# Self-complementary Graphs with Minimum Degree Two

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**Abstract**

In this paper, we shall classify the self-complementary graphs with minimum degree exactly 2.

# 1 Introduction

We consider only simple finite undirected graphs. We denote the vertex set and edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. Let $P_n$, $C_n$ and $K_n$ denote the path, the cycle and the complete graph with $n$ vertices, respectively. A graph $G$ is said to be *trivial* if $G$ has just one vertex, and $G$ is *empty* if $G$ has no vertex. For a graph $G$ and $S \subset V(G)$, let $\langle S \rangle$ denote the subgraph of $G$ induced by $S$. For two graphs $H$ and $K$, a graph $G$ is said to be obtained from $H$ and $K$ by *joining* $H$ and $K$ if $G$ is obtained by joining each vertex of $K$ to all vertices of $H$.

For a graph $G$, the *complement*, denoted by $\overline{G}$, of $G$ is defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. A graph $G$ is said to be *self-complementary* if $G$ and $\overline{G}$ are isomorphic. There exists a self-complementary graph with precisely $n$ vertices if and only if $n \equiv 0, 1 \pmod 4$. Since a self-complementary graph $G$ with $n$ vertices satisfies $|E(G)| = |E(\overline{G})|$ and $|E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2}$, the number $\frac{n(n-1)}{2}$ must be even, and hence we have $n \equiv 0, 1 \pmod 4$ [1]. Moreover, for every natural number $n \equiv 0, 1 \pmod 4$ with $n \geq 4$, we can construct self-complementary graphs with precisely $n$ vertices, as follows: Let $H$ be a graph which is either empty, trivial, or self-complementary, and let $P_4 = v_1v_2v_3v_4$. Join each of $v_2$ and $v_3$ to all vertices of $H$. The resulting graph $\tilde{H}$ with $|V(H)|+4$ vertices can easily be checked to be self-complementary. Thus, for each $n \equiv 0, 1 \pmod 4$ with $n \geq 4$, we can inductively construct self-complementary graphs with $n$ vertices from a self-complementary graph with $n-4$ vertices and $P_4$.

The self-complementary graphs constructed as above must have cut vertices. In every self-complementary graph $G$, it is easily checked that $G$ has cut vertices if and only if $G$ has vertices of degree 1. Surprisingly, every self-complementary graph with cut vertices have the above structure, as in the following theorem.

**THEOREM 1 (Kawarabayashi et al. [2])** *Let $G$ be a self-complementary graph with cut vertices. Then, $G$ can be obtained from a graph $H$ and $P_4 = v_1v_2v_3v_4$ by joining each of $v_2$ and $v_3$ to all vertices of $H$, where the graph $H$ is either empty, trivial or self-complementary.*

In this paper, we consider the self-complementary graphs with minimum degree exactly 2 and characterize them. Note that for the self-complementary graphs, the $k$-connectivity and the minimum degree $k$ are not equivalent for any integer $k \geq 2$, as the examples constructed below show. Consider a self-complementary graph obtained from $P_4 = v_1v_2v_3v_4$ by the following procedures: Let $B$ be any graph, and replace each of $v_1$ and $v_4$ by $B$, and each of $v_2$ and $v_3$ by $\overline{B}$, where two vertices $v_i$ and $v_j$ are adjacent in $P_4$ if and only if the corresponding $B$'s or $\overline{B}$'s to $v_i$ and $v_j$ are joined in the resulting graph. Putting $B = K_2$, we obtain the graph in Figure 1.

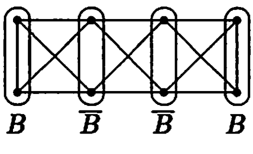


Figure 1: Self-complementary graph with connectivity 2 but no vertex of degree 2

Our main theorem is as follows:

**THEOREM 2** *Let $G$ be a self-complementary graph with minimum degree exactly 2. Then, $G$ has one of the structures of Type I, II(a), II(b), III and IV: (See Figure 2)*

*I* *G is obtained from H and $P_4 = v_1v_2v_3v_4$ by joining each of $v_1$ and $v_4$ to all vertices of H, where the graph H is either trivial or self-complementary,*

*II(a)* *G has precisely 2 vertices of degree 2 and G is obtained from H and $P_4 = v_1v_2v_3v_4$ by joining $v_1$ and $v_4$ to $x_1$ and $y_1$ respectively, joining $v_2$ to all vertices of H except $x'_1$ and joining $v_3$ to all vertices of H except $y'_1$, where H is a self-complementary graph and $x_1, y_1, x'_1, y'_1$ are 4 distinct vertices lying on an induced $P_4$ in H such that $\Psi_H(x_1) = x'_1$, $\Psi_H(y_1) = y'_1$, $\Psi_H(x'_1) = y_1$ and $\Psi_H(y'_1) = x_1$ by some isomorphism $\psi_H : H \to \overline{H}$,*

*II(b)* *G is obtained from H and $P_4 = v_1v_2v_3v_4$ by joining each of $v_1$ and $v_4$ to v and joining each of $v_2$ and $v_3$ to all vertices of H except v, where H is a self-complementary graph and v is a vertex of H such that $\Psi_H(v) = v$ by some isomorphism $\psi_H : H \to \overline{H}$,*

*III* *G is obtained from H by joining $K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$, adding four new vertices $x_1, x_2, x_3$ and $x_4$, and joining each $x_i$ to $v_i$ and $v_{i+1}$ (subscripts are taken modulo 4), where the graph H is either empty, trivial or self-complementary,*

*IV* *G is obtained from H by joining $K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$, adding four new vertices $x_1, x_2, x_3$ and $x_4$, and joining each of $x_1$ and $x_2$ to $v_1$ and $v_2$, and each of $x_3$ and $x_4$ to $v_3$ and $v_4$, where the graph H is either empty, trivial or self-complementary.*

## 2 Proof of the Theorem

In this section, we prove our main theorem. Before proving it, we give several lemmas.

Let $V_i$ and $\overline{V_i}$ denote the sets of vertices of degree $i$ in $G$ and $\overline{G}$, respectively. For a graph $T$ and disjoint subsets $P, Q \subset V(T)$, let $e_T(P, Q)$ be the number of edges $e$ of $T$ such that one of endpoints of $e$ belongs to $P$ and the other to $Q$.

**LEMMA 3** *There is just one self-complementary graph with 4 vertices, which is isomorphic to $P_4$. There are exactly two self-complementary graphs with 5 vertices, which are $C_5$ and $C_3^{++}$ shown in Figure 3.*

**Proof.** Let $G$ be a self-complementary graph with $n$ vertices. If $G$ has a cut vertex, then $G$ is isomorphic to either $P_4$ or $C_3^{++}$, by Theorem 1, when

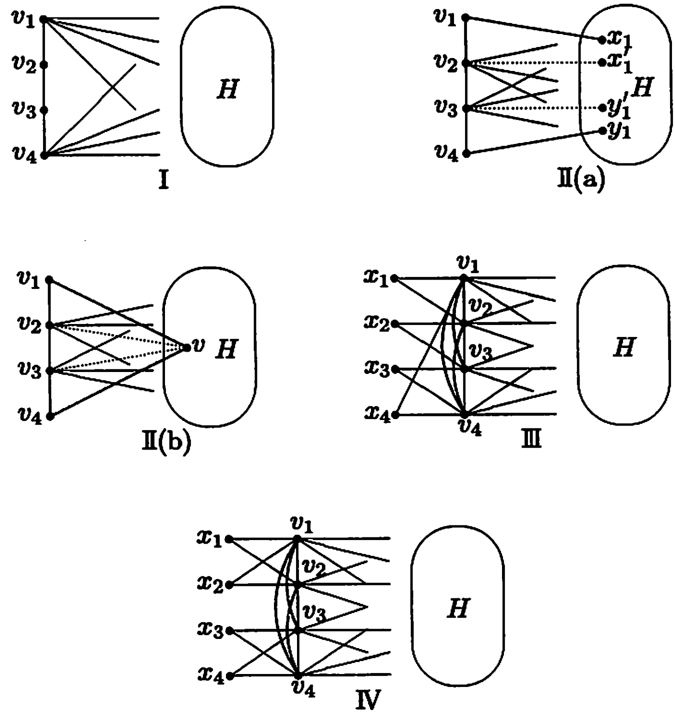


Figure 2: Structures of $G$

$n = 4, 5$. It is easy to see that $|E(G)| = \frac{1}{4}n(n-1)$. Thus, the average degree $\bar{d}(G)$ of $G$ is

$$\bar{d}(G) = \frac{2|E(G)|}{n} = \frac{1}{2}(n-1).$$

Hence, if $n = 4$, then $G$ must have a vertex of degree 1, and if $n = 5$, then either $G$ has a vertex of degree 1 or it is 2-regular. In the latter case, $G$ is isomorphic to $C_5$. ∎

**Lemma 4** *Let $G$ be a self-complementary graph and let $\Psi : G \to \overline{G}$ be an isomorphism. If $S$ is a subset of $V(G)$ such that for each $v \in S$, $\Psi(v) \in S$, then the subgraph $\langle S \rangle$ in $G$ induced by $S$ is either empty, trivial or self-complementary, depending on $|S| = 0, 1$ and $|S| > 1$.*

**Proof.** We may assume $|S| \geq 2$, since the lemma obviously holds when $|S| \leq 1$. Clearly, we have that $|S| = |\{\Psi(s) : s \in S\}|$. By the assumption of


Figure 3: $C_5$ and $C_3^{++}$

the lemma, for any vertices $x, y \in S$, we have $\Psi(x), \Psi(y) \in S$, and moreover, $xy \in E(G)$ if and only if $\Psi(x)\Psi(y) \notin E(G)$, or $xy \notin E(G)$ if and only if $\Psi(x)\Psi(y) \in E(G)$. Thus, we can define the isomorphism $\Psi_{\langle S\rangle} : \langle S\rangle \to \overline{\langle S\rangle}$. ■

**Lemma 5** *Let $G$ be a self-complementary graph with $n$ vertices. Then, for a fixed $i$,*

*(i) If $V_i \cap \overline{V_i} \neq \emptyset$, then $n = 2i+1$, and*

*(ii) If $V_i \cap \overline{V_i} = \emptyset$, then $|V_i|$ is even and $|V_i| \leq 2i$, and moreover, if $|V_i| = 2i$, then $V_i$ is independent in $G$ and the neighbors of each $x \in V_i$ are included in $\overline{V_i}$.*

**Proof.** We first prove (i). If we let $v \in V_i \cap \overline{V_i}$, then $i = \deg_G(v) = \deg_{\overline{G}}(v) = n-1-i$. Therefore, we have $n = 2i+1$.

Secondly we prove (ii). We first show that $|V_i|$ is even. Since $V_i \cap \overline{V_i} = \emptyset$, we have

$$e_G(V_i, \overline{V_i}) + e_{\overline{G}}(V_i, \overline{V_i}) = |V_i|^2. \tag{1}$$

Since $e_G(V_i, \overline{V_i}) = e_{\overline{G}}(V_i, \overline{V_i})$, $|V_i|^2$ must be even. Thus, $|V_i|$ is even.

Now suppose that $|V_i| > 2i$ for contradictions. Let $K$ be the bipartite subgraph of $G$ with partite sets $V_i$ and $\overline{V_i}$ and $E(K) = \{uv : u \in V_i, v \in \overline{V_i}\}$. By (1), we have

$$e_K(V_i, \overline{V_i}) = \sum_{v \in V_i} \deg_K(v) = \frac{1}{2}|V_i|^2.$$

Hence there exists $x \in V_i$ such that

$$\deg_G(x) \geq \deg_K(x) \geq \frac{1}{2}|V_i| > i.$$

This is contrary to the definition of $x \in V_i$. Therefore, we have $|V_i| \leq 2i$.

In the above paragraph, if we put $|V_i| = 2i$, then we must have that for any $x \in V_i$, $\deg_G(x) = \deg_K(x) = i$. Therefore, $V_i$ satisfies the required properties. ■

Now we shall prove Theorem 2.

**Proof of Theorem 2.** Let $G$ be a self-complementary graph with $n$ vertices and minimum degree 2. Let $\Psi : G \to \overline{G}$ be an isomorphism.

By Lemma 5(i), we may suppose that $V_2 \cap \overline{V_2} = \emptyset$ unless $n = 5$. By Lemma 3, there is only one self-complementary graph with at most 5 vertices whose minimum degree is 2, which is $C_5$. Obviously, $C_5$ belongs to Type I in our classification.

By Lemma 5(ii), there are two possibilities for the size of $V_2$. We consider these two cases separately.

<u>Case 1.</u> $|V_2| = 2$.

Let $V_2 = \{x, y\}$, where $x \neq y$. Then we can put $\overline{V_2} = \{x', y'\}$ and suppose that $\Psi$ maps $x$ and $y$ to $x'$ and $y'$, respectively. Let $A = V(G) - \{x, x', y, y'\}$.

<u>Subcase I.</u> $xy \in E(G)$.

Since $\deg_G(x') = \deg_G(y') = n - 3$ and $x'y' \notin E(G)$, both $x'$ and $y'$ are adjacent to all vertices $v \in A$. Since $\deg_G(x'), \deg_G(y') \geq 2$, we have $A \neq \emptyset$. Thus, by Lemma 4, the graph $\langle A \rangle = H$ is either trivial or self-complementary. Therefore, we can find the structure described in I.

<u>Subcase II.</u> $xy \notin E(G)$.

$xy \notin E(G)$ implies $x'y' \in E(G)$. Since $\{x, x', y, y'\}$ induces a self-complementary graph by Lemma 4, the graph induced by $\{x, x'y, y'\}$ must be isomorphic to $K_4$, in which we may put $xx', yy' \in E(G)$ without loss of generality. Then we have $\Psi(x') = y$ and $\Psi(y') = x$. Let $x_1$ and $y_1$ be the unique neighbors of $x$ and $y$ in $A$, respectively. Thus, we have $A \neq \emptyset$.

Let $x_1' = \Psi(x_1)$ and $y_1' = \Psi(y_1)$. Then we have $\Psi(x_1') = y_1$ and $\Psi(y_1') = x_1$. If we let $V = \{x_1, y_1, x_1', y_1'\}$, then $V$ satisfies the assumption of Lemma 4, and hence the graph $\langle V \rangle$ is self-complementary. Since we must have $|V| \equiv 0, 1 \pmod 4$, the size of $V$ is either 4 or 1. In the former case, $x_1, y_1, x_1', y_1'$ are distinct vertices lying on an induced $P_4$ in $H$, by Lemma 3, and the latter case is that $x_1 = y_1 = x_1' = y_1'$. These two cases are described in Type II(a) and II(b), respectively.

<u>Case 2.</u> $|V_2| = 4$.

In this case, by Lemma 5(ii), $V_2$ is independent in $G$, and hence $\overline{V_2}$ induces $K_4$ in $G$. Moreover, all neighbors of each $x \in V_2$ are contained in $\overline{V_2}$.

Let $V_2 = \{x_1, x_2, x_3, x_4\}$ and $\overline{V_2} = \{v_1, v_2, v_3, v_4\}$. Each $x_i$ is adjacent to exactly two vertices in $\overline{V_2}$, and hence each $v_i$ is adjacent to exactly two vertices in $V_2$. Since $\overline{V_2}$ induces $K_4$, we can define a map $\rho : V_2 \to E(K_4)$ such that for each $i = 1, 2, 3, 4$, $x_i$ is adjacent to the two endpoints of $\rho(x_i) \in E(K_4)$. Note that for each $v \in \overline{V_2}$, exactly two $x_i, x_j$ of $V_2$ are mapped to edges incident to $v$. Hence there exist essentially two possibilities of $\rho$;

(a) $E = \{\rho(x_i) : i = 1, 2, 3, 4\}$ forms a cycle of length 4 in $K_4$,

(b) $E = \{\rho(x_i) : i = 1, 2, 3, 4\}$ are independent in $K_4$ (that is, exactly two vertices of $V_2$ are mapped to one edge $e$ in $K_4$, and the other two in $V_2$ are mapped to the edge $e'$ in $K_4$ which is independent of $e$).

If we let $A = V(G) - (V_2 \cup \overline{V_2})$, then $\langle A \rangle$ is either empty, trivial or self-complementary, by Lemma 4. Moreover, since each $p \in \overline{V_2}$ has $n-3$ neighbors in $G$, $p$ is adjacent to all vertices in $A$. Therefore, we can find the structure of Type III and IV in these two cases (a) and (b), respectively. ■

# 3 Observation

In this section, we shall enumerate self-complementary graphs with minimum degree 2 of Type I, III and IV, and construct all self-complementary graphs of Type II(a) and II(b) with 8 and 9 vertices.

Let $\mathcal{N}(n)$ denote the number of the self-complementary graphs with exactly $n$ vertices, and in particular, let $\mathcal{N}(k, n)$ denote the number of such graphs with minimum degree exactly $k$. The number $\mathcal{N}(n)$ has been determined for all possible integers $n$ as in Table 1 [3]:

| $n$ | 4 | 5 | 8 | 9 | 12 | 13 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{N}(n)$ | 1 | 2 | 10 | 36 | 720 | 5600 | 703760 | 11220000 |

Table 1: The number of self-complementary graphs

Clarifying the structure of self-complementary graphs with vertices of degree 1, the following result has been obtained.

**Proposition 6** ([2]) $\mathcal{N}(1,4) = \mathcal{N}(1,5) = 1$. *For all* $n \geq 8$, $\mathcal{N}(1,n) = \mathcal{N}(n-4)$.

We extend this result with respect to the minimum degree. Let $\mathcal{N}_{\cdot}(2, n)$ denote the number of the self-complementary graphs of Type $\cdot$ with $n$ vertices and minimum degree 2.

**PROPOSITION 7** $\mathcal{N}_{\mathrm{I}}(2,4) = 0, \mathcal{N}_{\mathrm{I}}(2,5) = 1$. *For all* $n \geq 8$, $\mathcal{N}_{\mathrm{I}}(2,n) = \mathcal{N}(n-4)$.

**Proof.** By the property of Type I, $G$ includes the self-complementary subgraph $H$ obtained by removing a $P_4$. Clearly, we have $|V(H)| = |V(G)| - 4$. It is easy to see that from $m$ distinct self-complementary graphs as $H$, we can construct $m$ distinct self-complementary graphs of Type I. Thus, for all $n \geq 8$, $\mathcal{N}_{\mathrm{I}}(2,n) = \mathcal{N}(n-4)$. ■

**PROPOSITION 8** *For all positive integers* $n$, $\mathcal{N}_{\mathrm{III}}(2,n) = \mathcal{N}_{\mathrm{IV}}(2,n)$. *Moreover,* $\mathcal{N}_{\mathrm{III}}(2,8) = 0$, $\mathcal{N}_{\mathrm{III}}(2,9) = 1$. *For all* $n \geq 12$, $\mathcal{N}_{\mathrm{III}}(2,n) = \mathcal{N}(2, n-8)$.

**Proof.** Similar to Proposition 7. ■

By Propositions 6, 7 and 8, we can obtain Table 2.

| $n$ | 4 | 5 | 8 | 9 | 12 | 13 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{N}(n)$ | 1 | 2 | 10 | 36 | 720 | 5600 | 703760 | 11220000 |
| $\mathcal{N}(1,n)$ | 1 | 1 | 1 | 2 | 10 | 36 | 720 | 5600 |
| $\mathcal{N}_{\mathrm{I}}(2,n)$ | 1 | 1 | 1 | 2 | 10 | 36 | 720 | 5600 |
| $\mathcal{N}_{\mathrm{III}}(2,n)$ | 0 | 0 | 1 | 1 | 1 | 2 | 10 | 36 |
| $\mathcal{N}_{\mathrm{IV}}(2,n)$ | 0 | 0 | 1 | 1 | 1 | 2 | 10 | 36 |

Table 2: The numbers of various self-complementary graphs

Now we construct self-complementary graphs of Type II(a) and III(b) with 8 and 9 vertices, respectively.

**PROPOSITION 9** *There exist precisely* 2 *self-complementary graphs of Type II(a) with* 8 *vertices (See Figure* 4*). There exists no self-complementary graphs of Type II(b) with* 8 *vertices.*

**Proof.** By the property of Type II, a self-complementary graph $G$ considered here can be constructed from $P_4 = v_1v_2v_3v_4$ and $H$ by connecting suitably. Note that since $P_4$ is a unique self-complementary graph with 4 vertices, $H$ is also $P_4 = u_1u_2u_3u_4$ in this case. Since any isomorphism between $P_4$ and $\overline{P_4}$ fixes no vertex, there exists no self-complementary graphs of Type II(b). Thus, it suffices to choose distinct vertices $x_1, x_1', y_1, y_1'$ in

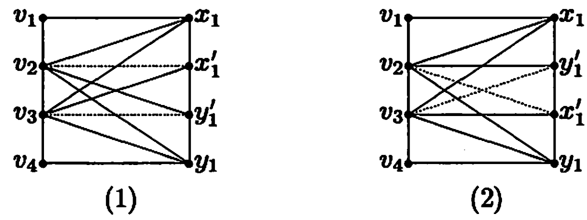


Figure 4: Self-complementary graphs of Type II with 8 vertices

$P_4 = u_1u_2u_3u_4$ so that $\Psi(x_1) = x_1'$, $\Psi(y_1) = y_1'$, $\Psi(x_1') = y_1$ and $\Psi(y_1') = x_1$ by some isomorphism $\Psi_H : H \to \overline{H}$.

Since $\Psi_H : H \to \overline{H}$ is an isomorphism, there are essentially two possibilities; (A) $\{x_1, y_1\} = \{u_1, u_4\}$, (B) $\{x_1, y_1\} = \{u_2, u_3\}$.

In case (A), we have $\{x_1, y_1\} = \{u_1, u_4\}$ and $\{x_1', y_1'\} = \{u_2, u_3\}$. Without loss of generality, we may suppose $x_1 = u_1$ and $y_1 = u_4$. Now we have two choices (1) $x_1' = u_2$ and $y_1' = u_3$, and (2) $x_1' = u_3$ and $y_1' = u_2$. From (1) and (2), we can obtain two required graphs, respectively, as in Figure 4.

In case (B), both $u_1$ and $u_4$ have degree 2, and hence the graphs constructed here have 4 vertices of degree 2. They are of Type III or IV. ■

**PROPOSITION 10** *There exist precisely* 6 *self-complementary graphs of Type II(a) with* 9 *vertices. There exist precisely* 2 *self-complementary graphs of Type II(b) with* 9 *vertices. See Figure* 5.

**Proof.** We proceed similarly to the above proof of Proposition 9. Note that for a candidate for the graph $H$, there are two self-complementary graphs with 5 vertices, which are $C_5$ and $C_3^{++}$ in Figure 3, by Lemma 3. Suppose that $C_5$ and $C_3^{++}$ are labeled as in Figure 3. We first specify 4 distinct vertices $x_1, y_1, x_1', y_1'$ in $H$ so that $x_1' = \Psi(x_1)$, $y_1' = \Psi(y_1)$, $x_1 = \Psi(y_1')$ and $y_1 = \Psi(x_1')$ by some isomorphism $\Psi : H \to \overline{H}$. In the following argument, we neglect the symmetry of $H$.

We first consider Type II(a). From $C_5$, there are essentially two choices of $x_1$ and $y_1$, depending on whether they are adjacent in $C_5$, or not. In the former case, there are two possibilities; (1) $x_1 = u_1$, $y_1 = u_2$, $x_1' = u_3$ and $y_1' = u_5$, and (2) $x_1 = u_1$, $y_1 = u_2$, $x_1' = u_5$ and $y_1' = u_3$. In the latter case, there are two possibilities; (3) $x_1 = u_1$, $y_1 = u_3$, $x_1' = u_4$ and $y_1' = u_5$, and (4) $x_1 = u_1$, $y_1 = u_3$, $x_1' = u_5$ and $y_1' = u_4$. From $C_3^{++}$, there are essentially two choices of $x_1$ and $y_1$, depending on the adjacency of $x_1$ and $y_1$. When $x_1$ and $y_1$ are not adjacent, we have two possibilities; (5)

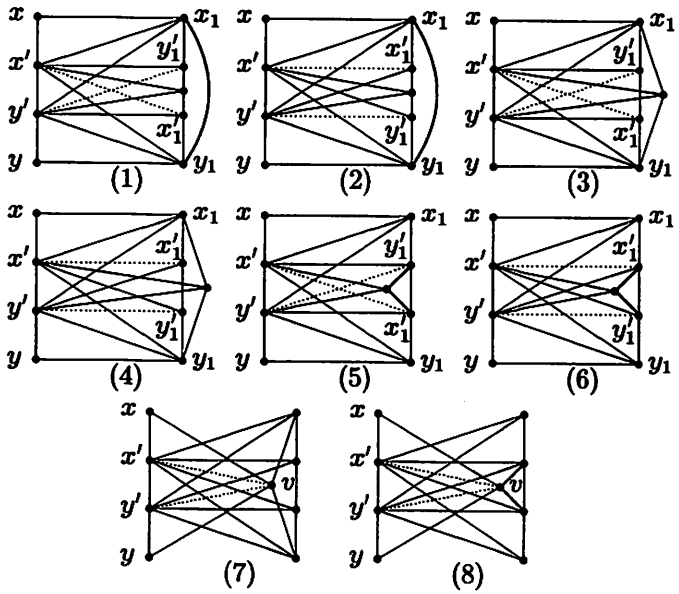


Figure 5: Self-complementary graphs of Type III(a) and III(b)

$x_1 = u_1$, $y_1 = u_4$, $x'_1 = u_2$ and $y'_1 = u_3$, and (6) $x_1 = u_1$, $y_1 = u_4$, $x'_1 = u_3$ and $y'_1 = u_2$. However, in the case when $x_1$ and $y_1$ are adjacent, the graph constructed has 4 vertices of degree 2. Hence this belongs to Type III or IV.

Now we consider Type II(b). From $C_5$, there is essentially one way to choose $v$ in $C_5$ such that $\Psi(v) = v$. This is Case (7). In $C_3^{++}$, no vertex other than $u_0$ is fixed by any isomorphism between $C_3^{++}$ and $\overline{C_3^{++}}$, and hence the only possibility is that $v = u_0$. This is Case (8). ■

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