On the existence of Aperiodic Perfect Maps for 2×2 Windows

SANG-MOK KIM

DEPARTMENT OF MATHEMATICS
SOGANG UNIVERSITY
SEOUL 121-742, KOREA

e-mail: smkim@math.sogang.ac.kr

June 11, 2002

Abstract. An aperiodic perfect map(APM) is an array with the property that each possible array of certain size, called a window, arises exactly once as a subarray in the array. In this article, we give some constructions which imply a complete answer for the existence of APMs with 2×2 windows for any alphabet size.

Footnotes:

2000 Mathematics Subject Classification : 94A55, 05B30.
Key words and phrases : de Bruijn Sequence, Aperiodic Perfect Map, Window property
This work was supported by the Brain Korea 21 Project.

1 Introduction

An Aperiodic Perfect Map(APM) is a c-ary $m \times n$ array such that for some $u \leq m$ and $v \leq n$ every c-ary $u \times v$ array occurs exactly once as a contiguous subarray or a window. This is called the perfect window property. An analogous perfect window property is also applicable to a periodic array: in a Periodic Perfect Map(PM), each window of a certain size occurs exactly once in a single period of the array.

The perfect window property was first studied mainly in the periodic case. De Bruijn considered periodic sequences [1] - a sequence of period n is called a *de Bruijn sequence* if for some $v \leq n$ every possible v-tuple array occurs exactly once in a period. Many construction methods for de Bruijn sequences have been devised [4], and whenever the parameters satisfy certain necessary conditions, the existence of de Bruijn sequences has been established ([1], [3], [10]). It is also shown by K.Paterson ([8], [9]) that a PM exists for every possible parameter set when the alphabet size c is a prime power.

According to S.Kanetkar and M.Wagh [6], a PM can be transformed into an APM of a slightly larger size by simple extension (this APM is called the *closure* of the PM). However, if the window size of APM is small enough or the alphabet size is not a prime power, there are still sets of parameters satisfying necessary conditions for existence of APM for which there does not exist a closure of any PM. This means that the general existence question for APM is still unsolved.

In this article, we give a complete answer for the question of the existence of APM with 2×2 windows for any alphabet size. We begin with some formal definitions and go on to give a brief introduction to the subject of aperiodic perfect maps and periodic perfect maps.
Most of terminologies follow C. Mitchell [7].

Let C be a finite set with $|C| = c$. Conventionally, we represent a c-ary $m \times n$ array as

$$A = (a_{i\,j}) \qquad (0 \leq i \leq m-1,\ 0 \leq j \leq n-1),$$

where each entry $a_{i\,j}$ satisfies $0 \leq a_{i\,j} \leq c-1$. For given integers s and t with $0 \leq s \leq m-1$, $0 \leq t \leq n-1$, we define the (s,t)-th $u \times v$ *window* of A to be the $u \times v$ subarray

$$A_{s\,t} = (\alpha_{i\,j}) \qquad (0 \leq i \leq u-1,\ 0 \leq j \leq v-1)$$

defined by

$$\alpha_{i\,j} = a_{i+s\ j+t},$$

where $i+s$ is computed modulo m and $j+t$ is computed modulo n. For integers m_0 and n_0 with $1 \leq m_0 \leq m$, $1 \leq n_0 \leq n$, we denote by $[A_{m_0 \times n_0}]_{u \times v}$

the set of $u \times v$ windows $A_{s\,t}$ for $0 \le s \le m_0 - 1$, $0 \le t \le n_0 - 1$, i.e.

$$[A_{m_0 \times n_0}]_{u \times v} = \{A_{s\,t} \mid 0 \le s \le m_0 - 1,\ 0 \le t \le n_0 - 1\}.$$

Note that following bounds

$$1 \le \left|[A_{m_0 \times n_0}]_{u \times v}\right| \le m_0 n_0$$

are immediate since the lower bound holds only if all windows $A_{s\,t}$, $0 \le s \le m_0 - 1$, $0 \le t \le n_0 - 1$, are identical and the upper bound holds if they are all distinct.

In the case of a finite sequence A of length n, A may be denoted by

$$A = (a_{0\,0}\ a_{0\,1}\ \dots\ a_{0\,n-1}) = (a_0\,a_1\,\dots\,a_{n-1}) = (a_i)$$

where $0 \le i \le n-1$. For integers n_0 and t with $1 \le n_0 \le n$, $0 \le t \le n_0 - 1$, we also simplify $A_{1\,t}$ as A_t and $[A_{m_0 \times n_0}]_{1 \times v}$ as $[A_{n_0}]_v$.

A c-ary span v de Bruijn sequence $B = (b_i)$ is defined as a periodic sequence of length $n = c^v$ with entries from $\{0, 1, \dots, c-1\}$ in which every distinct c-ary v-tuple occurs exactly once as a window B_t for some integer t with $0 \le t \le c^v - 1$. The well-known result on the existence of de Bruijn sequence is as follows.

Result 1 *([1], [3] and [10]) For any given $c \ge 2$ and $v \ge 1$, there exists a c-ary span v de Bruijn sequence.*

Let c, m, n, u and v be integers satisfying $c \ge 2$, $m \ge u \ge 1$, and $n \ge v \ge 1$. A c-ary $m \times n$ array $A = (a_{i\,j})$ is called a $c-ary$ $(m, n; u, v)$ Perfect Map (or simply PM) if each possible c-ary $u \times v$ array occurs exactly once as a $u \times v$ window $A_{s\,t}$ of A with $0 \le s \le m-1$ and $0 \le t \le n-1$. As has been observed, a c-ary span v de Bruijn sequence is a c-ary $(1, c^v; 1, v)$PM.

Now we immediately have the following necessary conditions for the existence of PM.

Result 2 *([7]) If A is a $c-ary$ $(m, n; u, v)$PM, then the parameters satisfy the following.*

(i) $m > u$ or $m = u = 1$,

(ii) $n > v$ or $n = v = 1$, and

(iii) $mn = c^{uv}$.

If the positive integers c, m, n, u and v satisfy the necessary conditions for the existence of a PM in Lemma 2, the set of ordered integers $(c; m, n; u, v)$ is called an *admissible parameter set* for PM.

When the window of size is 2×2, the complete answer for the existence of PM is given by G.Hurlbert, C.J.Mitchell, and K.G.Paterson [5], as follows.

Result 3 *The necessary conditions of Result 2 are sufficient for the existence of a* $c-ary$ $(m,n;2,2)$ *PM, i.e. for all* $m>2$, $n>2$ *and* c *with* $mn=c^4$, *there exists a* $c-ary$ $(m,n;2,2)$ *PM.*

We define an APM as follows. Let c, m, n, u and v be integers satisfying $c\geq 2$, $m\geq u\geq 1$, and $n\geq v\geq 1$. A $c-ary$ $m\times n$ *array* $A=(a_{i\,j})$ is called a c-ary $(m,n;u,v)$ *Aperiodic Perfect Map* (or simply APM) if each possible c-ary $u\times v$ array occurs exactly once as a window $A_{s\,t}$ of A with $0\leq s\leq m-u$ and $0\leq t\leq n-v$. We also define $(c;m,n;u,v)$ as an admissible parameter set for APM if it satisfies the conditions given in the following lemma.

Result 4 *([7]) If* A *is a* $c-ary$ $(m,n;u,v)APM$, *then the parameters satisfy the following.*

(i) $m\geq u$,

(ii) $n\geq v$ *and*

(iii) $(m-u+1)(n-v+1)=c^{uv}$.

In 1995, C.Mitchell [7] proved the existence theorem of 2-ary(binary) APMs as follows.

Result 5 *A* 2*-ary* $(m,n;u,v)APM$ *exists for every admissible parameter set* $(c;m,n;u,v)$ *for APM, i.e. the necessary condition for PM in Result 4 is sufficient when the alphabet size c is* 2.

Now we describe a construction of an APM from the simple extension of a PM. Note that the construction was suggested by S.Kanetkar and M.Wagh [6], and the notation follows C.Mitchell [7].

Definition 6 *Suppose* m,n,u,v *are positive integers satisfying* $1\leq u\leq m$ *and* $1\leq v\leq n$. *Suppose that* $A=(a_{i\,j})$ $(0\leq i\leq m-1,\ 0\leq j\leq n-1)$ *is an* $m\times n$ *array. Let* $E_{u\,v}(A)=(b_{i\,j})$ $(0\leq i\leq m+u-2,\ 0\leq j\leq n+v-2)$ *be the* $(m+u-1)\times(n+v-1)$ *array defined by*

$$b_{i\,j}=a_{s\,t}$$

where $s\equiv i\,(mod\,m)$ *and* $t\equiv j\,(mod\,n)$.

Then we can state the following lemma which follows immediately from the definitions.

Lemma 7 *If* A *is a* $c-ary$ $(m,n;u,v)PM$ *then* $E_{u\,v}(A)$ *is a c-ary* $(m+u-1,n+v-1;u,v)APM$.

Let $A = (a_{i\,j})$ be a given $c - ary$ $(m, n; u, v)$PM. Then we denote $E_{u\,v}(A) = (b_{i\,j})$ by $\bar{A} = (\bar{a}_{i\,j})$ where $b_{i\,j} = \bar{a}_{i\,j}$, and $\bar{A}$ is called the *closure* of a periodic perfect map A.

Remark 8 *Let $A = (a_{i\,j})$ be a c-ary $m \times n$ array. The transpose A^T of A is the c-ary $n \times m$ array $A^T = \left(a^T_{i\,j}\right)$ where $a^T = a_{j\,i}$. If A is a PM (or an APM) then A^T is also a PM (or an APM, respectively). Hence, without loss of generality, we need only consider those $c - ary$ (m, n, u, v)PM (or APM) with $n \geq m$ and we restrict our definition of admissible parameter set for PM (or APM) to those with $n \geq m$.*

Remark 9 *Note that a $c-ary$ (m, n, u, v)APM arises, by Result 7, from a $c-ary$ $(m-u+1, n-v+1, u, v)$PM. The latter must satisfy the conditions $u = m-u+1 = 1$ or $m-u+1 > u$, and $v = n-v+1 = 1$ or $n-v+1 > v$ i.e. the parameters of the APM that arises satisfy $m = u = 1$ or $m \geq 2u$, and $n = v = 1$ or $n \geq 2v$.*

2 Existence of APMs for 2×2 Windows

We give two constructions of APMs to show that they exist for all admissible parameter sets when the window size is 2×2, which means the necessary conditions in Result 4 are sufficient for the existence of an APM for those parameters. Note that the first construction is also given in [7] for the binary case.

Suppose that $u = v = 2$ for the admissible parameter set $(c; m, n; u, v)$ for APM. By the necessary condition for an APM in Result 4 (iii), we have

$$(m-1)(n-1) = c^4. \qquad (1)$$

We break the problem into three cases for possible values of m when $m = 2$, $m = 3$ and $m \geq 4$. Recall that by Remark 8 we need only consider $n \geq m$.

Consider the case $m \geq 4$. Since $n - 1 \geq m - 1 > 2 = u$, Result 3, [7] implies that c-ary $(m-1,\ n-1; 2, 2)$PM always exists if $(c;\ m-1,\ n-1; 2, 2)$ is an admissible set for PM. Hence, whenever $m > 3$ and $n > 3$, this PM gives rise to its closure which is a c-ary $(m, n; 2, 2)$APM as described in Definition 6 (see Remark 9). In order to complete our analysis of the existence of APM, therefore, it is sufficient to consider the other two cases, $m = 2$ and $m = 3$.

In the following two subsections, we deal with two general constructions which cover the cases $m = 2$ and $m = 3$, respectively. From these results, we conclude that there exists an APM for 2×2 windows if and only if the parameters form an admissible parameter set for APM. (i.e. the necessary conditions for APM (Result 4) are sufficient for the existence of c-ary $(m, n; 2, 2)$APM.

2.1 The existence of c-ary $(m, n; 2, 2)$APM when $m = 2$

Before we deal with the case $m = 2$, we consider a more general construction that implies the existence of c-ary $(2, n; 2, 2)$APM. The following construction is obtained directly from the existence of a c^u-ary span v de Bruijn sequence. Notice that for the binary case it was suggested by J.Burns and C.J.Mitchell [7]. From now on, we use the notation that $\mathbb{N}_l = \{0, 1, \ldots, l-1\}$ and $\mathbb{N}_l^n = \mathbb{N}_l \times \mathbb{N}_l^{n-1}$ for given positive integers l and n.

Construction 10 *(c-ary $(u, n; u, v)APM$)*

Suppose that $(c;\ m = u,\ n;\ u,\ v)$ is an admissible parameter set for APM (i.e. $n - v + 1 = c^{uv}$ from Result 4). We construct a c-ary $(u, n; u, v)APM$. Let $B = (b_k)$ be a c^u-ary span v de Bruijn sequence. Then, by Lemma 7, $\bar{B} = (\bar{b}_j)$ (the closure of B) is a c^u-ary $(1, c^{uv}+v-1; 1, v)APM$ where $\bar{b}_j = b_k$ if $k \equiv j \pmod{c^{uv}}$. Since $\bar{b}_j \in \mathbb{N}_{c^u}$, each $\bar{b}_j$ can be written as an integer representation in base c, i.e. let $\bar{b}_j = \sum_{i=0}^{u-1} b_{i\,j} c^i$ $(0 \le b_{i\,j} \le c-1)$. Using the one-to-one correspondence ϕ between $\mathbb{N}_{c^u}$ and $\mathbb{N}_c^u$, we can write $\bar{b}_j$ as a u-bit c-ary tuple $(b_{0\,j}, \ldots b_{u-1\,j})$, i.e.

$$\phi\left(\bar{b}_j\right) = \phi\left(\sum_{i=0}^{u-1} b_{i\,j} c^i\right) = (b_{0\,j}\ b_{1\,j}\ \ldots\ b_{u-1\,j}) \in \mathbb{N}_c^u.$$

We define a $u \times (c^{uv} + v - 1)$ array $A = (a_{i\,j})$ by $a_{i\,j} = b_{i\,j}$ where $0 \le i \le u-1$, $0 \le j \le c^{uv} + v - 2$, and $\bar{b}_j = b_k$ with $j = k \pmod{c^{uv}}$, i.e.

$$A = \left(\phi\left(\bar{b}_0\right)^T \mid \phi\left(\bar{b}_1\right)^T \mid \cdots \mid \phi\left(\bar{b}_{c^{uv}+v-3}\right)^T \mid \phi\left(\bar{b}_{c^{uv}+v-2}\right)^T\right)$$

where $(\cdot \mid \cdot)$ denotes the concatenation of column vectors.

Theorem 11 *Let $(c; m = u, n; u, v)$ be an admissible parameter for APM. Let B be a c^u-ary span v de Bruijn sequence. Then the $u \times (c^{uv} + v - 1)$ array constructed from B by Construction 10 is a c-ary $(u, n; u, v)APM$.*

Proof. Note that by Result 1 there always exists a c^u-ary span v de Bruijn sequence $B = (b_k)$. We prove that $A = (a_{i\,j})$ in Construction 10 is a c-ary $(u, n; u, v)$APM. Note that since we suppose $m = u$ and $n - v + 1 = c^{uv}$, any $u \times v$ window $A_{s\,t}$ in $[A_{1\times(n-v+1)}]_{u\times v}$ has $s = 0$ and $0 \le t \le c^{uv} - 1$. Let $A_{0\,t} = (\alpha_{i\,j})$ and $A_{0\,t'} = (\beta_{i\,j})$ be any two $u \times v$ windows in $[A_{1\times c^{uv}}]_{u\times v}$ where $0 \le t, t' \le c^{uv} - 1$. Suppose that $A_{0\,t} = A_{0\,t'}$. Then, for $i = 0, 1, \ldots, u-1$, $j = 0, 1, \ldots, v-1$, we have $\alpha_{i\,j} = a_{i\,j+t} = a_{i\,j+t'} = \beta_{i\,j}$ so that $b_{i\,j+t} = b_{i\,j+t'}$.

The one-to-one correspondence ϕ implies that $\bar{b}_{j+t} = \bar{b}_{j+t'}$ for $0 \le j \le v-1$. Hence, $B_t = B_{t'}$ in $[B_{c^{uv}}]_v$. From the definition of a de

Bruijn sequence, it follows that $t = t'$. Therefore, all windows $A_{0\,t}$ in $[A_{1\times(n-v+1)}]_{u\times v}$ are distinct so that $\left|[A_{1\times(n-v+1)}]_{u\times v}\right| = n-v+1$. Since $n-v+1 = c^{uv}$, it follows that each c-ary $u \times v$ array occurs exactly once in $[A_{1\times(n-v+1)}]_{u\times v}$ and so A is a c-ary $(u,n;u,v)$APM. ■

We have a simple example as follows.

Example 12 *For a given* 2^2 *– ary span* 2 *de Bruijn sequence*

$$B = (0\,0\,1\,1\,2\,2\,3\,3\,2\,1\,0\,3\,1\,3\,0\,2),$$

Construction 10 gives rise to a 2*-ary* $(2,17,2,2)$*APM which is*

$$A = \begin{pmatrix} 00001111100101010 \\ 00110011010111000 \end{pmatrix}$$

where the j-th column is $\phi(\bar{b}_j) = \begin{pmatrix} a_{0\,j} \\ a_{1\,j} \end{pmatrix}$ *where* $\bar{b}_j = a_{0\,j} + 2a_{1\,j}$ *with* $a_{0\,j}$, $a_{1\,j} \in \mathbb{N}_2$ *is the* j*-th entry of* $\bar{B}$.

Now we return to the main interest on the existence of APM for all 2×2 window size when $m=2$. From Construction 10 and Theorem 11, we obtain the following corollary which shows the existence of a c-ary $(2,n;2,2)$APM for arbitrary c.

Corollary 13 *For any given admissible parameter set* $(c; m = 2, n; u = 2, v = 2)$ *for APM, there exists a c-ary* $(2,n;2,2)$*APM.*

This establishes the case $m=2$ of APM for 2×2 windows.

2.2 The existence of c-ary $(m,n;2,2)$APM when $m=3$

Let $(c; m = 2u-1, n; u, v)$ be an admissible parameter set. We first deal with a new construction of c-ary $(2u-1,n;u,v)$APM which covers the case $m=3$.

Construction 14 *(c-ary* $(2u-1,n;u,v)$*APM)*

We construct an array with entries from $\mathbb{N}_c$ *where* $c \geq 2$. *Let* $(c; 2u-1, n; u, v)$ *be an admissible set of parameters for APM, i.e.* $n = c^{uv}/u + v - 1$. *Let* $B = (b_l)$ *be a c-ary span uv de Bruijn sequence and let* $\bar{B} = (\bar{b}_k)$ *be the closure of* B *which is a* $1\times(c^{uv}+uv-1)$ *array. Define a* $(2u-1)\times(\frac{c^{uv}}{u}+v-1)$ *array* $A = (a_{ij})$ *by*

$$a_{i\,j} = \bar{b}_{i+uj}$$

for $0 \le i \le 2u-2$ *and* $0 \le j \le \frac{c^{uv}}{u} + v - 2$. *Thus,*

$$A = (a_{i\,j}) = \begin{pmatrix} \bar{b}_0 & \bar{b}_u & \cdot & \cdot & \bar{b}_{c^{uv}+uv-2u} \\ \bar{b}_1 & \bar{b}_{u+1} & & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \bar{b}_{2u-2} & \bar{b}_{3u-2} & \cdot & \cdot & \bar{b}_{c^{uv}+uv-2} \end{pmatrix}$$

Note that $\bar{b}_k = b_l$ *for* $l \equiv k \pmod{c^{uv}}$.

Theorem 15 *Let* $(c;\ m = 2u-1,\ n;\ u,\ v)$ *be an admissible parameter for APM. Let B be a c-ary span uv de Bruijn sequence. Then the* $u \times (c^{uv}+v-1)$ *array constructed from B by Construction 14 is a c-ary* $(2u-1, \frac{c^{uv}}{u} + v - 1; u, v)$*APM.*

Proof. Note that there always exists a c-ary span uv de Bruijn sequence B. Let $A_{s\,t}$ and $A_{s'\,t'}$ be $u \times v$ windows for $0 \le s, s' \le u-1$ and $0 \le t, t' \le c^{uv}/u - 1$, and let $A_{s\,t} = (\alpha_{i\,j})$ and $A_{s'\,t'} = (\beta_{i\,j})$ where $0 \le i \le u-1$, $0 \le j \le v-1$. Suppose that $A_{s\,t} = A_{s'\,t'}$. Then, for all $0 \le i \le u-1$, $0 \le j \le v-1$, we have $\alpha_{i\,j} = \beta_{i\,j}$ i.e. $a_{s+i\,t+j} = a_{s'+i\,t'+j}$, and hence $\bar{b}_{(i+uj)+s+tu} = \bar{b}_{(i+uj)+s'+t'u}$. This implies that $\bar{b}_{s+tu+l} = \bar{b}_{s'+t'u+l}$ for all l with $0 \le l \le uv-1$. This implies that $B_{s+tu} = B_{s'+t'u}$. Since $0 \le s, s' \le u-1$ and $0 \le t, t' \le (c^{uv}/u)-1$ so that $0 \le s+tu,\ s'+t'u \le c^{uv}-1$, it follows from the definition of the given de Bruijn sequence that $s + tu = s' + t'u$ and hence $s = s'$ and $t = t'$. Therefore, we conclude that $A_{s\,t}$ and $A_{s'\,t'}$ are identical if and only if $s = s'$ and $t = t'$, which implies that

$$u\,(c^{uv} + v - 1) = \left|[A_{(m-u+1)\times(n-v+1)}]\right| = \left|[A_{u\times(\frac{c^{uv}}{u})}]\right| = c^{uv}.$$

It follows that every c-ary $u \times v$ array occurs exactly once as a window i.e. A is a c-ary $(2u-1, \frac{c^{uv}}{u} + v - 1; u, v)$APM. ■

The following example is established directly from Construction 14.

Example 16 *(A 2-ary* $(3, 9; 2, 2)$*APM)*

This is the case $u = v = 2$ *in Construction 14. Then* $uv = 4$ *and* $c^{uv} + uv - 1 = 19$. *We take* $\bar{B} = (\bar{b}_k)$ *the closure of a 2-ary span* $4(= uv)$ *de Bruijn sequence* $B = (b_l)$ *such as*

$$\bar{B} = (b_0\, b_1 \ldots b_{15}\, b_0\, b_1\, b_2) = (0\,0\,0\,0\,1\,1\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,0).$$

By Construction 14, we obtain the following 2-ary $(3, 9; 2, 2)$*APM* $A = (a_{i\,j})$ *where* $a_{i\,j} = \bar{b}_{i+2j}$.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

By Theorem 15, we have the following corollary concerning the existence for an APM when $m = 3$ and $u = 2$, $v = 2$.

Corollary 17 *For any given admissible parameter set* $(c; m = 3, n; u = 2, v = 2)$ *for* APM*, there exists a c-ary* $(3, n; 2, 2)APM$.

Proof. Clearly, $m = 3 = 2u - 1$ and the result follows by Theorem 15. ■

By Corollary 13, Corollary 17 and Result 3, therefore, the following result on the existence of APM for 2×2 window size is immediate.

Theorem 18 *For every admissible parameter set* $(c; m, n; 2, 2)$ *for* APM*, there exists a* $c - ary$ $(m, n; 2, 2)APM$*. (i.e. the necessary condition for* APM *(Result 4) is sufficient for the existence of a* $c - ary$ $(m, n; 2, 2)APM$*)*

Acknowledgement

Most of the work is a part of my Ph.D thesis in Royal Holloway, University of London. I would like to thank both my supervisor, Prof. Fred Piper, and my advisor, Prof. Peter Wild.

References

[1] N. de Bruijn, A combinatorial problem, Proc. Nederlandse Akademie van Wetenschappen, vol. 49, pp.758-764, 1946.

[2] J. Burns, C. Mitchell, Coding Schemes for two-dimensional position sending, Cryptography and Coding III, M.Ganley Ed. London, UK: Oxford Univ. Press, pp.31-61, 1993.

[3] I.J. Good, Normally recurring decimals, J. London Math. Soc., vol. 21, pp.167-169, 1946.

[4] H. Fredricksen, A survey of full length nonlinear shift register cycle algorithms, SIAM J. Algebraic and Discrete Methods, vol. 1, pp.107-113, 1980.

[5] G. Hurlbert, C.J. Mitchell, and K.G. Paterson, On the existence of de Bruijn tori with two by two windows, Journal of Combinatorial Theory (Series A) 76, pp.213-230, 1996.

[6] S. Kanetkar and M. Wagh, On Construction of Matrices with distict submatrices, SIAM J. Algebraic and Discrete Methods, vol. 1, pp.107-113, 1980.

[7] C.J. Mitchell, Aperiodic and semi-periodic perfect maps, IEEE transactions on Information Theory 41, pp.88-95, 1995.

[8] K.G. Paterson, New Classes of Perfect Maps I, Journal of Combinatorial Theory (Series A) 73, pp.302-334, 1996.

[9] K.G. Paterson, New Classes of Perfect Maps II, Journal of Combinatorial Theory (Series A) 73, pp.335-345, 1996.

[10] D. Rees, Note on a paper by I. J. Good, J. London Math. Soc., vol. 21, pp.169-172, 1946.