A Note on the Representation of Unit Interval Graphs: A Link Between Interval Graphs and Semiorders

Denise Sakai Troxell[1]
Mathematics and Science Division - Babson College

Abstract: A graph is a unit interval graph (respectively, an $\tilde{n}$-graph) if we can assign to each vertex an open interval of unit length (respectively, a set of n consecutive integers) so that edges correspond to pairs of intervals (respectively, of sets) that overlap. Sakai [14] and Troxell [18] provide a linear time algorithm to find the smallest integer n so that a unit interval graph is an $\tilde{n}$-graph, for the particular case of reduced connected graphs with chromatic number 3. This work shows how to obtain such smallest n for arbitrary graphs, by establishing a relationship with the work by Bogart and Stellpflug [1] in the theory of semiorders.

1. Introduction

In Sakai [14] and Troxell [18], graph theoretical concepts were used to study predicates that arise in attempting to describe the interrelation among perceived stimuli in an environment. In particular, two classes of graphs play a major role in the works above: unit interval graphs and $\tilde{n}$-graphs.

A graph[2] G is called *interval graph* (respectively, *$\tilde{n}$-graph*) if there is a function f assigning to each vertex in V an open interval on the real line (respectively, a set with n consecutive integers) so that for all $x,y \in V$, with $x \neq y$,

$$\{x,y\} \in E \Leftrightarrow f(x) \cap f(y) \neq \emptyset.$$

For an interval graph G, if the intervals can all be chosen with unit length, G is called *unit interval graph*. There is a close relationship between the class of unit interval graphs and the class of $\tilde{n}$-graphs as shown by the following result.

Theorem 1 [Roberts [13]] Every $\tilde{n}$-graph is a unit interval graph. Moreover, every unit interval graph is an $\tilde{n}$-graph for some positive integer n.

[1] **Denise Sakai Troxell**, Mathematics and Science Division, Babson College, Babson Park, MA 02457-0310, phone: (781) 239-6309, fax: (781) 239-6416, e-mail: troxell@babson.edu

[2] Except as otherwise defined, we are going to use the graph-theoretical terminology of Harary [6]; also in what follows, G is a simple graph with vertex set V and edge set E.

In view of Theorem 1, it is very natural to ask the following question:

> *If G is a unit interval graph, what is the smallest integer n so that G is an $\tilde{n}$-graph? We will denote this number by $n_{min}(G)$.*

This question has been partially answered in Sakai [14] and Troxell [18] for reduced connected graphs G with chromatic number 3, where a linear time algorithm to find $n_{\min}(G)$ is provided. This algorithm is based on a characterization of such graphs with a given $n_{\min}(G)$ in terms of excluded induced subgraphs. We repeat here, for completeness, the definition of reduced graphs, however this notion will not be used in the present work. Let us define the following binary relation R on the vertices of G as follows. For all $x,y\in V$, $(x,y)\in R$ if and only if $x=y$ or $[\{x,y\}\in E$ and for all $z\in V$, $\{x,z\}\in E \Leftrightarrow \{y,z\}\in E]$. It is easy to see that R is an equivalence relation and we can define the graph G^* whose vertices are the equivalence classes of V under R, and with an edge between two equivalence classes containing x and y, respectively, if and only if $\{x,y\}\in E$. We say that G is *reduced* if G is isomorphic to G^*.

In the remaining sections, we will relate Graph Theory and the Theory of Ordered Sets, and show that the question above can be completely answered even for graphs that are not reduced and for graphs with arbitrary chromatic number. The main tool in showing the preceding answer is the work by Bogart and Stellpflug [1] in the theory of semiorders. A binary relation S on a finite set A is called a *semiorder* if the following axioms are satisfied for all $a,b,c,d\in A$:

Axiom 1: $(a,a)\notin S$
Axiom 2: $[\ (a,b)\in S$ and $(c,d)\in S\ ] \Rightarrow [\ (a,d)\in S$ or $(c,b)\in S\ ]$
Axiom 3: $[\ (a,b)\in S$ and $(b,c)\in S\ ] \Rightarrow [\ (a,d)\in S$ or $(d,c)\in S\ ]$

In Section 2, we use the theory on comparability graphs to investigate the uniqueness of transitive orientations of complements of unit interval graphs. Section 3 establishes the connection between the question above and the work by Bogart and Stellpflug [1].

2. Transitive Orientations of Complements of Unit Interval Graphs

We begin this section by recalling some definitions and results that are essential in the proof of Theorem 6, where we show that the transitive orientation of the complement of a unit interval graph is unique up to reversal.

The *complement* of G, denoted by G^c, is defined to be that graph with the same vertex set as G, and for all $x,y \in V$, with $x \neq y$, $\{x,y\}$ is an edge in G^c if and only if $\{x,y\} \notin E$.

An *orientation* F for the graph G is an assignment of directions to the edges of G. More specifically, F is a binary relation on V such that, for all $x,y \in V$, $\{x,y\} \in E$ if and only if F contains exactly one of the pairs (x,y) and (y,x). The *reversal* of an orientation F, denoted by F^{-1}, is the orientation of G so that $(x,y) \in F^{-1} \Leftrightarrow (y,x) \in F$. An orientation F is said to be *transitive* if for all $x,y,z \in V$, with $x \neq z$, if $(x,y),(y,z) \in F$ then $(x,z) \in F$. Graphs with transitive orientations are called *comparability graphs* or *partially orderable graphs*.

The following characterization relates interval graphs to comparability graphs.

Theorem 2 [Gilmore and Hoffman [4]] A graph G is an interval graph if and only if G does not contain circuits of length four as induced subgraphs and G^c is a comparability graph.

The following Lemma is a straightforward consequence and will be used in the sequel.

Lemma 3 Let G be an interval graph. Then G^c has at most one connected component containing at least one edge.
Proof: If G^c has two connected components with at least one edge each, then the vertices incident to these two edges form an induced circuit of length four in G, contradicting Theorem 2. So, only one connected component of G^c can have more than one vertex and all the others are isolated vertices. ■

Theorem 4 gives a necessary and sufficient forbidden structure condition for an interval graph to be a unit interval graph. A *claw* is any graph isomorphic to the graph with vertices a,b,c,d and edges $\{a,b\},\{a,c\}$, and $\{a,d\}$.

Theorem 4 [Roberts [10]] Let G be an interval graph. Therefore G is a unit interval graph if and only if G contains no claw as an induced subgraph.

A comparability graph G is called *uniquely partially orderable* if it has exactly two transitive orientations, one being the reversal of the other. A characterization of uniquely partially orderable graphs is provided by Theorem 5 below. We need some preliminary definitions.

A subset Y of V is called *partitive* if for each $x \in V-Y$ either $Y \cap \mathrm{Adj}_G(x) = \emptyset$ or $Y \subseteq \mathrm{Adj}_G(x)$, where $\mathrm{Adj}_G(x)$ denotes the set of vertices adjacent to x in G. A partitive subset Y is said to be *nontrivial* if it has more than one vertex and is not V.

A subset W of V is called *stable* if whenever $x,y \in W$, $\{x,y\} \notin E$.

Theorem 5 [Shevrin and Filippov [16]; Trotter et al. [17]] Let G be a connected comparability graph. Therefore, G is uniquely partially orderable if and only if every nontrivial partitive set of G is a stable set.

For more about uniquely partially orderable graphs we refer the reader to the excellent book by Golumbic [5].

We are now ready to state the main result in this section. We should notice, however, that Theorem 6 is essentially proved in Roberts [12] for reduced unit interval graphs using a notion of compatibility between a graph and a simple order. We prefer the more graph-theoretical approach given here since the reduced assumption is not required and since it minimizes the introduction of too much Ordered Set terminology.

Theorem 6 Let G be a connected unit interval graph. If G^c has at least one edge, then G^c is uniquely partially orderable.
Proof: By Theorem 2, G^c is a comparability graph. In view of Lemma 3, let G_1 be the connected component of G^c with at least one edge (notice that all the other connected components are isolated vertices). Clearly, G_1 is also a comparability graph. We will show that every nontrivial partitive subset of G_1 is stable (in G^c). So, by Theorem 5, G_1 is uniquely partially orderable, and consequently G^c is uniquely partially orderable, by Lemma 3.

Let V_1 and E_1 be the vertex set and the edge set of G_1, respectively. Suppose by contradiction that G_1 has a nontrivial partitive subset that is not stable. Let Y be a maximal nontrivial partitive subset of G_1 that is not stable. By the definition of partitive subsets, if $z \in V_1 - Y$ then either z is adjacent in G_1 to all vertices in Y, or z is not adjacent in G_1 to any vertex in Y. Let $x,y \in Y$ such that $\{x,y\} \in E_1$. Consider

$$Y_1 = \{z \in V_1 - Y: \{z,x\} \in E_1\}$$
$$Y_2 = \{z \in V_1 - Y: \{z,x\} \notin E_1\}$$

Clearly $Y_1 \cap Y_2 = \varnothing$ and $Y_1 \cup Y_2 = V_1 - Y$. Also, if $z \in Y_1$ and $w \in Y_2$ then $\{z,w\} \in E_1$ since otherwise w,x,y,z would induce a claw in G, contradicting Theorem 4. So, if $Y_2 \neq \varnothing$ then $Y \cup Y_2$ would be a partitive subset because $z \in V_1 - (Y \cup Y_2) = Y_1$ implies $Y \cup Y_2 \subseteq \mathrm{Adj}_{G1}(z)$. By the maximality of Y, we must have $Y \cup Y_2 = V_1$. Therefore we would have $Y_1 = \varnothing$, and consequently G_1 would be disconnected, since every vertex in Y_2 is not adjacent in G_1 to any vertex in Y, a contradiction. Hence $Y_2 = \varnothing$. Since Y is nontrivial, $Y_1 \neq \varnothing$. Let $z \in Y_1$. Notice that $V=V_1$ since otherwise there is a $w \in V-V_1$ and w,x,y,z would induce a claw in G, again contradicting Theorem 4. But then G would be disconnected since every vertex in Y_I is adjacent in G_I to all vertices in Y, and $Y \cup Y_I = V$, a contradiction. So, every nontrivial partitive set of G_I is stable. ■

3. Unit Interval Graphs and Semiorders

Theorem 7 establishes the connection between unit interval graphs and semiorders.

Theorem 7 [Roberts [11]] A graph G is a unit interval graph if and only if G^c is a comparability graph and every transitive orientation of G^c is a semiorder.

Before we proceed, a more convenient equivalent definition of semiorders and some notation will be introduced.

Theorem 8 Let S be a binary relation on a finite set A. The following statements are equivalent.

- **i.** S is a semiorder on A.
- **ii.** There exists a function f from A to a set of equal length nonempty open intervals of real numbers so that for all $x,y \in A$, $(x,y) \in S$ if and only if the interval $f(x)$ is to the left of the interval $f(y)$, that is, the right-hand endpoint of $f(x)$ is less than or equal to the left-hand endpoint of $f(y)$.
- **iii.** For any positive number δ, there exists a real-valued function g on A such that for all $a,b \in A$, $(a,b) \in S \Leftrightarrow g(a) > g(b) + \delta$.

Proof: For a proof for the equivalence **i** $\Leftrightarrow$ **iii** the reader is referred to Scott and Suppes [15]. The proof for the implications **ii** $\Rightarrow$ **i** and **iii** $\Rightarrow$ **ii** are straightforward and they are left to the reader. ■

We will use the definition of semiorders given in item **ii.** of Theorem 8. In this definition, the function f is called a *representation* of the semiorder. A representation f is said to be *discrete* if all the intervals in the image of f have integer endpoints. A semiorder is *k-representable* if it has a discrete representation using intervals all of length k. It should be observed that if a semiorder is k-representable, then it is also $(k+1)$-representable. (For a constructive proof of this claim see Bogart and Stellpflug [1].) It makes sense to have the following definition: a semiorder has *representation length k* if it is k-representable but not $(k-1)$-representable.

The *reversal* of a semiorder S on A, denoted by S^{-1}, is the binary relation on A so that for all $x,y \in A$, $(x,y) \in S^{-1} \Leftrightarrow (y,x) \in S$.

Lemma 9 Let S be a semiorder on a finite set A and let k be a positive integer. Therefore, S^{-1} is a semiorder on A; furthermore S is k-representable if and only if S^{-1} is k-representable.
Proof: The results follow by reversing the signs and the order of endpoints of intervals in a representation of S obtaining a representation of S^{-1}. ■

We are finally ready to state the result that will allow us to find $n_{\min}(G)$, that is, the smallest integer n so that a unit interval graph G is an $\tilde{n}$-graph.

Theorem 10 Let G be a connected unit interval graph. If G^c has at least one edge, then there exists a unique positive integer $k(G)$ so that every transitive orientation of G^c is a semiorder with representation length $k(G)$. Moreover, $k(G) = n_{\min}(G)$.
Proof: By Theorem 6, G^c is uniquely partially orderable. Let S and S^{-1} be the only two transitive orientations of G^c. By Theorem 7, S and S^{-1} are semiorders.

Let us show first that either S or S^{-1} is k-representable for some positive integer k. Since G is a unit interval graph, Theorem 1 implies that there is a positive integer n so that G is an $\tilde{n}$-graph. Let f be a function from V to a family of sets of n consecutive integers so that for all $x,y \in V$, with $x \neq y$, $\{x,y\} \in E \Leftrightarrow f(x) \cap f(y) \neq \varnothing$. Define the function f' from V to a set of equal length nonempty open intervals of real numbers, so that for each $x \in V$, $f'(x)$ is the interval with left-hand endpoint as the smallest integer in $f(x)$ minus one, and right-hand endpoint as the largest integer in $f(x)$. It is not difficult to see that f' is a discrete representation of either S or S^{-1} using intervals all of length n. Therefore, by Lemma 9, both S and S^{-1} are k-representable for $k=n$. Choose $k(G)$ to be the smallest positive integer k so that S and S^{-1} are k-representable, and the result holds.

It remains to be shown that $k(G) = n_{\min}(G)$. Notice that, if n is chosen to be equal to $n_{\min}(G)$, an argument similar to the one presented in the previous paragraph shows that $k(G) \leq n_{\min}(G)$. To show that $k(G) \geq n_{\min}(G)$, let f be a discrete representation of S using intervals of length $k(G)$. Define the function f' from V to the family of sets of $k(G)$ consecutive integers, so that for each $x \in V$, $f'(x)$ is the set of consecutive integers starting with the left-hand endpoint of the interval $f(x)$ plus one and ending with the right-hand endpoint of $f(x)$. It is not difficult to see that for all $x,y \in V$, with $x \neq y$, $\{x,y\} \in E \Leftrightarrow f'(x) \cap f'(y) \neq \varnothing$. Therefore G is an $\tilde{n}$-graph with $n=k(G)$, and this shows that $k(G) \geq n_{\min}(G)$. ■

Theorem 10 is the key result in answering the question mentioned in Section 1. Bogart and Stellpflug [1] find k so that a given semiorder has representation length k by looking for certain forbidden structures in the semiorder. Therefore, in view of Theorem 10, if G is a connected unit interval graph and G^c has at least one edge, we can find $n_{\min}(G)$ by looking at the semiorder given by a transitive orientation of G^c, and finding $k(G)$ such that the semiorder has representation length $k(G)$. On the other hand, if G^c has no edges, then G is a complete graph and it is easy to see that $n_{\min}(G) = 1$. We can finally find $n_{\min}(G)$ if G is a unit interval graph, not necessarily connected, by observing that $n_{\min}(G)$ is the largest $n_{\min}(G_i)$ over all connected components G_i of G.

4. Further Research

There are several related works on semiorder representations besides the work by Bogart and Stellpflug [1]. For example, Doignon [2,3] and Pirlot [8,9] use a variant of the usual definition of semiorders and semiorder representations, and establish the existence of a minimal representation. Mitas [7] provides an order-theoretic proof of the same fact and present a linear time algorithm for determining the representation length of a semiorder. A comprehensive summary of previous results on semiorder representation as well as further developments in the area can be found in Pirlot and Vincke [10].

One direction for further research would be to use the results on this paper to bridge the works above on semiorder representations and the graph-theoretic counterparts to look for alternative algorithms to find $n_{\min}(G)$ in terms of graph theoretic features, and to obtain a better interpretation of $n_{\min}(G)$.

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