphs and two families of graphs were shown to have Hall number two, and six particular graphs and eight families of graphs were shown to be Hall-2^+-critical. In a related work [3], the line graphs which have Hall number ≤ 2 have been completely characterized.

In the next section we give the results from [4] and [8] that are relevant to the new results presented in section 3. Theorem 1 of section 3 answers a question posed in [4] (in [4] the special case of Theorem 1 when $n = 3$ was proved, with considerable difficulty). Theorem 2 solves two problems posed in [4] and the special case of $W(a, 1, 1)$, $a \geq 2$, in Theorem 3 solves another problem posed in [4]. Theorem 4 gives some more small Hall-2^+-critical graphs and answers an obvious question related to Theorem 3. Proofs are given in section 4.

In [4] it was opined that if the problems posed there were solved, as they are here, then the Hall-number-two problem would be close to solution. We feel that this opinion is correct, i.e., that the Hall-2^+-critical graphs described in [4], in [8], and here come close to forming a complete catalog of Hall-2^+-critical graphs, and that any strays that have not yet been discovered will be rounded up in the process of trying to prove that the catalog is complete. The straightforward way to start that proof is to attempt to show that any graph G with none of the catalogued Hall-2^+-critical graphs as an induced subgraph must have Hall number ≤ 2. We recommend breaking into cases according to the greatest order of a clique in G.

The forbidden-induced-subgraph characterization of the graphs with Hall number 1 (see Theorem A) was greatly facilitated by the other characterization, every-block-a-clique, which was discovered first. It is worth noting that while the graphs with choice number ≤ 2 have a forbidden-induced-subgraph characterization, there is a more satisfying characterization of another sort, given in [2], from which the forbidden-induced-subgraph characterization can be derived. Do the graphs with Hall number ≤ 2 have some sort of alternative characterization, analogous to those of the other two classes of graphs mentioned? This is a vague question, since we cannot specify the sort of characterization we are fishing for, before it is found; but it would be very helpful to find such a characterization. One last question, out of pure curiosity, before getting to work: does every Hall-2^+-critical graph necessarily have Hall number 3? Of course, a similar question could be asked with 2 replaced by any positive integer k, and 3 replaced by $k+1$.

2 Old results

In this section we give the results from [4] and [8] that we need, together with a few we do not need, for background.

Theorem A ([6] and [8]). *The following are equivalent:*

(a) $h(G) = 1$;

(b) every block of G *is a clique;*

(c) G *contains none of* C_n, $n \geq 4$, *nor* K_4*-minus-an-edge, as an induced subgraph.*

The graphs in (c) are the Hall-1^+-critical graphs.

If $m_1, \ldots, m_k$ are positive integers, at most one of them equal to 1, let $\theta(m_1, \ldots, m_k)$ be the graph constructed by joining two vertices by k internally disjoint paths of lengths $m_1, \ldots, m_k$. [This notation was introduced by Erdös, Rubin, and Taylor [2] for $k = 3$; note the appearance of the Greek letter theta.] Thus $\theta(m_1, m_2) = C_{m_1+m_2}$ and $\theta(2,2,1) = K_4$-minus-an-edge.

Theorem B ([4] and [8]). *The following have Hall number 2:*

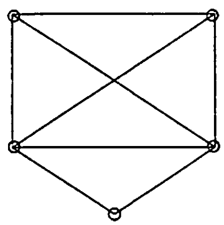
(a) C_n, $n \geq 4$;

(b) $\theta(m, 2, 1)$, *for all* $m \geq 2$;

(c) $\theta(m, 2, 2)$, *for all* $m \geq 2$;

(d) $\theta(3, 3, 2)$;

(e) K_4 *with an "ear" of length* 2,



Theorem C ([4] and [8]). *The following are Hall-2^+-critical:*

(a) Two cycles, not both of length 3, *joined by a path, possibly of length zero.*

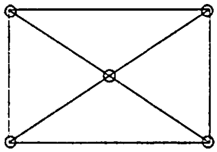
(b) $\theta(m_1, m_2, m_3)$, $m_1 \geq m_2 \geq m_3$, *provided* $m_2 \geq 3$ *and* $(m_1, m_2, m_3) \neq (3, 3, 2)$;

(c) $\theta(m, 2, 2, 1)$ *and* $\theta(m, 2, 2, 2)$ *for any* $m \geq 2$;

(d) $\theta(3, 3, 2, 2)$;

(e) any cycle together with two triangles based on incident edges of the cycle;

(f)



Comments:

(i) Theorems A and B imply that each Hall-1^+-critical graph has Hall number 2.

(ii) $\theta(3,3,2)$ has Hall number 2 (Theorem B(d)), yet $\theta(3,3,1)$ is Hall-2^+-critical (Theorem C(b)). This is the sort of unnerving surprise one encounters with systems-of-distinct-representatives results associated with the Hall number.

Incidentally, letting $G = \theta(3,3,1)$, we have $c(G) \le 3$ by Brooks' theorem for the choice number ([2] and [11]), whence $2 < h(G) \le c(G) \le 3$ implies that $h(G) = c(G) = 3$. Notice that $\chi(G) = 2$. In fact, G (also known as $K_{3,3}$ minus two independent edges) is the smallest graph whose choice number is greater than its chromatic number.

(iii) Not wishing to swamp the reader's attention, we have omitted most of the results from [4] (although these will probably have to be looked up by anyone attempting a full assault on the Hall-number-2 problem). Here are two more, though, that may satisfy curiosity aroused by Theorem B(e): both K_5 with an ear of length 2, and K_4 with an ear of any length greater than 2, are Hall-2^+-critical. Incidentally, it is then easy to see that these graphs have Hall number 3, from the fact that Hall's condition suffices for a proper list-coloring of a clique.

3 New results

Theorem 1. *Suppose that $h(G) = 2$, $v_o \in V(G)$, and, for each integer $m \ge 0$, $G(m)$ is obtained by tethering a clique to G at v_o, by a path of length m. If $h(G(0)) = 2$, then $h(G(m)) = 2$ for each $m \ge 0$.*

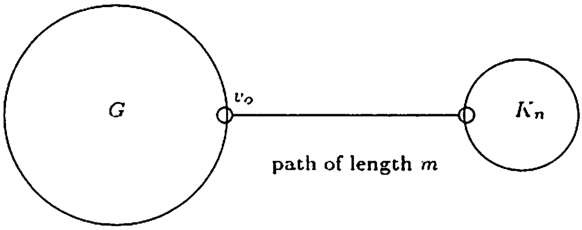


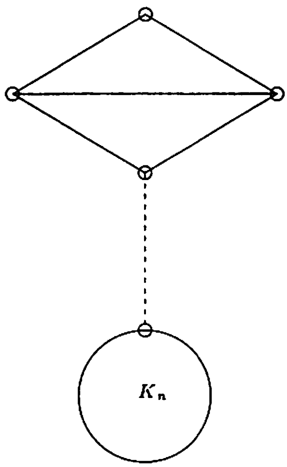
Figure 1: $G(m)$

As mentioned in the Introduction, this result was proven in [4] only for the special case when the clique being attached is K_3. For that case, a converse is proven in [4], which also holds here: if $h(G(m)) = 2$ for some $m > 0$ then $h(G(0)) = 2$ (and thus $h(G(m)) = 2$ for all $m \geq 0$). However, since we do not foresee making any use of this converse, we will not bother to state and prove it formally.

Theorem 2. *The following have Hall number 2:*

(a) any graph obtained by tethering a clique to one of the vertices of degree 2 in $\theta(2,2,1)$, by a path, possibly of length 0;

(b) any graph obtained by tethering a clique to the unique vertex of degree 2 in K_4 with an ear of length 2, by a path, possibly of length 0.



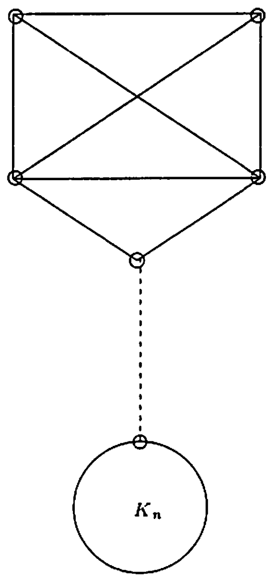


Figure 2

It is shown in [4] that $\theta(2,2,1)$, in Theorem 2(a), cannot be replaced by $\theta(m,2,1)$ for any $m \geq 3$. If $\theta(2,2,1)$ is replaced by any such $\theta(m,2,1)$, and the order of the tethered clique is three, then the resulting graph is Hall-2^+-critical.

For integers $k \geq 2$ and $a_1, \ldots, a_k \geq 1$ (where, for $k = 2$, $a_1 + a_2 \geq 3$), we define the partial wheel graph $W(a_1, \ldots, a_k)$ to be the graph obtained by making a vertex outside the cycle C_m, $m = \sum_{j=1}^{k} a_j$, adjacent to k vertices on the cycle in such a way that the lengths of the paths around the cycle between the vertices of degree 3, in one orientation of the cycle, are $a_1, \ldots, a_k$. Thus, $W(a_1, \ldots, a_k) = W(a_k, \ldots, a_1) = W(a_k, a_1, \ldots, a_{k-1})$ and, when $k = 2$, $W(a_1, a_2) = \theta(a_1, a_2, 2)$. Notice also that $W(1,1,1) = K_4$, and $W(1,1,1,1)$ is the graph in Theorem C(f).

Theorem 3. *The partial wheel graphs which are Hall-2^+-critical are:*

(a) $W(a_1, a_2)$, $a_1 \geq 4$, $a_2 \geq 3$;

(b) $W(a_1, a_2, 1)$ *for any* (a_1, a_2) *except* $(1,1)$; *and*

(c) $W(1,1,1,1)$.

The partial wheel graphs $W(a_1, a_2, a_3)$, $a_1 \geq a_2 \geq a_3 \geq 2$, *and* $W(a_1, \ldots, a_k)$, $k \geq 4$, *all have Hall number greater than 2, but only* $W(1,1,1,1)$, *among these, is Hall-2^+-critical.*

For integers $k \geq 2, r_1, \ldots, r_k, a_1, \ldots . a_k \geq 1$, let $WL(r_1, \ldots, r_k; a_1, \ldots, a_k)$ denote the "partial-wheel-like" graph obtainable from $W(a_1, \ldots, a_k)$ by replacing the "radial" edge that strikes between the arcs of lengths a_{j-1}, a_j (subtraction is mod k) by a path of length $r_j, j = 1, \ldots, k$. Thus $WL(1, \ldots, 1; a_1, \ldots, a_k) = W(a_1, \ldots, a_k)$.

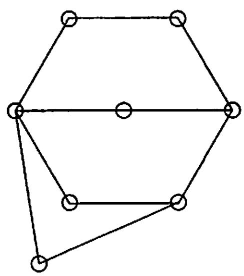
Theorem 4. *The following are Hall-2^+-critical (all a_i, r_j are understood to be positive integers):*

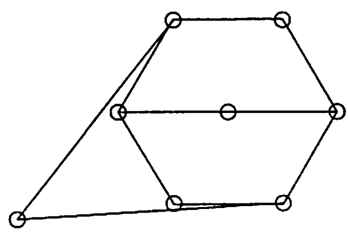
(a) $WL(r_1, r_2, r_3; 1, 1, 1), r_1 \geq 2$;

(b) $WL(r, 1, 1; a_1, 1, a_3), r \geq 2, a_1 + a_3 \geq 3$;

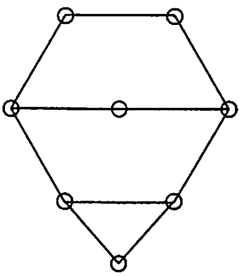
(c) $WL(r, 1, 1; 1, a, 1), r \geq 2, a \geq 2$;

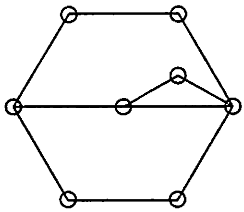
(d) $WL(2, 2, 1; 1, 2, 2)$;

(e) 

(f) 

(g) (h)





The graphs in (e)-(h), above, are obtained by fastening an "ear" of length two in various ways to $\theta(3,3,2)$. In each case, if a longer ear were attached, the resulting graph would have Hall number > 2, but would not be Hall-2^+-critical, since it would have as a proper induced subgraph one of the other Hall-2^+-critical theta graphs. See Theorem C(b). The only thing these graphs have to do with the partial-wheel-like graphs is that we stumbled upon them while struggling with the latter.

There are three other graphs obtainable by attaching an ear of length 2 to $\theta(3,3,2)$. Of these, two have proper Hall-2^+-critical induced subgraphs, and the third is $\theta(3,3,2,2)$, already known to be Hall-2^+-critical (Theorem C(d)).

For $k \geq \chi(G)$, the restricted Hall number $h_k(G)$ is defined as the Hall number is defined, except that the lists are formed from a stock of only k colors. It is straightforward to see that $h_k(G) \leq h_{k+1}(G) \leq h(G)$, and that $h_k(G) \leq k$. There is a good deal of information in [8] about the restricted Hall numbers, especially of the theta graphs. We mention the restricted Hall numbers here to alert those interested to the fact that the proof of Theorem 3 shows that $h_3(W(a_1,a_2,1)) > 2$ if $(a_1,a_2) \neq (1,1)$ or $(2,1)$ or $(1,2)$, and, therefore, that $h_3(W(a_1,a_2,1)) = 3 = h(W(a_1,a_2,1)) = c(W(a_1,a_2,1))$ in these cases (by Brooks' theorem, again). The case of $(2,1,1)$ is singular; we will see that $h_4(W(2,1,1)) > 2$, and brute force checking shows that $h_3(W(2,1,1)) = 2$. Meanwhile, the proof in [4] that $h(W(1,1,1,1)) > 2$ shows that $h_4(W(1,1,1,1)) > 2$. Again, brute force checking shows that $h_3(W(1,1,1,1)) = 2$.

Of the Hall-2^+-critical graphs G listed in Theorem 4, all have $h_3(G) > 2$ except $G = WL(2,1,1;1,1,1)$. For this G we have $h_4(G) > 2$ and brute force checking shows that $h_3(G) = 2$.

4 Proofs and intermediate results

Lemma 1. *Suppose that $h(G_o) = m$ and that G is obtained by attaching a clique of order m to a vertex of G_o. Then $h(G) = m$.*

Proof: Since G_o is an induced subgraph of G, $m = h(G_o) \leq h(G)$.

Now suppose that L is a list assignment satisfying Hall's condition with G, and $|L(z)| \geq m$ for all $z \in V(G)$. Let v be the vertex of G_o at which the clique $K = K_m$ is attached. Since $h(G_o) = m$, there is a proper L-coloring ϕ of G_o. Each of the lists $L'(z) = L(z) \setminus \{\phi(v)\}$, $z \in V(K) \setminus \{v\}$, has at least $m-1$ elements; since the choice number of the $(m-1)$-clique $K - v$ is $m-1$, $K - v$ has a proper L'-coloring. Putting this together with ϕ results in a proper L-coloring of G. Thus $h(G) \leq m$. □

Corollary 1. *Attaching a pendant vertex to a graph with Hall number* 2 *results in a graph with Hall number* 2.

Corollary 2. *Attaching a tree at one vertex of a graph with Hall number* 2 *results in a graph with Hall number* 2.

Definition: A subgraph H of G is L-tight if and only if the inequality (1) is an equality, i.e. $|V(H)| = \sum_{\sigma \in C} \alpha(\sigma, L, H)$. Observe that if H is L-tight, then in every proper L-coloring of H, each $\sigma \in C$ appears as a color exactly $\alpha(\sigma, L, H)$ times on the vertices of H.

We will use $< >$ or $< >_G$ to stand for "the subgraph induced (in G) by ..." The following is a special case of a lemma from [1].

Lemma 2. *Suppose that G and L satisfy Hall's condition, K is a clique in G, $\tau \in L(K)$, and removing τ from every $L(v)$, $v \in K$, on which it appears results in a new list assignment which does not satisfy Hall's condition with G. Then there is an L-tight induced subgraph H of G such that each maximum independent set of vertices in $< \{u \in V(H);\ \tau \in L(u)\} >_H$ contains a vertex of K.*

Proof: Let L' denote the new list assignment, i.e. $L' = L$ on $V(G) \setminus V(K)$ and $L'(v) = L(v) \setminus \{\tau\}$ for each $v \in V(K)$. Let H be an induced subgraph of G such that $\sum_{\sigma \in C} \alpha(\sigma, L', H) \leq |V(H)| - 1$. That H fulfills the claims of the Lemma follows from the observations that $|V(H)| \leq \sum_{\sigma \in C} \alpha(\sigma, L, H)$, that $\alpha(\sigma, L, H) = \alpha(\sigma, L', H)$ for all $\sigma \in C \setminus \{\tau\}$, and that $\alpha(\tau, L, H) \leq \alpha(\tau, L', H) + 1$, with equality only if each maximum independent set of vertices among those vertices of H with τ on their L-lists contains a vertex of K. □

Lemma 3. *Suppose that G is a clique, G and L satisfy Hall's condition, and H, K are L-tight subcliques of G. Then $H \cap K$ and $< H \cup K >_G$ are L-tight.*

Remark: in case H and K have no vertices in common, the conclusion applies only to $< H \cup K >_G$.

Proof: Recall that for a clique K, $\sum_{\sigma \in C} \alpha(\sigma, L, K) = |L(K)| = |\bigcup_{v \in V(K)} L(v)|$. We have, by the assumed L-tightness of H and K, and the assumption that G and L satisfy Hall's condition,

$$
\begin{aligned}
|V(H) \cup V(K)| + |V(H) \cap V(K)| &= |V(H)| + |V(K)| \\
&= |L(H)| + |L(K)| \\
&= |L(H) \cup L(K)| + |L(H) \cap L(K)| \\
&\geq |L(< H \cup K >)| + |L(H \cap K)| \\
&\geq |V(H) \cup V(K)| + |V(H) \cap V(K)|,
\end{aligned}
$$

which implies equality throughout, and, in particular, equality in each of the inequalities $|L(< H \cup K >)| \geq |V(H) \cup V(K)|$ and $|L(H \cap K)| \geq |V(H) \cap V(K)|$, which are special cases of (1). □

The proof of Lemma 3 proves more, namely that $L(H) \cap L(K) = L(H \cap K)$, under the hypotheses of the Lemma. That is, any symbol appearing on a list on H, and also on a list of K, must appear on a list on a common vertex of H and K. However, we shall make no use of this extra conclusion here.

Suppose that P is a path with vertices $v_o, \ldots, v_m$, $m \geq 1$, in order along the path; suppose L is a list assignment to P, and $\sigma_o \in L(v_o)$, $\sigma_m \in L(v_m)$. We will say that σ_o at v_o *forces σ_m at v_m along P* if and only if there is a proper L-coloring of P with v_o colored σ_o, and for every such coloring, v_m is colored σ_m. The following lemma. easily provable by induction on m, also appears in [4] and in [8].

Lemma 4. *Let P, $v_o, \ldots, v_m$, L, $\sigma_o \in L(v_o)$, and $\sigma_m \in L(v_m)$ be as above, and suppose that $|L(v_i)| \geq 2$, $i = 1, \ldots, m$. Then σ_o at v_o forces σ_m at v_m along P if and only if there exist $\sigma_1, \ldots, \sigma_{m-1}$ such that $L(v_i) = \{\sigma_{i-1}, \sigma_i\}$, $i = 1, \ldots, m$.*

Corollary 3. *If P, $v_o, \ldots, v_m$, L, $\sigma_o \in L(v_o)$, and $\sigma_m \in L(v_m)$ are as above, if $|L(v_i)| \geq 2$, $i = 1, \ldots, m$, and if σ_o at v_o forces σ_m at v_m along*

P, then no $\sigma_o' \in L(v_o)$ different from σ_o forces σ_m at v_m along P.

Lemma 5. *Suppose that G and L satisfy Hall's condition, and L' is obtained by replacing some particular $\beta \in C$ on some of the L-lists on G in which it appears by another symbol $\tau \in C \setminus L(G)$. Then G and L' satisfy Hall's condition.*

Proof: The fact that τ did not formerly appear on any list before the replacement implies that for any subgraph H of G, $\alpha(\tau, L, H) = 0$ and $\alpha(\beta, L, H) = \alpha(\beta, L, H) + \alpha(\tau, L, H) \leq \alpha(\beta, L', H) + \alpha(\tau, L', H)$. Meanwhile, for any $\sigma \in C \setminus \{\beta, \tau\}$, $\alpha(\sigma, L, H) = \alpha(\sigma, L', H)$. The conclusion follows. □

Proof of Theorem 1. We proceed by induction on the order n of the clique being tethered to G. Corollary 2 implies that for $n = 1$ or 2 there is nothing to prove, so assume $n \geq 3$, $h(G(0)) = 2$, and $m \geq 1$. Suppose L is a list assignment to $G(m)$ satisfying Hall's condition with $G(m)$ and $|L(z)| \geq 2$ for all $z \in V(G(m))$. We suppose that there is no proper L-coloring of $G(m)$. Let the vertices of the tethering path P be $v_o, \ldots, v_m$, going along P from G out to the clique K_n. Let $K = K_n - v_m$, an $(n-1)$-clique.

By Corollary 2, $G(m) - K$ is properly L-colorable. Let φ be any proper L-coloring of $G(m) - K$, and let $\beta = \varphi(v_m)$. Let L' be the list assignment to K obtained by removing β from all lists on which it appears. Since $G(m)$ is not properly L-colorable, it must be that K is not properly L'-colorable. Since K is a clique, it must be that K and L' do not satisfy Hall's condition. Since K and L do satisfy Hall's condition, by Lemma 2 there is an L-tight subclique $H(\beta)$ of K with β appearing somewhere on its lists.

By Lemma 3 the subclique K' of K induced by the union of the L-tight subcliques $H(\varphi(v_m))$, over all proper L-colorings φ of $G(m) - K$, is L-tight itself. Suppose there is a vertex $u \in V(K) \setminus V(K')$. $G(m) - u$ is not properly L-colorable, since a proper L-coloring φ of $G(m) - u$ would properly L-color $G(m) - K$, and K', and yet the color $\varphi(v_m)$ must also appear somewhere as a color on the L-tight clique $H(\varphi(v_m))$; thus φ could not be proper.

But $G(0) - u$ has Hall number 2, being an induced subgraph of $G(0)$ with G as an induced subgraph of it. By the induction hypothesis on n, $G(m) - u$ ought to have Hall number 2, and thus a proper L-coloring.

We conclude that there is no $u \in V(K) \setminus V(K')$, that is, that $K = K'$. So K itself is L-tight.

Let $A = L(K) \cap L(v_m)$. Since K_n and L satisfy Hall's condition, and $K = K_n - v_m$ is L-tight, there must be some color $\tau \in L(v_m) \setminus A$. In fact, there is exactly one such color: if there were two, then we could follow a proper L-coloring of $G(m) - K_n$ with a proper L-coloring of K, and put one of two colors in $L(v_m)$ not in $L(K)$ on v_m, to obtain a proper L-coloring of $G(m)$.

So $L(v_m) = \{\tau\}\dot{\cup}A$ (the dot indicates disjoint union), and it must be that for every proper L-coloring φ of G, $\varphi(v_o)$ at v_o forces τ at v_{m-1} along the path $P - v_m$. [In case $m = 1$, this means that $\varphi(v_o) = \tau$.] By Corollary 3 and Lemma 4, there is a color $\sigma_0 \in L(v_o)$ such that $\varphi(v_o) = \sigma_0$ for every proper L-coloring φ of G; $\sigma_0 = \tau$ if $m = 1$ and, if $m > 1$, there exist $\sigma_1, \ldots, \sigma_{m-1} \in C$ such that $L(v_i) = \{\sigma_{i-1}, \sigma_i\}$, $i = 1, \ldots, m-1$, and $\sigma_{m-1} = \tau$. By Lemma 5, or really by its proof, we may assume that $\sigma_0, \ldots, \sigma_{m-1}$ are distinct, and that $L(K)$ and $L(G(m) - K_n)$ are disjoint: After renaming $\sigma_0, \ldots, \sigma_{m-2}$ so that they are distinct and not in $L(K) \cup \{\tau\}$ (we already have $\sigma_{i-1} \neq \sigma_i$, $i = 1, \ldots, m-1$, since $|L(v_i)| \geq 2$), rename σ_0 throughout G with its new name at v_o, and then rename any $\sigma \in L(K)$ that appears in $L(G)$ with a new name, throughout G. After all this, we still have that in every proper L-coloring of G, v_o is colored σ_0, and σ_0 at v_o forces $\sigma_{m-1} = \tau$ at v_{m-1} along $P - v_m$; by Lemma 5, $G(m)$ and L still satisfy Hall's condition, yet $G(m)$ is not properly L-colorable.

Since $|L(v_o)| \geq 2$, $L(v_o)$ contains a color other than σ_0. We claim that $L(v_o)$ contains only one such color. Otherwise, let L' be the list assignment to G obtained by removing σ_0 from $L(v_o)$ and leaving all other L-lists as they are. Then we have $|L'(z)| \geq 2$ for all $z \in V(G)$ (assuming $L(v_o)$ contains at least two colors other than σ_0), and G is not properly L'-colorable (since a proper L'-coloring would be a proper L-coloring with a color other than σ_0 on v_o). Since G has Hall number 2, it follows that G and L' do not satisfy Hall's condition. By Lemma 2 there is an L-tight induced subgraph $\tilde{H}$ of G such that v_o is in every maximum independent set of vertices of $\tilde{H}$, among those with σ_0 on their L-lists.

Let $H = \langle \tilde{H} \cup P \cup K_n \rangle$. Since $\sigma_0, \ldots, \sigma_{m-1}$ are distinct and not in $L(K)$, and v_o is in every maximum independent set of vertices in $\tilde{H}$, among those with σ_0 on their L-lists, we have that $\alpha(\sigma_0, L, H) = \alpha(\sigma_0, L, \tilde{H})$; to see this, observe that $\langle \{u \in V(H); \sigma_0 \in L(u)\} \rangle$ is obtainable from $\langle \{u \in V(\tilde{H}); \sigma_0 \in L(u)\} \rangle$ by adding the pendant vertex v_1, attached by an edge to v_o, and the fact that every maximum independent set of vertices in the smaller graph contains v_o implies that the vertex independence numbers of the two graphs are equal.

By the tightness of $\tilde{H}$ and K we have that

$$\begin{aligned}\sum_{\sigma\in C}\alpha(\sigma,L,H) &= \sum_{\sigma\in L(G(m)-K_n)}\alpha(\sigma,L,H)+\sum_{\sigma\in L(K)}\alpha(\sigma,L,H)\\ &= \sum_{\sigma}\alpha(\sigma,L,\tilde{H})+m-1+|V(K)|\\ &= |V(\tilde{H})|+|V(K)|+m-1\\ &< |V(\tilde{H})|+|V(K)|+m=|V(H)|.\end{aligned}$$

[Remark: $L(v_m)=\{\sigma_{m-1}\}\dot{\cup}A$, $A\subseteq L(K)$, implies that $L(G)=L(G(m)-K_n)\cup L(K)$.] That is, Hall's condition is not satisfied.

We conclude that $L(v_o)=\{\sigma_0,\beta\}$ for some $\beta\in C$. Let us define the list assignment L_0 on $G(0)$ by $L_0=L$ on $V(G(0))\setminus\{v_o\}$ and $L_0(v_o)=A\dot{\cup}\{\beta\}$. Then $|L_0(z)|\geq 2$ for all $z\in V(G(0))$. (Note that $\beta\notin A\subseteq L(K)$ because arrangements were made so that $L(K)$ and $L(G(m)-K_n)$ are disjoint; furthermore, $L(v_m)=\{\tau\}\dot{\cup}A$ implies $|A|\geq 1$).

If $G(0)$ were properly L_0-colorable with v_o colored β, then G would be properly L-colorable with v_o colored β, which is not the case. On the other hand, every proper L-coloring of K uses every color in $L(K)$, so no color of A could be used on v_o in a proper L_0-coloring of $G(0)$. Therefore, $G(0)$ is not properly L_0-colorable. Since $G(0)$ has Hall number 2, by hypothesis, it follows that there is an induced subgraph H_0 of $G(0)$ such that $\sum_{\sigma\in C}\alpha(\sigma,L_0,H_0)<|V(H_0)|$. If $v_o\notin V(H_0)$, then H_0 can be thought of as a subgraph of $G(m)$, with the same list assignment, namely L. Therefore, $v_o\in V(H_0)$, because $G(m)$ and L satisfy Hall's condition.

Define $H=<V(H_0)\cup V(P)>_{G(m)}$. Then $|V(H)|=|V(H_0)|+m$, and $\sum_{\sigma\in C}\alpha(\sigma,L,H)=\sum_{\sigma\in C}\alpha(\sigma,L_0,H_0)+m$. (The occurrences of $\sigma_0,\ldots,\sigma_{m-1}=\tau$ along the path add m to the "α-sum", and, since arrangements have been made in the original assignment so that $L(K)\cap L(G)=\emptyset$, the colors of A in $L(v_m)$ contribute no more to the α-sum than they did as elements of $L_0(v_o)$.) Thus $\sum_{\alpha\in C}\alpha(\sigma,L,H)<|V(H)|$, so $G(m)$ and L do not satisfy Hall's condition, after all. This contradiction establishes the result. □

The preceding proof nowhere uses the assumption that the lists on $K=K_n-v_m$ have cardinality ≥ 2. The slightly stronger result actually proven is awkward to state, and does not advance our current project, so we will not bother the reader with it, beyond this note.

Proof of Theorem 2 By Theorem 1, it suffices to prove that the graph has Hall number 2 when the tethering path has length zero. We will consider both cases together, with the graphs as in Figure 3.

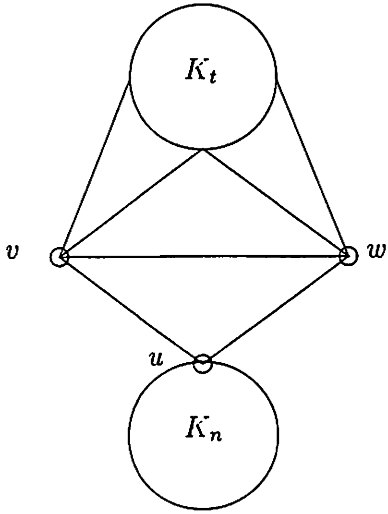


Figure 3: $t = 1$ or 2

Let G be one of the graphs depicted in Figure 3 and suppose that L is a list assignment to G satisfying Hall's condition, and $|L(z)| \geq 2$ for all $z \in V(G)$. Suppose that there is no proper L-coloring of G.

Let $K = K_n - u$, a clique of order $n-1$. Going by induction on n, as in the first part of the proof of Theorem 1, we can assume that $n \geq 3$ and, further, deduce that K is L-tight. We also have that in every proper L-coloring of $G - K$, u is colored with some element of $L(K)$.

Because K is L-tight, for (1) to hold with $H = K_n = < K \cup \{u\} >$, it must be that $L(u) \setminus L(K)$ is non-empty. We claim that $|L(u) \setminus L(K)| = 1$. If not, then $L'(u) = L(u) \setminus L(K)$ has at least two elements. Define $L' = L$ on $G - K_n$ and we have a list assignment L' to $G - K$ satisfying $|L'(z)| \geq 2$ for all $z \in V(G-K)$, such that there is no proper L'-coloring of $G - K$. (If there were such a coloring, it would constitute a proper L'-coloring of $G - K$ with u colored with a symbol not in $L(K)$.) Since $G - K$ has Hall number 2, by Theorem B, (b) and (e), it must be that $G - K$ and L' do not satisfy Hall's condition. Let H' be an induced subgraph of $G - K$ such that $\sum_{\sigma \in C} \alpha(\sigma, L', H') < |V(H')|$. Let $H = < H' \cup K >_G$. Observe that for $\sigma \in C \setminus L(K)$, $\alpha(\sigma, L, H) = \alpha(\sigma, L', H')$, and for $\sigma \in L(K)$, $\alpha(\sigma, L, H) = \alpha(\sigma, L', H') + 1$. Therefore,

$$\begin{aligned}\sum_{\sigma \in C} \alpha(\sigma, L, H) &= \sum_{\sigma \in C \setminus L(K)} \alpha(\sigma, L', H') + \sum_{\sigma \in L(K)} \alpha(\sigma, L', H') + |L(K)| \\ &= \sum_{\sigma \in C} \alpha(\sigma, L', H') + |V(K)| \qquad (K \text{ is } L\text{-tight}) \\ &< |V(H')| + |V(K)| = |V(H)|,\end{aligned}$$

contradicting the assumption that G and L satisfy Hall's condition.

Thus $L(u) \setminus L(K) = \{\tau\}$, for some $\tau \in C$. Since u can never be colored with τ in any proper L-coloring of $G - K$, it must be that in any proper L-coloring of the clique $G - K_n$ (which is either K_3 or K_4), τ must appear as a color on either v or w. Therefore, if we modify the list assignment to $G - K_n$ by removing τ from $L(v)$ and $L(w)$, $G - K_n$ is not properly colorable from the new lists. Since $G - K_n$ is a clique, it follows that $G - K_n$ and the new list assignment fail to satisfy Hall's condition. By Lemma 2, there is an L-tight subclique $\tilde{H}$ of $G - K_n$ such that no vertex in $V(\tilde{H}) \setminus \{v, w\}$ has τ on its L-list; and, certainly, $\tau \in L(\tilde{H} \cap < v, w >)$.

Let $H =< \tilde{H} \cup K_n >$. Then for $\sigma \in C \setminus (L(K) \cup \{\tau\})$, $\alpha(\sigma, L, H) = \alpha(\sigma, L, \tilde{H})$, $\alpha(\tau, L, H) = \alpha(\tau, L, \tilde{H})$, and, for $\sigma \in L(K)$, $\alpha(\sigma, L, H) = \alpha(\sigma, L, \tilde{H}) + 1$. Thus

$$\begin{aligned}\sum_{\sigma \in C} \alpha(\sigma, L, H) &= \sum_{\sigma \in C \setminus L(K)} \alpha(\sigma, L, \tilde{H}) + \sum_{\sigma \in L(K)} \alpha(\sigma, L, \tilde{H}) + |L(K)| \\ &= \sum_{\sigma \in C} \alpha(\sigma, L, \tilde{H}) + |V(K)| \\ &= |V(\tilde{H}| + |V(K)| \\ &= |V(\tilde{H}| + |V(K_n)| - 1 = |V(H)| - 1,\end{aligned}$$

contradicting the assumption that G and L satisfy Hall's condition. This contradiction descends from the assumption that G has no proper L-coloring, and establishes the result. □

As in Theorem 1, something more has been proven than was stated. With K as in the proof, for the existence of a proper L-coloring it suffices that Hall's condition be satisfied, and that all the lists on $G - K$ have cardinality at least two.

Proof of Theorem 3. The claim in (a) follows from Theorem B, (b), (c), and (d), Theorem C (b), and the remark that $W(a_1, a_2) = \theta(a_1, a_2, 2)$.

That $W(1, 1, 1, 1)$ is Hall-2^+-critical is the claim of Theorem C (f). Every other $W(a_1, \ldots, a_k)$, $k \geq 4$, contains a proper induced subgraph of one of the following forms:

(i) $\theta(m_1, m_2, 1)$, $m_1 \geq m_2 \geq 3$, which is Hall-2^+-critical according to Theorem C (b); or

(ii) one of the graphs described in Theorem C (e) (the cycle on which the triangles are based may be a triangle itself).

The partial wheel graphs $W(a_1, a_2, a_3)$, $(a_1, a_2, a_3) \neq (1, 1, 1)$, remain to be classified. If $a_1 \geq a_2 \geq a_3 \geq 2$ then $W(a_1, a_2, a_3)$ has induced subgraphs of the form $\theta(m_1, m_2, 1)$, $m_1 \geq m_2 \geq 3$; thus $W(a_1, a_2, a_3)$ has Hall number greater than 2, but is not Hall-2^+-critical.

To show that $G = W(a_1, a_2, 1)$, $(a_1, a_2) \neq (1, 1)$, is Hall-2^+-critical, it suffices to show that the Hall number of the graph is greater than 2, since by Corollary 2, Theorem A, and Theorem B, (a) and (b), removing any vertex from G results in a graph with Hall number ≤ 2. We shall show that $h(G) > 2$ in each case by describing a list assignment L satisfying $|L(z)| \geq 2$ for all $z \in V(G)$, such that G and L satisfy Hall's condition, yet there is no proper L-coloring of G. We shall leave to the reader the verification of Hall's condition, and of the non-colorability of G, in each case. The non-colorability will be obvious, but verifying Hall's condition will be a chore. In each case, however, the task of checking (1) for every induced subgraph H of G can be accomplished by verifying that $G - v$ is properly L-colorable for each $v \in V(G)$, and then verifying that (1) holds when $H = G$.

The assignments to $W(2, 1, 1)$ and $W(2, 2, 1)$ are given in Figure 4, with numbers representing colors, and lists given in "word", rather than "set", form.

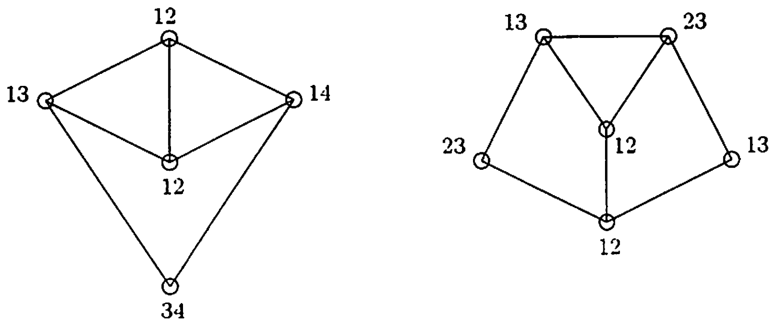


Figure 4: $W(2, 1, 1)$ and $W(2, 2, 1)$

A propos the remarks on the restricted Hall numbers at the end of section 3, observe that the list assignment to $W(2, 1, 1)$ used 4 colors, while the assignment to $W(2, 2, 1)$ uses 3. The remaining assignments in this proof will use only 3 colors.

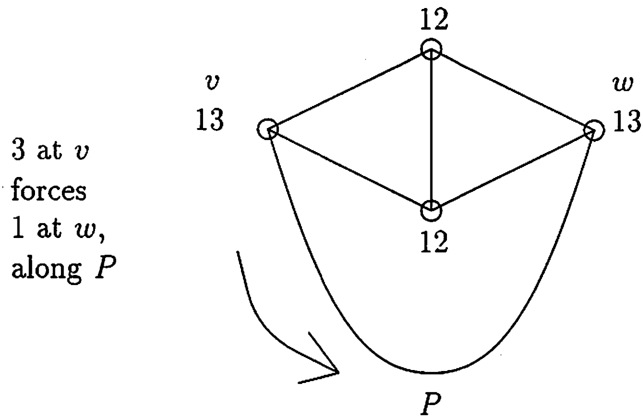


Figure 5: $W(a,1,1), a \geq 3$

In Figure 5, dealing with the cases $W(a,1,1)$, $a \geq 3$, the path P from v to w has length a, and an assignment is to be made to the internal vertices of P so that 3 at v forces 1 at w, along P, and so that Hall's condition is satisfied. If a is odd, let 23 be assigned to the internal vertices of P. If a is even, $a \geq 4$, let the next vertex after v along P be assigned 23, the next vertex after that 12, and then the remaining internal vertices of P are assigned 13.

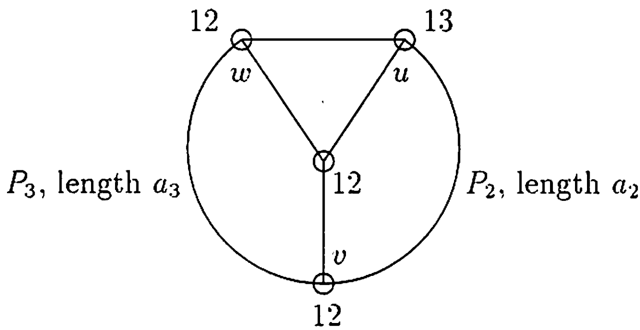


Figure 6: $W(1,a_2,a_3), 2 \leq a_2 \leq a_3, 3 \leq a_3$

Figure 6 depicts the remaining cases. $W(1,a_2,a_3)$, $2 \leq a_2 \leq a_3$, $3 \leq a_3$. The plan here is to assign lists of length 2 to the internal vertices of the paths P_i, of lengths a_i, $i = 2,3$, so that Hall's condition is satisfied and 3 at u forces 1 at v, along P_2, and 1 at v forces 2 at w, along P_3.

If a_2 is odd, assign 13 to each internal vertex of P_2 except the last before v, to which 12 is assigned. (Note $a_2 \geq 3$.) If a_2 is even, assign 23 to the next vertex of P_2 after u, and then 12 to the remaining internal vertices of P_2, if any (i.e., if $a_2 \geq 4$).

If a_3 is odd, $a_3 \geq 3$, assign 13 to each of the two vertices of P_3 immediately following v, and then 12 to the remainder of P_3. If a_3 is even, $a_3 \geq 4$, assign 13 to the next vertex of P_3 after v, then 23 to the vertex after that, and then 12 to the remaining internal vertices of P_3. □

Proof of Theorem 4. We leave it to the reader to verify that removing any vertex from any of the graphs claimed to be Hall-2^+-critical results in a graph with Hall number ≤ 2. Apply Theorems A and B, and Corollary 2.

To prove Hall-2^+-criticality, therefore, it suffices to supply list assignments, with lists of "length" ≥ 2, satisfying Hall's condition, from which no proper coloring is possible. As in the proof of Theorem 3, we shall supply the list assignments and leave the verification of the satisfaction of Hall's condition, and of non-colorability, to the reader. The remarks in the proof of Theorem 3 on the former verification are applicable here.

For case (a), we may as well suppose that $r_1 \geq r_2 \geq r_3, r_1 \geq 2$. This case breaks into several subcases.

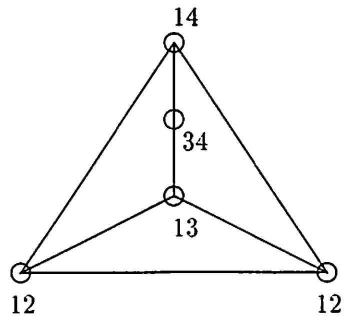


Figure 7: $WL(2,1,1;1,1,1)$

The list assignment in Figure 7 uses four symbols. As claimed earlier, four is the smallest number of symbols permitting such a list assignment. From here on, all list assignments will use only three symbols.

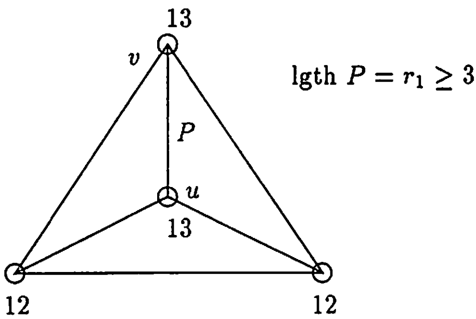


Figure 8: $WL(r_1, 1, 1; 1, 1, 1), r_1 \geq 3$

With reference to Figure 8: if r_1 is odd, put 23 at every internal vertex of P; otherwise, if $r_1 \geq 4$ is even, let the internal vertices of P, from u to v, be assigned 13, then 12, then all 23.

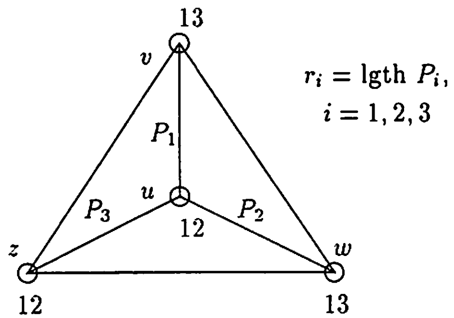
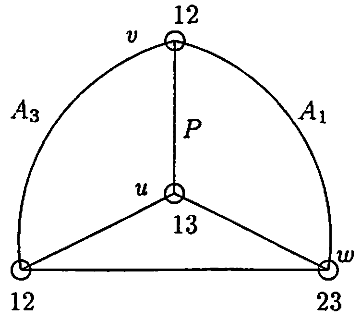


Figure 9: $WL(r_1, r_2, r_3; 1, 1, 1), r_1 \geq r_2 \geq r_3, r_2 \geq 2$.

With reference to Figure 9, assign 12 to the internal vertices of P_3, if any. If r_3 is odd, then assign to the internal vertices of P_i, $i = 1, 2$, in order along the path from u, either 12 and then all 23, if r_i is odd ($r_1 \geq 3$), or, if r_i is even, 13 and then all 23.

If r_3 is even, then assign to the internal vertices. of P_i, $i = 1, 2$, in order along the path from u, either 12, 13, and then all, 23, if r_i is odd, or, if r_i is even, all 23.

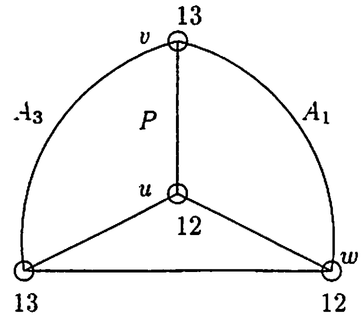
The proof of (b) also breaks into cases. We may as well assume that $a_1 \geq a_3$ and therefore that $a_1 \geq 2$.



lgth $P = r \geq 2$
lgth $A_i = a_i, i = 1, 3,$
$a_1 \geq 2$

Figure 10: $WL(r, 1, 1; a_1, 1, a_3), r \geq 2, a_1 \geq a_3, a_1 \geq 2, a_3$ odd

With reference to Figure 10, assuming a_3 is odd, put 12 on the internal vertices of A_3, if any (i.e., if $a_3 \geq 3$). Lists are assigned to the internal vertices of P, and A_1 as follows. If r is even, put 23 on the internal vertices of P; if r is odd, put 13, 12, and then all 23, on the internal vertices of P, going from u to v. If a_1 is even, put 13 on the internal vertices of A_1; if a_1 is odd, put 23, 12, and then all 13 on the internal vertices of A_1, going from w to v.



lgth $P = r \geq 2$
lgth $A_i = a_i, i = 1, 3$
$a_1 \geq a_3 \geq 2$

Figure 11: $WL(r, 1, 1; a_1, 1, a_3), r \geq 2, a_1 \geq a_3 \geq 2, a_3$ even

With reference to Figure 11, assuming a_3 is even, put 13 on the internal vertices of A_3. Lists are assigned to the internal vertices of P and of A_1 as follows. If r is even, assign 13 to the internal vertices of P; if r is odd, $r \geq 3$, then, going from v to u, assign 23 and then all 12 to the internal vertices of P. Assign lists to the internal vertices of A_1 by the same rules, with a_1 replacing r, w replacing u, and A_1 replacing P in the directions.

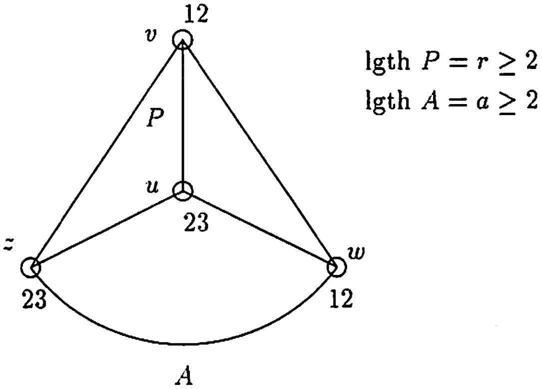


Figure 12: $WL(r,1,1;1,a,1), r,a \geq 2$, r,a not both odd

We prove (c) with reference to Figures 12 and 13. If r is even, assign 13 to each internal vertex of P. If r is odd, assign 12 to the first vertex along P coming from v, and then 23 to the rest. If a is even, assign 13 to the internal vertices of A. If a is odd, assign 23 to the first vertex along A coming from z, and then 12 to the rest.

The assignment described does not satisfy Hall's condition when r and a are both odd. For that case alone, with reference to Figure 13, assign 13 to the internal vertices of P and 23 to the internal vertices of A.

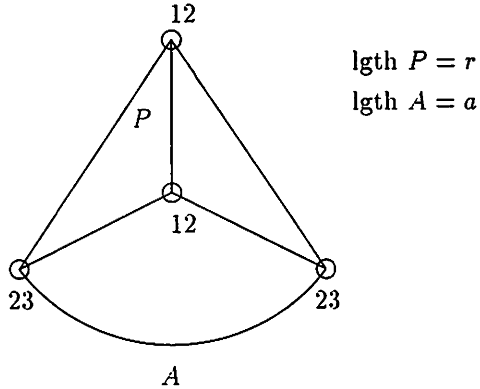


Figure 13: $WL(r,1,1;1,a,1), r,a \geq 3$, both odd

We prove (d)-(h) of Theorem 4 with diagrams alone.

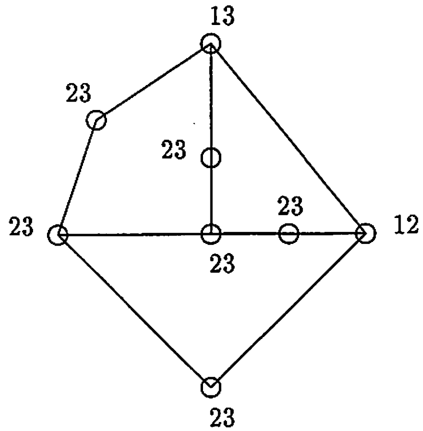


Figure 14: $WL(2,2,1;1,2,2)$

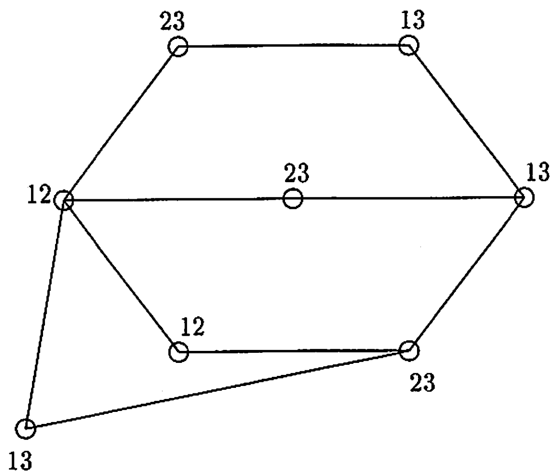


Figure 15: Theorem 4(e)

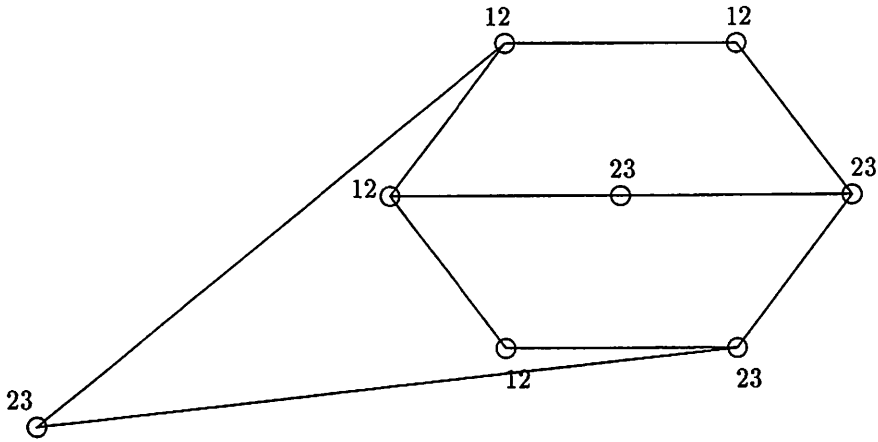


Figure 16: Theorem 4(f)

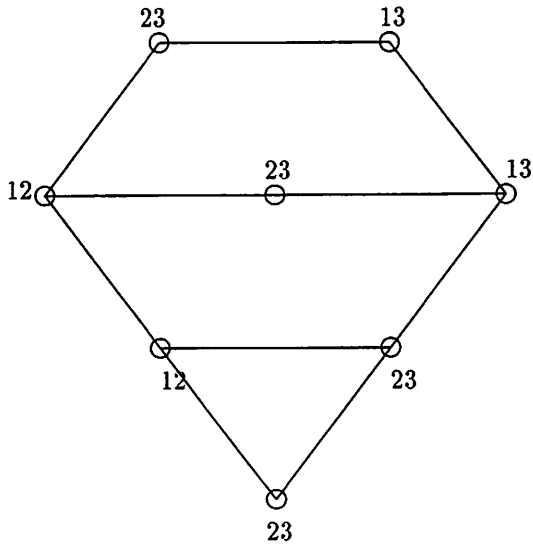


Figure 17: Theorem 4(g)

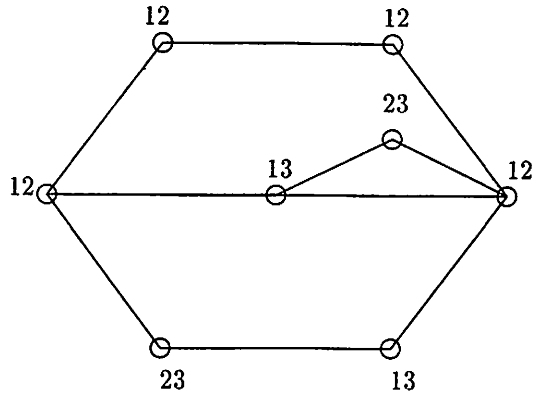


Figure 18: Theorem 4(h)

All that remains is the verification of the last assertion, that the graphs $WL(r_1,\ldots,r_k;a_1,\ldots,a_k)$, $k\geq 3$, $r_1\geq 2$, all have Hall number greater than 2, but that only those mentioned in (a)-(d) are Hall-2^+-critical.

First, suppose that $k=3$. Since $r_1\geq 2$, we can remove vertices from the path with length r_1 to obtain $\theta(a_1+a_3,r_2+r_3,a_2)=H$ as an induced subgraph of $G=WL(r_1,r_2,r_3;a_1,a_2,a_3)$. By Theorems B and C, H is Hall-2^+-critical unless H is one of $\theta(m,2,1)$, $\theta(m,2,2)$, for some $m\geq 2$, or $\theta(3,3,2)$. Thus, unless (a_1+a_3,r_2+r_3,a_2) is some reordering of one of $(m,2,1)$, $(m,2,2)$, for some $m\geq 2$, or $(3,3,2)$, G has a proper Hall-2^+-critical induced subgraph, and so has Hall number greater than 2, but is not Hall-2^+-critical itself.

Now, the analysis of these possibilities is sort of fun, but rather lengthy; we will leave it to the reader to verify that the only $WL(r_1,r_2,r_3;a_1,a_2,a_3)$, $r_1\geq 2$, that do not have a proper Hall-2^+-critical induced subgraph are the graphs in (a)-(d) of Theorem 4. (In carrying out this verification, keep in mind that if any of r_2,r_3,a_1,a_2,a_3 is ≥ 2, you can remove vertices to destroy a path in G, and obtain another proper induced θ-subgraph.)

Suppose now that $k=4$ and $G=WL(r_1,r_2,r_3,r_4;a_1,a_2,a_3,a_4)$, $r_1\geq 2$. Removing the radial path of length r_1, we see that G has $WL(r_2,r_2,r_4;a_2,a_3,a_1{+}a_4)$ as a proper induced subgraph. If $\max(r_2,r_3,r_4)>1$, this has Hall number >2 by the $k=3$ case. Otherwise, because $a_1+a_4\geq 2$, it has Hall number >2 by Theorem 3.

Now suppose that $k\geq 5$, and we proceed by induction on k. Since $r_1>1$ we see that the graph $WL(r_1,\ldots,r_k;a_1,\ldots,a_k)$ has $W(r_2,\ldots,r_k;a_2,\ldots,a_{k-1},a_1+a_k)$ as a proper induced subgraph. If $\max(r_2,\ldots,r_k)>1$, this has Hall number >2 by the induction hypothesis. Otherwise, it has Hall number >2 by Theorem 3. □

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