Realizability of p-Point, q-Line Graphs with Prescribed Minimum Degree and Line Connectivity

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Abstract

It is an established fact that some graph-theoretic extremal questions play an important part in the investigation of communication network vulnerability. Questions concerning the realizability of graph invariants are generalizations of these extremal problems. We define a (p, q, λ, δ) graph as a graph having p points, q lines, line connectivity λ and minimum degree δ. An arbitrary quadruple of integers (a, b, c, d) is called (p, q, λ, δ) realizable if there is a (p, q, λ, δ) graph with $p = a$, $q = b$, $\lambda = c$, and $\delta = d$. Inequalities representing necessary and sufficient conditions for a quadruple to be (p, q, λ, δ) realizable are derived. In recent papers, the author gave necessary and sufficient conditions for (p, q, κ, Δ), (p, q, λ, Δ), (p, q, δ, Δ) and (p, q, κ, δ) realizability, where Δ denotes the maximum degree for all points in a graph and κ denotes the point connectivity of a graph. Boesch and Suffel gave the solutions for (p, q, κ), (p, q, λ), (p, q, δ), $(p, \Delta, \delta, \lambda)$ and $(p, \Delta, \delta, \kappa)$ realizability in earlier manuscripts.

Introduction

Here we consider an undirected graph $G = (V, X)$ with a finite point set V and a set X whose elements, called lines, are two point subsets of V. The number of points is denoted by p, and the number of lines $|X|$ is denoted by q(G) or q. This paper uses the notation and terminology of Harary [13]; however a few basic concepts are now reproduced.

The line connectivity of a graph G (denoted by λ or $\lambda(G)$) is the minimum number of lines whose removal results in a disconnected graph. A graph is called trivial if it has just one point. The point connectivity (denoted by $\kappa(G)$ or κ) is the minimum number of points whose removal results in a disconnected or trivial graph. The number of lines connected to a point v of G is the degree of that point, denoted by $d_v(G)$ or d_v. The minimum degree is denoted by δ or $\delta(G)$ and the maximum degree is denoted by Δ. If $\delta = \Delta$, the graph is called regular. A p point graph with $\delta = p - 1$ is called complete and is denoted by K_p. A set of λ lines whose removal disconnects G is called a minimum line disconnecting set.

It is an established fact that some graph-theoretic extremal questions play an important part in the investigation of communication network vulnerability [1-12,15]. Harary [14] found the maximum point connectivity among all graphs with a given number of points and a given number of lines. Questions concerning the realizability of graph invariants are generalizations of these extremal problems. We define a (p, q, λ, δ) graph as a graph having p points, q lines, line connectivity λ and minimum degree δ. An arbitrary quadruple of integers (a, b, c, d) is called (p, q, λ, δ) realizable if there is a (p, q, λ, δ) graph with $p = a$, $q = b$, $\lambda = c$ and $\delta = d$. Inequalities representing necessary and sufficient conditions for a quadruple to be (p, q, λ, δ) realizable (or, more briefly, realizable) are derived. The author derived necessary and sufficient conditions for (p, q, κ, Δ), (p, q, λ, Δ), (p, q, δ, Δ) and (p, q, κ, δ) realizability in recent papers [9-11]. In [1-3] the conditions for (p, q, κ), (p, q, λ), (p, q, δ), $(p, \Delta, \delta, \lambda)$ and $(p, \Delta, \delta, \kappa)$ realizability were given by Boesch and Suffel.

Preliminaries

We start by reviewing some known results that are pertinent to the realizability question.

Lemma 1 [8]: If $\delta \geq \lfloor \frac{1}{2} p \rfloor$, then $\lambda = \delta$.

Lemma 2 [14]: If $2 \leq \delta \leq p - 1$, then there is a graph on p points with $q = \lceil \frac{1}{2} p\, \delta \rceil$ and $\lambda = \delta = \kappa$. (This graph is a power of cycle and is usually called the Harary graph on p points).

Lemma 3: If $\lambda < \delta$, then after the removal of any minimum line disconnecting set, each component must have at least $\delta + 1$ points.

This lemma follows from Chartrand's work concerning Lemma 1.

We now give some new results.

Lemma 4: If $\lambda < \delta$, then

$$q \leq \lambda + \tfrac{1}{2}\,\delta(\delta + 1) + \tfrac{1}{2}(p - \delta - 1)(p - \delta - 2).$$

Proof: Let G be a graph with $\lambda < \delta$, W be a minimum line disconnecting set and A, B be the two components of G - W. Let $|A| = N$, thus $|B| = p - N$. We then have $q(G - W) \leq \tfrac{1}{2} N(N - 1) + \tfrac{1}{2}(P - N)(P - N - 1)$. We wish to maximize the right side of this inequality on $\delta + 1 \leq N \leq p - \delta - 1$ (recall Lemma 3). Since this quantity is a quadratic in N with a leading term of N^2, the maximum must take place at one of the bounds of the interval. It is easily verified that the value of the right side of the inequality is the same at each bound. Since $q(G) = q(G - W) + \lambda$ the result follows easily.

Lemma 5: If $q = \lceil \tfrac{1}{2}\, p\, \delta \rceil$ and λ is odd, then δ is odd.

Proof: Suppose there is a graph G with $q = \lceil \tfrac{1}{2}\, p\, \delta \rceil$, λ odd and δ even. Let W denote a minimum line disconnecting set of G and let A be one of the two components of G - W. As G is regular of degree δ we have $\sum_{i \in A} d_i(G - W) = \delta |A| - \lambda$, which is odd. Since A is a component of G - W, this is impossible and the result is proven.

Lemma 6: If $p = 2\,\delta + 2$, $q = \tfrac{1}{2}\, p\, \delta$ and λ is odd, then $\lambda = \delta$.

Proof: We assume there is a graph with $p = 2\delta + 2$, $q = \tfrac{1}{2}\, p\, \delta$, λ odd and $\lambda < \delta$ and proceed as in the proof of Lemma 5. (Here we note that Lemma 3 implies $|A| = \delta + 1$).

The (p, q, λ, δ) realizability theorem

Theorem. A quadruple of non-negative integers (p, q, λ, δ) is realizable if and only if exactly one of the following conditions holds:

(I) $p \leq 2\,\delta + 1$, $\lceil \tfrac{1}{2}\, p\, \delta \rceil \leq q \leq \delta + \tfrac{1}{2}(p - 1)(p - 2)$, and $\lambda = \delta \leq p - 1 \leq q$.

(II) $p \geq 2\,\delta + 2$ and if $\lambda > 0$ then $q \geq p - 1$
- (A) $\lceil \tfrac{1}{2}\, p\, \delta \rceil + 1 \leq q \leq \delta + \tfrac{1}{2}(p - 1)(p - 2)$, $\lambda \leq \delta$, and if $\lambda < \delta$, then $q \leq \lambda + \tfrac{1}{2}\,\delta(\delta + 1) + \tfrac{1}{2}(p - \delta - 1)(p - \delta - 2)$.
- (B) $q = \lceil \tfrac{1}{2}\, p\, \delta \rceil$
  - (a) $\lambda \leq \delta$ and λ is even.
  - (b) $\lambda \leq \delta$, λ and δ both odd, and if $p = 2\,\delta + 2$, then $\lambda = \delta$.

Proof: The conditions in (I) follow from Lemma 1 together with well known facts concerning graphs. The conditions in (II) are a consequence of Lemmas 4, 5 and 6, and some obvious facts about graphs.

We now provide constructions to prove sufficiency.

Case 1. Suppose that $\delta \geq 2$, $\lambda = \delta \leq p - 1$, and $\lceil \frac{1}{2} p\delta \rceil \leq q \leq \delta + \frac{1}{2}(p-1)(p-2)$. Let H denote the Harary graph on p points with $\lceil \frac{1}{2} p\delta \rceil$ lines and $\lambda = \delta$, and let v denote one of the points in H. Adding $q - \lceil \frac{1}{2} p\delta \rceil$ lines to H in such a way that none of the added lines are incident to v yields the desired graph. Thus any quadruple satisfying this case is realizable.

Case 2. Suppose that $\delta \geq 2$, $\lambda < \delta$, λ is even, $p \geq 2\delta + 2$ and

$$\lceil \tfrac{1}{2} p\delta \rceil \leq q \leq \lambda + \tfrac{1}{2}\delta(\delta+1) + \tfrac{1}{2}(p-\delta-1)(p-\delta-2).$$

Let H_1 denote the Harary graph on $p - \delta - 1$ points with $\lceil \frac{1}{2}(p-\delta-1)\delta \rceil$ lines and $\lambda(H_1) = \delta(H_1) = \delta$. Take the union of H_1 and $K_{\delta+1}$ to form a single graph. If $\lambda = 0$, add $q - \lceil \frac{1}{2} p\delta \rceil$ lines to H_1 and we are done. If $\lambda > 0$, denote the points in H_1 by A and the points in $K_{\delta+1}$ by B. We note that every graph with minimum degree δ has a path containing at least $\delta + 1$ points. Let P_1 be a path in H_1 containing λ points. Starting at one endpoint of P_1, travel along P_1 labeling the points $1,2,3,\ldots,\lambda$. In a like manner, take a path in $K_{\delta+1}$ containing λ points and denote these points by $1', 2', 3', \ldots, \lambda'$. Add the lines $\{1, 1'\}, \{2, 2'\}, \{3, 3'\}, \ldots, \{\lambda, \lambda'\}$ and delete the lines $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{\lambda-1, \lambda\}, \{1', 2'\}, \{3', 4'\}, \{5', 6'\}, \ldots, \{(\lambda-1)', \lambda'\}$. Adding $q - \lceil \frac{1}{2} p\delta \rceil$ lines, none of which join points in A to points in B yields the desired graph. There is at least one point in B which is not adjacent to a point in A, denote such a point by v_1. Each line deleted from H_1 can be replaced by a path containing v_1. This, together with the fact that no two lines deleted from $K_{\delta+1}$ were adjacent shows that our graph has the desired line connectivity and satisfies the needed conditions.

Case 3. Suppose that $\delta \geq 2$, $\lambda < \delta$, λ is odd, $p \geq 2\delta + 2$ and

$$\lceil \tfrac{1}{2} p\delta \rceil + 1 \leq q \leq \lambda + \tfrac{1}{2}\delta(\delta+1) + \tfrac{1}{2}(p-\delta-1)(p-\delta-2).$$

First repeat the construction used in Case 2 with $\lambda - 1$ in place of λ and $q - \lceil \frac{1}{2} p\delta \rceil - 1$ in place of $q - \lceil \frac{1}{2} p\delta \rceil$. Adding a line which joins a point in A to a point in B finishes the construction.

Case 4. Suppose that $\delta \geq 2$, $\lambda < \delta$, $p \geq 2\delta + 2$, $q = \lceil \frac{1}{2} p\delta \rceil$ and λ, p and δ are all odd.

Note that $p \geq 2\delta + 3$. Take the Harary graph on $\delta + 2$ points with $\lceil \frac{1}{2}(\delta+2)\delta \rceil$ lines and $\lambda = \delta$, and delete one of the lines incident to the point of degree $\delta + 1$. Denote the resulting graph by C. Let H_2 denote the Harary graph on $p - \delta - 2$ points with $\lceil \frac{1}{2}(p-\delta-2)\delta \rceil$ lines and $\lambda(H_2) = \delta(H_2) = \delta$. Take the union of H_2 and C to form a single graph. There is a path in C which contains λ points and has, as an endpoint, the point of degree $\delta - 1$. As we did in Case 2, we label the points in this path $1,2,3,\ldots,\lambda$ (with the point of degree $\delta - 1$ labeled 1). We now take a path containing λ points in H_2 and, in the usual manner, label its points $1', 2', 3', \ldots, \lambda'$. Add the lines $\{1,1'\}, \{2,2'\}, \{3,3'\}, \ldots, \{\lambda,\lambda'\}$ and delete the lines $\{2,3\}, \{4,5\}, \{6,7\}, \ldots, \{\lambda-1,\lambda\}, \{1',2'\}, \{3',4'\}, \{5',6'\}, \ldots, \{(\lambda-2)', (\lambda-1)'\}$. Each line deleted in C can be replaced by a path containing two points in H_2, and vice versa. We therefore have the desired line connectivity and this case is finished.

Case 5. Suppose that $\delta \geq 2$, $\lambda < \delta$, $p \geq 2\delta + 2$, $q = \lceil \frac{1}{2} p \delta \rceil$, p is even and λ and δ are both odd. Note that Lemma 6 implies $p \geq 2\delta + 4$. Let C be defined as it was in Case 4. Take the Harary graph on $p - \delta - 2$ points with $\lceil \frac{1}{2}(p - \delta - 2)\delta \rceil$ lines and $\lambda = \delta$, and delete one of the lines incident to the point of degree $\delta + 1$. Denote the resulting graph by D. Forming the union of C and D and continuing in a manner similar to that used in Case 4 will give us the desired graph (which is regular of degree δ).

Case 6. Suppose that $\delta \leq 1$. First, we consider the possibility that $\delta = \lambda = 0$ and $q \leq \frac{1}{2}(p - 1)(p - 2)$. Let G be the graph with p points and no lines and denote one of the points in G by v_2. Adding q lines to G, none of which are incident to v_2, finishes the construction.

Next we assume that $\delta = 1$, $\lambda = 0$, $p \geq 4$ and $\lceil \frac{1}{2} p \rceil \leq q \leq 1 + \frac{1}{2}(p - 2)(p - 3)$. If p is even, let A be the graph composed of $\frac{1}{2}(p - 2)$ copies of K_2. However, if p is odd let A be the union of $K_{1,2}$ and $\frac{1}{2}(p - 5)$ copies of K_2.

Take the union of A and K_2 and add $q - \lceil \frac{1}{2} p \rceil$ lines incident only to points in A. The resulting graph has the desired properties.

On the other hand, it may be that $\lambda = \delta = 1$, $p \geq 2$ and $p - 1 \leq q \leq 1 + \frac{1}{2}(p - 1)(p - 2)$. Consider a path on p points and denote one of its endpoints by v_3. Adding $q - (p - 1)$ lines, none of which are incident to v_3, yields the desired graph and we are finished with Case 6.

We are done with our constructions and will now show sufficiency. Case 1 shows the conditions of the theorem are sufficient if we also have $\delta \geq 2$ and $\lambda = \delta$. If we assume $\delta \geq 2$, $\lambda < \delta$ and $\lceil \frac{1}{2} p \delta \rceil + 1 \leq q$ then Cases 2 and 3 show the sufficiency of the conditions in the theorem. Similarly, if we assume $\delta \geq 2$, $\lambda < \delta$ and $\lceil \frac{1}{2} p \delta \rceil = q$ then Cases 2, 4 and 5 are adequate. Case 6 shows the sufficiency of the conditions of the theorem for $\delta < 2$ and our proof is done.

Conclusion

The (p, q, λ, δ) realizability theorem in this paper solves several extremal problems. If any three of the parameters p, q, λ and δ are given we can find the range of values for the unknown parameter. Let max $(p \mid q,\lambda,\delta)$ denote the maximum value of p among all (q,λ,δ) graphs and min $(p \mid q,\lambda,\delta)$ denote the minimum value of p among all (q,λ,δ) graphs.

Corollary 1. For all (q,λ,δ) graphs the following results hold:

$$(1)\ \max(p \mid q,\lambda,\delta) = \begin{cases} \lfloor (2q - 2)/\delta \rfloor, & \text{if } \delta \geq 2 \text{ and one of the following holds:} \\ & \text{(a) } \delta \text{ is even and } \lambda \text{ is odd, or} \\ & \text{(b) } \lambda \text{ is odd, } \lambda < \delta \text{ and } p = 2\delta + 2 \\ q + 1, & \text{if } \delta = 1 \\ \infty, & \text{if } \delta = 0 \\ \lfloor 2q/\delta \rfloor, & \text{otherwise.} \end{cases}$$

$$(2)\ \min(p \mid q,\lambda,\delta) = \begin{cases} \max(M, 2\delta + 2, N), & \text{if } \lambda < \delta \\ \max(M, \delta + 1), & \text{if } \lambda = \delta \text{ and } q > \delta \\ \delta + 1, & \text{if } \lambda = \delta = q; \end{cases}$$

where $M = \lceil [3 + (1 + 8q - 8\delta)^{1/2}]/2 \rceil$ and
$N = \lceil \delta + \{3 + [1 + 8q - 8\lambda - 4\delta(\delta + 1)]^{1/2}\}/2 \rceil$.

Proof: First we will prove equation (1). If δ is even and λ is odd, Lemma 5 implies $p \le \lfloor (2q-2)/\delta \rfloor$. From Lemma 6 we have if λ is odd, $\lambda < \delta$ and $p = 2\delta + 2$ then $p \le \lfloor (2q - 2)/\delta \rfloor$. To complete our list of upper bounds of p note that $p \le \lfloor 2q/\delta \rfloor$ and if $\lambda > 0$ then $p \le q + 1$. If we assume $\delta \ge 2$, it follows that $\lfloor 2q/\delta \rfloor < q + 1$. On the other hand, $\delta = 1$ implies $\lfloor 2q/\delta \rfloor \ge q + 1$. The result follows. We will now prove equation (2). From the realizability theorem $q \le \delta + \frac{1}{2}(p - 1)(p - 2)$ holds for all graphs. Therefore $p^2 - 3p + 2(1 + \delta - q) \ge 0$ and, as a result, $p \ge M$ if $q > \delta$. If $\lambda < \delta$ then Lemma 4 states that

$$q \le \lambda + \tfrac{1}{2}\delta(\delta + 1) + \tfrac{1}{2}(p - \delta - 1)(p - \delta - 2).$$

Thus

$$p^2 - (2\delta + 3)p + 2(\lambda + \delta^2 + 2\delta + 1 - q) \ge 0$$

and, consequently, $p \ge N$ or

$$p \le \lfloor \delta + \{3 - [1 + 8q - 8\lambda - 4\delta(\delta + 1)]^{1/2}\}/2 \rfloor.$$

We denote the last quantity by Q. As a result of the realizability theorem we have $p \ge \delta + 1$ and if $\lambda < \delta$ then $p \ge 2\delta + 2$. We note that $2\delta + 2 \ge Q$ and therefore equation (2) holds.

Corollary 2. For all (p,q,λ) graphs the following results hold:
(3) $\max(\delta \mid p,q,\lambda) = \min(\lfloor 2q/p \rfloor, \max(\lambda, \lfloor (p - 2)/2 \rfloor))$;
(4) $\min(\delta \mid p,q,\lambda) = \lambda$.

Proof: First we will prove equation (3). From Lemma 1 we see that if $\lambda < \delta$ then $\delta \le \lfloor (p - 2)/2 \rfloor$. Thus $\delta \le \max(\lambda, \lfloor (p - 2)/2 \rfloor)$. Other upper bounds for δ include both $\lfloor 2q/p \rfloor$ and $p - 1$. The result follows from the fact that $\max(\lambda, \lfloor (p - 2)/2 \rfloor) \le p - 1$. Next we will prove equation (4). We note that $\delta \ge q - \frac{1}{2}(p - 1)(p - 2)$ and $\delta \ge \lambda$. In [3] it was shown that $q \le \lambda + \frac{1}{2}(p - 1)(p - 2)$. Thus $\lambda \ge q - \frac{1}{2}(p - 1)(p - 2)$ and equation (4) has been proven.

The following two corollaries are given without proof.

Corollary 3. For all (p,q,δ) graphs the following results hold:

$$\max(\lambda \mid p,q,\delta) = \begin{cases} 0, & \text{if } q < p - 1 \\ \delta, & \text{if } q \ge p - 1. \end{cases}$$

$$\min(\lambda \mid p,q,\delta) = \begin{cases} \delta, & \text{if } p \le 2\delta + 1 \text{ or } q > \delta + ½\,\delta(\delta + 1) + ½\,(p - \delta - 1)(p - \delta - 2) \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 4. For all (p,λ,δ) graphs the following results hold:

$$\max(q \mid p,\lambda,\delta) = \begin{cases} \lambda + ½\,\delta(\delta + 1) + ½\,(p - \delta - 1)(p - \delta - 2), & \text{if } \lambda < \delta \\ \delta + ½\,(p - 1)(p - 2), & \text{if } \lambda = \delta. \end{cases}$$

$$\min(q \mid p,\lambda,\delta) = \begin{cases} \lceil p\delta/2 \rceil + 1, & \text{if } \delta \ge 2 \text{ and one of the following holds:} \\ & \quad \text{(a) } \delta \text{ is even and } \lambda \text{ is odd, or} \\ & \quad \text{(b) } \lambda \text{ is odd, } \lambda < \delta \text{ and } p = 2\delta + 2 \\ p - 1, & \text{if } \delta = 1 \\ \lceil p\delta/2 \rceil, & \text{otherwise.} \end{cases}$$

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