Induced Graph Ramsey Theory

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Abstract

We say that a graph F *strongly arrows* (G, H) and write $F \rightarrowtail (G, H)$ if for every edge-coloring of F with colors red and blue a red G or a blue H occurs as an induced subgraph of F. *Induced Ramsey numbers* are defined by $r^*(G, H) = \min\{|V(F)| : F \rightarrowtail (G, H)\}$. The value of $r^*(G, H)$ is finite for all graphs, and good upper bounds on induced Ramsey numbers in general, and for particular families of graphs are known. Most of these results, however, use the probabilistic method, and therefore do no yield explicit constructions. This paper provides several constructions for upper bounds on $r^*(G, H)$ including $r^*(C_n) = r^*(C_n, C_n) \leq c^{(\log n)^2}$, $r^*(T, K_n) \leq |T|n^{|T| \log |T|}$, $r^*(B, C_n) \leq |B|^{\lceil \log n \rceil + 4}$, where T is a tree, B is bipartite, K_n is the complete graph on n vertices and C_n a cycle on n vertices. We also have some new upper bounds for small graphs: $r^*(K_3 + e) \leq 21$, and $r^*(K_4 - e) \leq 46$.

1 Introduction

Any edge-coloring of a K_5 with red and blue will contain a monochromatic P_4 (path on four vertices). The proof is a standard Ramsey theory argument: fix one vertex u of the K_5. There are at least two neighbors v and w of u such that uv and uw have the same color, say red. Consider the four edges leaving v and w other than uv and uw. If one of these edges is red it forms a red P_4 with uv and uw, otherwise the four edges are blue and form a blue C_4 (cylce on four vertices) which, obviously, contains a blue P_4.

What happens if we require the monochromatic P_4 subgraph to be induced? The complete graph will no longer do, since it does not contain P_4

ARS COMBINATORIA 66(2003), pp. 3-21

as an induced subgraph at all, but the graph in Figure 1 will work, as we will show in Section 6.1.

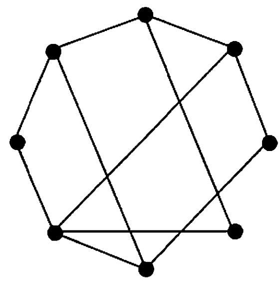

Figure 1: Graph with monochromatic, induced P_4 in every two-coloring

It is not at all obvious that given graphs G and H there will always be a graph F which for all two-colorings will either contain a red induced G or a blue induced H (in this case we say that F *strongly arrows* G and H, and write $F \rightarrowtail (G, H)$). This result is known as the Induced Graph Theorem and was proved independently by Deuber, by Erdős, Hajnal and Pósa, and by Rödl in the seventies [Die97]. In a later analysis, Erdős and Hajnal showed that the graph F can be assumed to have at most $2^{2^{n^{1+\epsilon}}}$ vertices [KPR98], where n is the numer of vertices in G and H. The smallest order $|F|$ (number of vertices) of a graph F for which $F \rightarrowtail (G, H)$ is called $r^*(G, H)$, or $r^*(G)$ in the diagonal case $G = H$. If we omit the condition that subgraphs must be induced, we get the ordinary Ramsey numbers $r(G, H)$ which are well investigated [Rad99, GRS90].

Erdős and Rödl conjectured $r^*(G) \leq c^{|G|}$, where c is a constant [CG98]. A recent paper by Kohayakawa, Prömel, and Rödl established $r^*(G) \leq 2^{|G|(\log|G|)^2}$ [KPR98].

While this result tells us something about the order of induced Ramsey graphs F for G, it does not allow us to construct F explicitly. Most constructions in Ramsey theory are randomized, using the probabilistic method for proving existence. However, there are situations in which one needs explicit constructions: fault-tolerant networks, for example [AC88], or the study of the computational complexity of $\rightarrowtail$, the arrowing relation [Sch99].

A look at the proofs of the Induced Graph Theorem shows that most of them have relied on the probabilistic method. The exceptions are the proofs by Deuber and by Nešetřil and Rödl [Die97].[1] However, both proofs

[1] Diestel [Die97] includes both proofs. Deuber's is the first proof of Theorem 9.3.1, and Nešetřil's and Rödl's is the second proof.

construct graphs of enormous size (worse than taking repeated exponents, but better than Ackermann's function). Perhaps we cannot expect better results for the general case. In this paper we concentrate on explicit constructions for special families of graphs: complete graphs, bipartite graphs, cycles, and trees. We conclude the introduction with a survey of our results in comparison to previously known constructive and nonconstructive results in the area.

Let us start with small graphs. We restrict ourselves to connected graphs on four vertices. Smaller graphs are trivial, and we do not know of any results for particular larger graphs that are not either trivial (stars) or belong to noninduced Ramsey theory (complete graphs). P_4 is the path on four vertices, $K_{1,3}$ a star with three edges, C_4 the cycle on four vertices, $K_3 + e$ a triangle with an additional edge attached to one of the vertices, and $K_4 - e$ a complete graph on four vertices with one of its edges removed. Figure 2 collects the known results.

G	$r^*(G)$	Reference
P_4	8	Harary, Nešetřil, Rödl [HNR83], Section 6.1
$K_{1,3}$	6	Harary [HNR83]
K_3	6	Gleason and Greenwood [GRS90]
C_4	≤ 10	Harary, Nešetřil, Rödl [HNR83]
$K_3 + e$	21	Section 6.2
$K_4 - e$	46	Section 6.3
K_4	18	Gleason and Greenwood [GRS90]

Figure 2: Induced Diagonal Ramsey Numbers for Small Graphs

We also show that $r^*(K_3+e, K_3) \leq 18$ (Section 6.2), $r^*(K_4-e, K_3) \leq 16$ (Section 6.3), and $r^*(C_3, C_4) \leq 14$ (Section 6.4). These are the only bounds for nondiagonal Ramsey numbers we are aware of (excluding trivial cases and pairs of complete graphs).

The only true lower bound for induced Ramsey numbers we know of is $r^*(P_4) \geq 8$ from the paper by Harary, Nešetřil, and Rödl [HNR83]. And this is not only true for small graphs, but for the asymptotic case as well. All available lower bounds simply use $r^*(G, H) \geq r(G, H)$.

Let us now turn to the asymptotic results. As we mentioned earlier, there are several proofs of the Induced Graph Ramsey Theorem. The current best upper bound is given in a paper by Kohayakawa, Prömel, Rödl [KPR98]:

$$r^*(G, H) \leq |H|^{c|G| \log \chi(H)},$$

where c is a constant, $\chi(H)$ is the chromatic number of H, and $|G| \leq |H|$. This comes close to the Erdős and Rödl conjecture that $r^*(G) \leq c^{|G|}$.

The Erdős and Rödl conjecture is true for bipartite graphs by a result of Rödl from his 1973 master's thesis.[2] If one of the graphs is a tree, then r^* behaves polynomially: $r^*(T,H) \le c|T|^2|H|^4(\log|T||H|^2)^2$ [KPR98] (the actual bound is slightly better, but more complicated). If both graphs are trees, then $r^*(T) \le |T|^3(\log|T|)^4$ as Beck showed [Bec90]. The case of induced paths and induced cycles was settled by a result of Haxell, Kohayakawa, Luczak [HKL95] who showed that $r^*(C_n) \le cn$, where C_n is the cycle on n vertices. As a matter of fact they showed more: even the size Ramsey number (counting the number of edges rather than vertices) is linear in n. Furthermore the result is true for any number of colors (only the constant c depends on the number of colors). Luczak and Rödl [LR96] showed that for graphs of bounded maximum degree $r^*(H) \le |H|^c$ (where c depends on the degree bound).

The only construtive result in the preceding list is Rödl's result for bipartite graphs. It is also tight in the sense that there is an exponential lower bound to match it (for complete bipartite graphs, even in the non-induced case). Figure 3 sums up our result while comparing them to the randomized results. In the table T is a tree, B a bipartite graph.

G	H	$r^*(G,H)$ constructive	random	References
T	K_n	$\|T\|n^{\|T\|\log\|T\|}$	$c\|T\|^2n^4(\log\|T\|n^2)^2$	Theorem 3.1, [KPR98]
T	$K_{n,n}$	$\|T\|^2n$	$c\|T\|^2(2n)^4(\log\|T\|(2n)^2)^2$	Theorem 3.2, [KPR98]
P_n	B	$\|B\|^{\lceil\log n\rceil}$	$cn^2\|B\|^4(\log n\|B\|^2)^2$	Corollary 4.8, [KPR98]
T	B	$\|B\|^{(6\|T\|\log\|T\|)^{1/2}}$	$c\|T\|^2\|B\|^4(\log\|T\|\|B\|^2)^2$	Theorem 4.10, [KPR98]
T	T	$6^{\|T\|^{1/2}\log^{3/2}\|T\|}$	$\|T\|^3(\log\|T\|)^4$	Corollary 4.11, [Bec90]
B	C_n	$\|B\|^{\lceil\log n\rceil+4}$	?	Theorem 5.2
T	C_n	$\|T\|^{\lceil\log n\rceil+1}$	$c\|T\|^2n^4(\log\|T\|n^2)^2$	Corollary 5.3, [KPR98]
C_n	C_m	$4^{(\log(n)+\log(m))^2}$	cn (for $m \approx n$)	Theorem 5.4, [HKL95]

Figure 3: Asymptotic Induced Ramsey Numbers

2 Definitions

A *graph* $F = (V,E)$ consists of a *vertex set* V and an edge set E. The *order* of F (written $|F|$) is the number of vertices, $|V|$, the *size* of F is the number of edges, $|E|$. For the purposes of this paper, all graphs are finite, undirected, and simple (without loops or multiple edges). We say

[2]The only published version of this result we could locate is in Diestel's book [Die97, Lemma 9.3.3].

$F' = (V', E')$ is a *subgraph* of $F = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We call F' an *induced subgraph* of F if $E' = \{uv \in E : u, v \in V'\}$.

A graph $F = (V, E)$ is *bipartite*, if its vertex set can be partitioned into two sets such that all edges of F are between the two sets that is $V = V_1 \dot\cup V_2$, and $E \subseteq \{uv : u \in V_1, v \in V_2\}$.

We use the following notation for particular graphs: K_n is the complete graph on n vertices, $K_{n,m}$ the complete bipartite graph on n and m vertices (and similarly $K_{n,m,o}$ the complete tripartite graph on n, m and o vertices), C_n the cycle on n vertices, and P_n the path on n vertices (of length $n-1$). We will call $K_{1,n}$ a *star*. The star $K_{1,n}$ has one *center* and n *outer* vertices.

Connected, acyclic graphs are called trees. A *rooted tree* is a tree with one of its nodes designated as a *root*. We think of the edges as being oriented away from the root, and, therefore, will talk about parents and children with regard to nodes in a rooted tree. A rooted tree is called *full* if all its vertices (with the exception of the leaves) have the same number of children and all the leaves are at the same level. If the number of children is d and the leaves are at level h, we speak of the full *d-ary tree of height h*.

Definition 2.1 *We say that a graph F* arrows (G, H) *and write $F \to (G, H)$ if for every edge-coloring of F with colors red and blue, a red G or a blue H occurs as a subgraph. We say F* strongly arrows (G, H) *and write $F \rightarrowtail (G, H)$ if the subgraph is induced (as a subgraph of F). We define the* generalized Ramsey numbers

$$\begin{aligned} r(G, H) &= \min\{n : K_n \to (G, H)\}, \\ r^*(G, H) &= \min\{|V(F)| : F \rightarrowtail (G, H)\}. \end{aligned}$$

The (induced) size Ramsey numbers *are defined as*

$$\begin{aligned} r_e(G, H) &= \min\{|E(F)| : F \to (G, H)\}, \\ r_e^*(G, H) &= \min\{|E(F)| : F \rightarrowtail (G, H)\}. \end{aligned}$$

Definition 2.2 *The* composition $F[G]$ *of two graphs $F = (V, E)$ and $G = (V', E')$ is a graph on $V \times V'$ with edges between points (v_1, v_1') and (v_2, v_2') if $v_1 v_2 \in E$ or $v_1 = v_2$ and $v_1' v_2' \in E'$, i.e. the vertices of F are replaced with copies of G and the edges of F with complete bipartite graphs (or conversely).*

As references we use Diestel [Die97] for graph theory, and Graham, Rothschild, Spencer [GRS90] for Ramsey theory.

3 Trees versus Complete Graphs

Theorem 3.1 *Given a tree T and $n \geq 2$ we can construct a graph F of order at most $|T| n^{2|T| \log |T|}$ such that $F \rightarrowtail (T, K_n)$.*

Proof. Let T be a tree of order t. We will construct graphs F_n inductively such that $F_n \rightarrowtail (T, K_n)$.

For $n = 2$ we can let $F_2 = T$ which serves as the base of the construction. For the inductive step we assume that we have built a graph F_n such that $F_n \rightarrowtail (T, K_n)$. Consider the graph $F = T[F_n]$, the composition of T and F_n. We will show that $F \rightarrowtail (T, K_{(1+\beta)n})$ for some $\beta > 0$ depending on T only.

Fix a coloring of F in which F does not contain a red induced subgraph T. Then each copy of F_n will contain a blue clique of order at least n. Restrict F and its coloring to the union of these vertices, i.e. we get a graph of the form $F' = T[K_n]$ where each copy of K_n is colored blue and F' does not contain a red T. Let us call the different copies of K_n the layers of F' and call these layers adjacent if the corresponding vertices in T they replace are adjacent. For a vertex v of T let $L(v)$ denote the vertices of F' which belong to the layer associated with v. We claim that there are subsets A and B of vertices of some two adjacent layers such that all edges between A and B are blue and $|A| + |B| \geq (1 + \beta)n$ (we will determine β later). Then the vertices in A and B form a blue clique of order $(1 + \beta)n$, hence F itself contains a blue clique of this order. That means we can increase the order of cliques by a factor of $(1 + \beta)$ by multiplying the order of the graph by t. Hence $r^*(T, K_n) \leq t^{1+\log_{1+\beta} n} \leq tn^{1/(\log_t(1+\beta))}$.

We still have to prove the claim, so assume for a contradiction that for any two adjacent layers and any two subsets A and B of these layers with $|A| + |B| \geq (1 + \beta)n$ there is a red edge between A and B. Consider a vertex v in T with children $v_1, \ldots, v_d$ for each of which there is a set $R(v_i)$ of vertices from the layer associated with v_i in F' such that $|R(v_i)| \geq (1 - k\beta)n$ for some $k \geq 0$. We claim that there is a subset $R(v)$ of the layer associated with v such that each vertex $w \in R(v)$ is connected by a red edge to some vertex in $R(v_i)$ for each $1 \leq i \leq d$ and $|R(v)| \geq (1 - d(k + 1)\beta)n$. The basic observation is that if R is a subset of a layer and L a subset of an adjacent layer, then L contains a subset of at least $|L| - (n(1 + \beta) - |R|)$ vertices each of which is connected by a red edge to a vertex in R; this follows by taking subsets A of L of order $n(1 + \beta) - |R|$. Letting $R = R(v_1)$ and $L = L(v)$ we obtain a set R' of $(1 - (k + 1)\beta)n$ vertices in $L(v)$ all of which are connected by a red edge to some vertex in $R(v_1)$. Repeating this argument with $L = R'$ and $R = R(v_2)$, etc. will give us the set $R(v)$ we were looking for.

Now traverse the graph T in a breadth-first way, letting $R(v) = L(v)$ for all the leaves v of T and using the procedure described above to compute $R(v)$ for inner nodes. An easy computation shows that if v is the root of the tree then $|R(v)| \geq (1 - (t - 1)\beta)n$. Hence choosing $\beta = 1/t$, we know that there is a red subtree isomorphic to T in F which contradicts the assumption thereby establishing the claim.

Hence $F \rightarrowtail (T, K_{(1+\beta)n})$, and $|F| \leq |T| * |F_n|$. This implies that in general $|F_n| \leq |T|^{\lceil \log_{1+\beta} n \rceil} \leq |T| n^{2|T| \log |T|}$ where we use the fact that $\log_t(1 + 1/t) \geq 1/(2t \log t)$. □

We can use the construction of the Theorem 3.1 to obtain results for trees versus complete bipartite graphs. Remember that the induced size Ramsey number $r_e^*(G, H)$ is the smallest number of edges of a graph F fulfilling $F \rightarrowtail (G, H)$.

Theorem 3.2 *If T is a tree, the following hold (constructively):*

- $r^*(T, K_{n,n}) \leq |T|^2 n$.
- $r_e^*(T, K_{n,n}) \leq |T|^3 n^2$.
- $r^*(T, C_4) \leq 2|T|^2$.

The theorem follows from the following lemma.

Lemma 3.3 *Given a tree T of order t and an integer n we can construct a graph F of order at most $t^2 n$ and size at most $t(tn)^2$ such that $F \rightarrowtail (T, K_{n,n})$ for every n.*

Proof. Use $F = T[\overline{K_{tn}}]$. As in the proof above one can show that there is either a red T or a complete bipartite graph on $2tn/t$ vertices. We do not need an inductive construction here since we can force the absence of edges within the two partitions of $K_{n,n}$ by using the complement of a complete graph. Note that F has at most $t^2 n$ vertices and $t(tn)^2$ edges. □

4 Bipartite Graphs versus Trees

Rödl showed in his master's thesis that $r^*(G) \leq 2^{c|G|}$ for a bipartite graph G using an explicit construction [Die97, Lemma 9.3.3]. Since $r^*(G) \geq r(G) \geq 2^{|G|}$ by a result of Chvátal and Harary [GRS90], this is a reasonably tight bound. In this section we consider the restricted version in which one of the graphs is a tree. We will give explicit constructions showing that

- $r^*(G, P_n) \leq |G|^{\lceil \log n \rceil}$ if G is bipartite,
- $r^*(G, T) \leq |G|^{\lceil (d-1) \log |T| \rceil}$, where d is the maximum degree of T, and G is bipartite,
- $r^*(G, T) \leq |T| * |G|^{\log |T|}$ for a full tree T and bipartite G,
- $r^*(G, T) \leq |G|^{(2|T| \log |T|)^{1/2}}$ if G is bipartite.

Note that the last result implies an upper bound of $4^{|T|^{1/2}\log^{3/2}|T|}$ for $r^*(T)$.

To simplify the presentation in this section we introduce an extension of the composition notation.

Definition 4.1 *Let $F = (V_1 \dot\cup V_2, E)$ be a bipartite graph, $F_1 = (W_1, E_1)$, $F_2 = (W_2, E_2)$ two graphs, and $A_1 \subseteq W_1$, $A_2 \subseteq W_2$. We define $F[V_1 \to F_1|A_1, V_2 \to F_2|A_2]$ with vertex set $V_1 \times W_1 \dot\cup V_2 \times W_2$ to include the following edges:*

(i) an edge between (v, w) and (v, w') for each $ww' \in E_i$ and $v \in V_i$ (where $i = 1, 2$),

(ii) an edge between (v, w) and (v', w') for $vv' \in E$, $v \in V_1$, $v' \in V_2$, $w \in A_1$, $w' \in A_2$.

We will drop A_1, A_2, or both in the notation if they are maximal (V_1, or V_2). We will drop $V_1 \to F_1|A_1$ or $V_2 \to F_2|A_2$ if no substitution on V_1 or V_2 takes place (i.e. F_1 or F_2 is a single vertex).

That is, we build $F[V_1 \to F_1|A_1, V_2 \to F_2|A_2]$ from F by substituting each vertex of F by a copy of F_1 or F_2 (depending on whether it is from V_1 or V_2) and include complete bipartite graphs between the vertices from A_1 and A_2.

The next two results will show two ways to build trees (versus a fixed bipartite graphs). For these (and later results) we will need a notion of locating vertices of an arrowed tree in a coloring.

Definition 4.2 *If $F \rightarrowtail (G, T)$ we call T* located *in F, if for each $v \in V(T)$ there is a $A_v \subseteq V(F)$, such that the A_v are pairwise disjoint, and if a coloring of F does not contain a red induced G, then it contains a blue induced copy of T with the copy of v in the set A_v (we will also say: v is located in F).*

The first lemma allows us to build trees along an edge.

Lemma 4.3 *Let T be a tree, and T_1, T_2 be two subtrees of T obtained by removing one edge from T. If $F_1 \rightarrowtail (G, T_1)$, and $F_2 \rightarrowtail (G, T_2)$ such that T_i is located in F_i, then we can build an F such that $F \rightarrowtail (G, T)$, T is located in F, and $|F| \leq |G| * \max\{|F_1|, |F_2|\}$.*

Proof. Let $G = (V_1 \dot\cup V_2, E)$. Fix $v_1 \in T_1$, and $v_2 \in T_2$ such that v_1v_2 is the edge removed from T to obtain T_1 and T_2. Since T_1 and T_2 are located in F_1 and F_2, rsp., there are sets A_1 and A_2 such that the copies of v_1 and v_2 in F_1 and F_2 will occur in these sets. Consider an arbitrary coloring of the graph $F = G[V_1 \to F_1|A_1, V_2 \to F_2|A_2]$. We can assume that all the copies of F_i contain a blue induced T_i with v_i in A_i (otherwise we are done). More precisely for each $v \in V_i$ there is a blue, induced tree $T_{v,i}$ in F, its vertices from $\{v\} \times F_i$, containing the vertex $\{v, v_i\}$ in $\{v\} \times A_i$. The graph F restricted to these $\{v, v_i\}$ $(v \in V)$ is isomorphic to G. Hence this restricted graph it has to contain a blue edge (otherwise we have a copy of a red induced G) which (by construction) connects a blue induced T_1 to a blue induced T_2. Again, more precisely, there are $u \in V_1$, and $w \in V_2$ such that the edge $\{\{u, v_1\}, \{w, v_2\}\}$ is blue. Together with $T_{u,1}$, and $T_{w,2}$ this completes a blue induced T. The bound on the size of F is immediate. Furthermore T is located in F. This follows from the assumption about T_1 and T_2 and the construction. □

The second lemma deals with vertices.

Lemma 4.4 *Let T be a tree rooted in v and F be a graph such that $F \rightarrowtail (G, T)$ and v is located in F. Given d, define a tree T' by taking d copies of T and identifying their roots v with the outer vertices of a $K_{1,d}$. Then we can construct F' such that $F' \rightarrowtail (G, T')$ and the center of the $K_{1,d}$ is located in F'. Furthermore $|F'| \leq d|G|^2 * |F|$.*

Before we prove the lemma we need to establish the following claim.

Claim 4.5 *For every bipartite graph G there is a bipartite graph $G' = (V_1 \dot\cup V_2, E)$ of order at most $d|G|^2$ such that every coloring of G' either contains a red induced G or a blue induced star $K_{1,d}$ with its root in V_1, and its children in V_2.*

Proof. Let $G = (W_1 \dot\cup W_2, E')$ be a bipartite graph. Construct G' by substituting every vertex in W_2 by $n := |W_1|(d-1) + 1$ new vertices, that is $V_1 = W_1$, $V_2 = W_2 \times \{1, \dots, n\}$, and $E = \{a(b, i) : a \in V_1, b \in V_2, ab \in E, \text{ and } 1 \leq i \leq n\}$. Assuming that there is no blue induced $K_{1,d}$ with center in V_1 one easily constructs a red induced G. Furthermore $|G'| \leq |W_1| + |W_2|n = |W_1| + |W_2| + |W_2||W_1|(d-1) \leq |G|^2 d$. □

Proof of Lemma 4.4. Fix G, F, v, T, d and T' as in the assumption of the lemma, and select $A \subseteq V(F)$ such that v will lie in A if a coloring of F contains a blue induced T. Choose $G' = (V_1 \dot\cup V_2, E)$ as constructed in the preceding claim. Let $F' = G'[V_2 \rightarrow F|A]$. Each copy of F contains a blue induced copy of T with v in A (otherwise we are done). If we restrict F' to the copies of these v we get a graph isomorphic to G'. Hence we can assume that F' contains a blue induced $K_{1,d}$ with center in V_1 and all of its children copies of v vertices. This completes a blue induced T'. We have $|F'| \leq |G'| * |F| \leq d|G|^2|F|$, and the center of the star is located in F'. □

Let us see how to apply these lemmas. The first lemma gives us a good handle on trees of small maximum degree. The reason is the following well-known result.

Lemma 4.6 *For every tree T of maximum degree d there is a node of the tree such that the subtree rooted in that node is of order between $(|T|-1)/d$ and $(|T|-1)(d-1)/d$.*

Proof. Root the tree in one of its leaves. Starting from there keep selecting a child v of the current node such that the subtree T_v rooted in v has more than $(|T|-1)(d-1)/d$ vertices (the unique child of the root certainly fulfills this property). Eventually we find a vertex v for which there is no such child, i.e. for all children w of v the subtree T_w rooted in w has at most $(|T|-1)(d-1)/d$ vertices. Since $|T_v| \geq (|T|-1)(d-1)/d$ we know that one of the (at most $d-1$) children has at least $(|T|-1)/d$ vertices, completing the proof. □

Theorem 4.7 *Let G be a bipartite graph, and T a tree on n vertices with maximum degree d ($d \geq 2$). Then we can construct a graph F of order at most $|G|^{\lceil (d-1)\log n \rceil}$ such that $F \rightarrowtail (G,T)$ (and T is located in F).*

Proof. We use Lemma 4.6 to split T into two parts each of size at most $(d-1)/d(|T|-1)$. Recursively we can build T from these subtrees using Lemma 4.3. If $V(T) = \{v\}$, we let F be the graph on a single vertex, and A_v contain that single vertex. This will do as a base for the recursion. If $k \geq (d-1)\log n$ we have $(d/(d-1))^k = (1+1/(d-1))^k \geq 2^{\log n} \geq n$, hence the construction will take at most $\lceil (d-1)\log(n) \rceil$ steps, hence $|F| \leq |G|^{\lceil (d-1)\log n \rceil}$. □

One specific result seems worth mentioning.

Corollary 4.8 *For every bipartite graph G we can build a graph F of order at most $|G|^{\lceil \log n \rceil}$ such that $F \rightarrowtail (G, P_n)$ (and P_n is located in F).*

Lemma 4.4 works well with full trees. Suppose T is a full d-ary tree of height h. Then we can apply the lemma recursively to get an F with $F \rightarrowtail (G,T)$ and $|F| \leq d^h * |G|^{2h} \leq |T||G|^{2\log_d |T|}$.

Theorem 4.9 *For every full d-ary tree T and every bipartite graph G we can construct a graph F of order at most $|T| * |G|^{2\log_d |T|}$ such that $F \rightarrowtail (G,T)$.*

The previous best constructive upper bound on bipartite graphs versus trees is given by Rödl's construction for bipartite graphs. We can improve that bound by combining the two ideas above: we use Lemma 4.4

for vertices of high degree, and Lemma 4.3 for vertices of low degree. Suppose we are given a tree T on n vertices with a vertex v of degree $d \geq n/(n \log n/3)^{1/2}$. Let $T_1, \ldots, T_d$ be the trees rooted in the children $v_1, \ldots, v_d$ of v. There is a single tree T' with root v' such that every T_i is a subgraph of T' where v_i is mapped to v', and the degree sequence of T' is a subsequence of the degree sequence of $T - \{v\}$. (To see this, consider all trees T' with root v' such that every T_i is a subgraph of T' and v_i is mapped to v'. Assume every node in T' has smallest possible degree, where we minimize the degrees of the vertices in a breadth-first order. Any vertex in that tree T' which does not coincide with a vertex of the same degree from one of the T_i could be substituted by a vertex of smaller degree all of whose grandchildren are the same tree, namely the tree obtained from identifying the roots of all the grandchildren of the original vertex.) Hence if we take d copies of T' and identify the satellite vertices of a $K_{1,d}$ with the roots of the copies of T', the resulting graph will contain T as a subgraph. Therefore we can apply the construction of Lemma 4.4 for vertex v. We now repeat the construction recursively with T' until all vertices have degree at most $n/(n \log n/3)^{1/2}$. This will take at most $(n \log n/3)^{1/2}$ steps (by the condition on the degree sequence of T' we are not introducing new vertices of high degree, and we are removing at least one vertex of high degree at each step). At this point we apply the construction from Theorem 4.7 which will give us a graph of size $|G|^{(n/(n \log n/3)^{1/2}) \log n} = |G|^{(3n \log n)^{1/2}}$. This graph we now use as the base for the $(n \log n/3)^{1/2}$ steps of the Lemma 4.4 construction which gives us an upper bound of $(|G| * |G|^2)^{(n/(n \log n/3)^{1/2})} |G|^{(3n \log n)^{1/2}} = |G|^{(6n \log n)^{1/2}}$ which is what we set out to prove.

Theorem 4.10 *Given a bipartite graph G and a tree T we can construct a graph F of order at most $|G|^{(6|T| \log |T|)^{1/2}}$ such that $F \rightarrowtail (G, T)$.*

Corollary 4.11 *Given a tree T we can build a graph F of order at most $6^{|T|^{1/2} \log^{3/2} |T|}$ such that $F \rightarrowtail (T, T)$.*

5 Cycles

This section contains constructions showing the following bounds:

- $r^*(G, C_n) \leq |G|^{\lceil \log n \rceil + 4}$ if G is bipartite.
- $r^*(T, C_n) \leq |T|^{\lceil \log n \rceil + 1}$ if T is a tree.
- $r^*(C_n, C_m) \leq 4^{(\log(n) + \log(m))^2}$.

There does not currently seem to be a randomized result better than our upper bound on $r^*(G, C_n)$.

5.1 Cycles and Bipartite Graphs

Lemma 5.1 *If G has maximum degree d, then $G[\overline{K_{d+1}}] \rightarrowtail (G, K_{1,2})$.*

Proof. Let $F = G[\overline{K_{d+1}}]$ and fix a coloring in which F does not have a blue induced $K_{1,2}$. We will show how to recursively embed a red induced G in F. Each vertex of G has $d+1$ possible representatives to choose from. We start with any vertex, and choose any of its representatives. Assume now that we have built a part of G in F and now want to add vertex $v \in V(G)$. We have $d+1$ choices. However we might have already embedded some of the neighbors of v. Since v has at most d neighbors, and each representative chosen for one of these neighbors is incident to at least d red edges to the representatives of v (otherwise there would be a blue $K_{1,2}$) there is one representative of v which is connected by red edges to all representatives of neighbors of v chosen so far. □

Theorem 5.2 *Given a bipartite graph G and a cycle C_n ($n \geq 3$) we can construct a graph F of order at most $|G|^{\lceil \log n \rceil + 4}$ such that $F \rightarrowtail (G, C_n)$.*

Proof. Fix $G = (V_1 \dot\cup V_2, E)$ with maximum degree d. Construct a tree T from a $K_{1,d+1}$ by replacing each edge with a P_{n-1}. Combining the ideas from Corollary 4.8 and Lemma 4.4, we obtain a graph H of order at most $(d+1)|G|^2 * |G|^{\lceil \log n \rceil} \leq |G|^{3+\lceil \log n \rceil}$ and a set $A \subseteq V(H)$ such that $H \rightarrowtail (G, T)$. Furthermore if H does not contain a red induced G, it contains a blue induced T with its $d+1$ leaves in the set A (since T is induced this in partiular means that there are no edges between those leaves).

Consider the graph $F = G[V_1 \rightarrow H|A, V_2 \rightarrow H|A]$. Fix a coloring of F without a red induced G subgraph. Then each copy of H contains a blue induced H with its $d+1$ leaves in A. Restrict F and its coloring to these $(d+1)|G|$ leaves. By Lemma 5.1 this graph has to contain a blue induced $K_{1,2}$. This blue induced $K_{1,2}$, however, completes a blue induced C_n in F, and we are done. □

The use of Lemma 5.1 in the last theorem was necessary to deal with cycles in the graph G. If G is acyclic, we can obtain slightly better bounds.

Corollary 5.3 *For every tree T and every $n \geq 3$ we can build a graph F of order at most $|T|^{\lceil \log n \rceil + 1}$ such that $F \rightarrowtail (T, C_n)$.*

5.2 Cycles versus Cycles

Theorem 5.4 *For every C_n and C_m we can construct a graph F of order at most $4^{(\log(n)+\log(m))^2}$ such that $F \rightarrowtail (C_n, C_m)$.*

We split the proof into two cases: one of n and m is even (Lemma 5.5), or both are odd (Lemma 5.7). The case that one of the cycles is even is covered by Theorem 5.2, but we improve the construction to get a slightly better bound.

Lemma 5.5 *For every m and every even n we can build a graph F of order at most $3/2n^{\lceil \log m \rceil + 1}$ such that $F \rightarrowtail (C_m, C_n)$.*

Proof. Since n is even we can apply Corollary 4.8 to obtain a graph $F_{m,n}$ of order at most $n^{\lceil \log m \rceil}$ such that $F \rightarrowtail (C_n, P_m)$.

Construct a graph G by taking two points and connecting them by three three vertex-disjoint paths of length $n/2$ (see Figure 4).

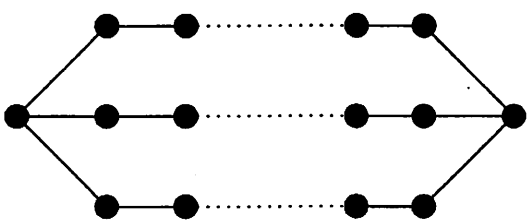

Figure 4: The three paths of G

Consider an arbitrary coloring of $F = G[F_{m,n}]$. If none of the copies of $F_{m,n}$ contain a red C_n, they all contain a blue induced path of length $m-2$. For every vertex v of G fix the two endpoints a_v and b_v of such a path. Let v and w be adjacent in G. If we restrict F to the $K_{2,2}$ on the vertices $\{a_v, b_v, a_w, b_w\}$ we see that either both $a_v a_w$ and $b_v b_w$ or both $a_v b_w$ and $b_v a_w$ are red (otherwise one of the blue induced paths is extended to a blue induced C_m). In the second case we say that the edge vw in G switches (meaning that the red edges switch from the a to the b vertices, and vice versa). Of the three paths of length $n/2$ that connect the two special points in G we can select two such that the parity of the total number of switching edges on the resulting cycle is even. Following this cycle of length n in F (starting with an arbitrary a on the cycle) and switching from a_v to b_w or b_v to a_w whenever the corresponding edge is switching, or remaining on the a or b side if the edge is not switching, yields a red induced cycle of length n in F. □

For the following lemma and theorem let $T_{h,\ell}$ be the following tree: take a full ternary tree of height h, and 3^h paths of length ℓ. Pair up the paths with the leaves of the ternary tree, and identify one end of each path with the corresponding leaf of the ternary tree. Note that $T_{h,\ell}$ contains 3^h paths of length $h + \ell$ from its root to its leaves.

Lemma 5.6 *For every n, m, h (with n odd) we can build a graph F of order at most $(nm)^{2h+\lceil \log \ell \rceil}$ and pairwise disjoint $A_v \subseteq V(F)$ such that F contains either a red induced C_n, a blue induced C_m, or a blue induced $T_{h,\ell}$ with each leaf v in A_v (for all 3^h leaves).*

Proof. Fix n (odd) and m. If we could apply Lemma 4.3, we could build a graph strongly arrowing $(C_n, T_{h,\ell}$, however, C_n is not bipartite. Fortunately, it is nearly bipartite.

Consider the graph G obtained by linking a central point to each point of a C_m by a path of length $(n-1)/2$. If a coloring of G does not contain a red induced C_m, one of the edges of the C_m has to be blue, which forces one of the edges on the two paths from that edge to the central point to be red. Splitting up the vertices of G into sets V_1 and V_2 according to whether their distance from the central point is odd or even gives us a graph G which in any coloring either contains a red induced C_m, a blue induced C_n, or a blue edge between the two disjoint sets V_1 and V_2. We can now use G, V_1, and V_2 in the construction from Lemma 4.3 to build a graph which in any coloring either contains a red induced C_m, a blue induced C_n, or a blue induced $T_{h,\ell}$. We first apply the lemma $\lceil \log \ell \rceil$ many times to obtain a graph such that any coloring of that graph contains either a red induced C_n, a blue induced C_n, or a path of length ℓ whose endpoints are located. It now takes us another $2h$ steps to complete the ternary structure connecting the paths (two steps for each level of the full ternary tree). Hence $|F| \leq |G|^{2h+\lceil \log \ell \rceil}$, implying, together with $|G| \leq nm$, the upper bound. □

We need to take a closer look at the construction. Suppose v and w are two leaves of $T_{h,\ell}$ in Lemma 5.6, and A_v and A_w their associated sets in F. Consider a coloring of F without a blue induced $T_{h,\ell}$. F either contains a red induced C_n, or a blue induced C_m. This does not change if we add edges between A_v and A_w (since they belong to different stages of the construction).

Lemma 5.7 *Suppose n and m are odd, then we can build a graph F of order at most $(nm)^{2\log(n)+\log(m)}$ such that $F \rightarrowtail (C_n, C_m)$.*

Proof. Fix $n \leq m$, both odd. Let $h = \lceil \log_3(n) \rceil$, $\ell = (m-1)/2 - h$, and choose F and A_v be as in Lemma 5.6. We can choose a sequence $v_0, \ldots, v_{n-1}$ of n (distinct) leaves of $T_{h,\ell}$ such that the path from v_i to $v_{i+1 \bmod n}$ in $T_{h,\ell}$ has length precisely $2((m-1)/2 - h) + 2h = m - 1$ for all i (we only have to make sure that v_i and v_{i+1} are not children of the same child of the root; since n is odd we need three children of the root to achieve this). To F we add complete bipartite graphs between A_{v_i} and $A_{v_{i+1 \bmod n}}$ for all i. Call the resulting graph F'. Fix a coloring of F' without a red induced C_n, or a blue induced C_m. By Lemma 5.6 there is a blue induced

$T_{h,\ell}$ with endpoint v_i in A_{v_i}. Restrict F' to $\{v_0, \ldots, v_{n-1}\}$. The resulting graph is an (induced) C_n. If all of its edges are red, we are done. Hence at least one edge, $v_i v_{i+1}$ say, is blue. This edge, however, completes a blue induced C_m, because we already have a blue P_m from v_i to v_{i+1} which as a subgraph of F (rather than F') is induced.

We get the upper bound on F by observing that $2h \leq \log n$, and $\lceil \log \ell \rceil \leq \log m$. □

6 Small Graphs

A fair amount of research effort has been directed towards determining Ramsey numbers precisely. Currently all the numbers $r(K_i, K_j)$ for $i+j \leq 9$ are known, and we also know that $43 \leq r(K_5) \leq 49$ [Rad99]. It seems as if mathematicians want to prepare for the invasion of Erdös's hypothetical alien force which will come down to earth and ask us for the value of $r(K_5)$. They will be quite surprised if the aliens ask for induced Ramsey numbers instead. It appears there has been only one paper so far that investigates induced Ramsey numbers for small graphs [HNR83]. The paper shows that $r^*(C_4) \leq 10$, since both $K_{3,7} \rightarrowtail C_4$ and $K_{5,5} \rightarrowtail C_4$, and that $r^*(P_4) \geq 8$. The proof of $r^*(P_4) \leq 8$ is wrong, but we manage to get the same result here. We show upper bounds for $r^*(K_3 + e)$, $r^*(K_4 - e)$, $r^*(K_3, K_3 + e)$, $r^*(K_3, K_4 - e)$, and $r^*(C_3, C_4)$. Although all of these seem to be rather far away from what we would expect the actual numbers to be, there have not been any previous upper bounds for these problems at all, so they should be considered as a challenge rather than the last word.

6.1 P_4

In this section we will show that $r^*(P_4) \leq 8$. It was claimed earlier that the Möbius ladder M_8 strongly arrows P_4 [HNR83], but this is not the case (color the outer edge alternatingly red and blue, and two of the spokes red, and the other two blue such that a blue induced C_4 results).

We claim that the graph in Figure 5 contains a monochromatic P_4 in every two-coloring. For a contradiction assume there is a coloring in which it does not. Fix this coloring. At least two of the edges 1, 2, and 3 have to have the same color, say red. If 1 and 2 are red, then 4, 5, and 10 have to be blue. Now 8, 9 and 11 are forced to be red (8, and 9 because 4 and 5 are blue, and 11 because 10 and 5 are blue) completing a red induced P_4. In case that 1 and 3 are red we know that 4, 5, 6, and 7 all have to be blue. This in turn forces 8 and 9 to be red. If 11 is red, then 8, 11,9 form a red induced P_4, otherwise 5,11,6 is a blue induced P_4. Finally, we can assume that 2 and 3 are red, and 1 is blue. This forces 7 to be blue. Then 6 has to

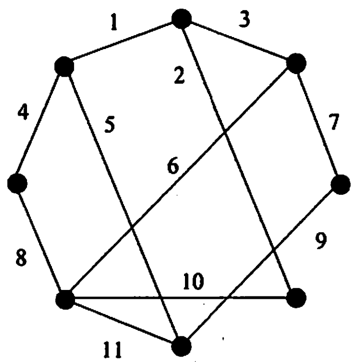

Figure 5: Graph strongly arrowing P_4

be red (otherwise 8, 10, and 2 are a red induced P_4), forcing 8 and 11 to be blue which in turn forces 4, 5 and 9 to form a red induced P_4.

6.2 $K_3 + e$

We will prove the following two upper bounds involving $K_3 + e$:

(*i*) $r^*(K_3 + e) \leq 21$,

(*ii*) $r^*(K_3 + e, K_3) \leq 18$.

The best lower bounds are given by $r(K_3 + e, K_3) = r(K_3 + e) = 7$ (Chvátal, Harary [Rad99]).

Let us start with the first result, the second will be an easy modification. Let the graph H consist of a $K_3 + e$ together with an isolated vertex. Take three copies H_1, H_2, and H_3 of H, and a copy of a K_6 on vertices a, b, c, d, e and f. Include complete bipartite graphs between $\{a, b\}$ and H_1, $\{c, d\}$ and H_2, and $\{e, f\}$ and H_3. Call the resulting graph F. We claim that $F \rightarrowtail K_3 + e$. Fix a coloring of F. The central K_6 contains a monochromatic triangle, say in red. At least one of the sets $\{a, b\}$, $\{c, d\}$, or $\{e, f\}$ intersects the triangle in exactly one vertex, suppose it is $\{a, b\}$, and the vertex is a. Now all edges between a and the vertices of H_1 have to be blue, since otherwise we would have completed a red induced $K_3 + e$. Within H_1 is an induced copy of $K_3 + e$. If all the edges red, we are done, hence we can assume that one of the edges is blue. This edge, together with a, and the isolated vertex of H_1 form a blue induced $K_3 + e$.

For the second result we can save three vertices by letting H be $K_3 + e$ without an additional isolated vertex.

6.3 $K_4 - e$

We will show two upper bounds involving $K_4 - e$ in this section:

(*i*) $r^*(K_3, K_4 - e) \leq 16$,

(*ii*) $r^*(K_4 - e) \leq 46$.

We note that the best lower bounds are through $r(K_3, K_4 - e) = 7$, and $r(K_4 - e) = 10$ (both by Chvátal, Harary [Rad99]).

To prove (*i*), take a $K_{1,3,4,4}$ and a copy of $K_4 - e$ and include all edges between them. For a contradicion assume that there is a coloring of the resulting graph which has neither a red triangle nor a blue, induced $K_4 - e$. Because of the later the copy of $K_4 - e$ will contain at least one red edge, call it f with endpoints u and v. Consider the following situation: u is incident to two red edges leading to one partition (with endvertices x_1, x_2), and one red edge leading to another partition (with endvertex y) of the $K_{1,3,4,4}$. Since we do not have any red triangle the following edges are forced to be blue: yv, yx_1, yx_2, vx_1, vx_2 completing a blue induced $K_4 - e$ (since there is no edge x_1x_2). Hence this cannot occur, and we get one of the following two cases: u is incident to two red edges to one partition, and no red edge to any other partition, or u is incident to at most one red edge in each partition. In either case, the graph contains a blue $K_{1,7}$ (not induced) centered in u the edges of which split up as $1+3+3$ among the partitions. Call the single vertex w, and the groups of three vertices A and B. If there were two blue edges from w to either A or B this would complete a blue induced $K_4 - e$, hence w is incident to at least two red edges to each of A and B. Let A' be the red neighbors of w in A, and B' be the red neighbors of w in B. As before all edges between A' and B' are forced to be blue. Since all the vertices in A' and B' are connected to u by blue edges, this again completes a blue induced $K_4 - e$.

For (*ii*) we first note that $K_{1,3,6} \rightarrowtail (K_4 - e, K_{1,3})$. Suppose this was false. Let the partitions of $K_{1,3,6}$ be A, B and C with $|A| = 1$, $|B| = 3$, $|C| = 6$. Then there is a red edge uv from A to B (since there is no blue induced $K_{1,3}$). Again the absence of a blue induced $K_{1,3}$ forces at least four red edges from u to C and four red edges from v to C. Hence u and v have at least two common red neighbors in C completing a red induced $K_4 - e$.

Construct a graph F as follows: take a K_6 and four copies of $K_{1,3,6}$ and include all edges from the K_6 to the other four graphs. Consider a coloring which contains neither a red, nor a blue induced $K_4 - e$. Now K_6 contains a monochromatic triangle, say in red. Suppose that every copy of $K_{1,3,6}$ contains a vertex which has at least two red edges leading to that triangle. Then for two such vertices the red neighbors in the red triangle must be identical (since there are four copies, and only three two element subsets of the three vertices) completing a red induced $K_4 - e$. Hence in one of the copies of $K_{1,3,6}$ all vertices have at most one red edge to the red triangle. Since $K_{1,3,6}$ does not contain a red induced $K_4 - e$ it does contain a blue induced $K_{1,3}$. Each of its four vertices is connected by at least two blue edges to the red triangle. It follows that one of the triangle points has at least three outgoing blue edges, one to the center of the $K_{1,3}$ and two to outer vertices of the $K_{1,3}$. This, however, would complete a blue induced $K_4 - e$.

6.4 C_3 versus C_4

We show that $r^*(C_3, C_4) \leq 14$. The best lower bound is through $r(C_3, C_4) = 7$ (by Faudree, Schelp, Rosta [Rad99]). To show the upper bound construct a graph F as follows: take a $K_{4,4,4}$ together with two isolated vertices u and v, and include a complete bipartite graph between $\{u, v\}$ and the $K_{4,4,4}$. We claim that $F \rightarrowtail (C_3, C_4)$. Fix a coloring of F. If there are two paths (of length two) consisting of blue edges only from u to v whose midpoints belong to the same partition of the $K_{4,4,4}$ we get a blue induced C_4. Hence there is at most one blue path from u to v through each partition. We can therefore restrict F to u, v, and a $K_{3,3,3}$ such that there is no blue path from u to v. Hence each vertex of the $K_{3,3,3}$ has a red neighbor in $\{u, v\}$. Label each vertex in $K_{3,3,3}$ with such a neighbor. In each partition of the $K_{3,3,3}$ one of the labels occurs at least twice, and since we have three partitions, there are two partitions in which the same label occurs at least twice. Hence we have four vertices from the $K_{3,3,3}$ with the same label (say u) which split across partitions as $2 + 2 + 0$. If any of the edges between these four vertices is red it completes a red C_3 (with u). Otherwise the four vertices induce a blue C_4.

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The $L(2,1)$-labeling problem on ditrees

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Abstract

An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d_G(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d_G(x,y)=2$. The $L(2,1)$-labeling problem is to find the smallest number $\lambda(G)$ such that there exists a $L(2,1)$-labeling function with no label greater than $\lambda(G)$. Motivated by the channel assignment problem introduced by Hale, the $L(2,1)$-labeling problem has been extensively studied in the past decade. In this paper, we study this concept for digraphs. In particular, results on ditrees are given.

Keywords. $L(2,1)$-labeling, $L(2,1)$-labeling number, ditree.

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