A Symmetrical Beauty. A Non-simple 7-Venn Diagram With A Minimum Vertex Set

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Abstract

In this short note using the method developed in [10] and [11], we construct a highly symmetrical, non-simple, attractive 7-Venn diagram. This diagram has the minimum number of vertices, 21. The only similar two, published in [1] and [11], differ from ours in many ways. One of them was found by computer search ([1]). Both of them are "necklace" type Venn diagrams (see [14] for definition), but ours is not.

1 Introduction

First we briefly repeat the necessary definitions and notations. They can be found in more detail in [6], [7], [8], [2] and [10].

An *n-Venn diagram* consists of a family $\mathcal{F}$ of n simple closed Jordan curves in the plane so that all possible intersections (2^n many) of the interiors and the exteriors of these curves are nonempty and connected.

We note that each of the 2^n cells can be described by an n-tuple of zeros and ones where the ith coordinate is 0 if X_i is the unbounded exterior of C_i, otherwise it is 1, $i = 1, 2, \ldots, n$.

It is clear that there is a one-to-one correspondence between the 2^n sets of a Venn diagram and the vertices of the n-dimensional hypercube. If $A = X_1 \bigcap X_2 \bigcap \ldots \bigcap X_n$ is a set in a Venn diagram then the corresponding n-tuple in the hypercube is the ***description*** of A.

A Venn diagram is ***simple*** if at most two curves intersect (transversely) at any point in the plane. Among the nonsimple Venn diagrams, we will consider only those in which any two curves meet (not necessarily transversely) in isolated points and not in segments of curves.

With each Venn diagram one can associate two graphs. The Venn diagram itself can be viewed as a planar graph $V(\mathcal{F})$ where all the intersection points of the curves in $\mathcal{F}$ are the vertices of $V(\mathcal{F})$ and the segments of the curves with vertices as end points are the edges of $V(\mathcal{F})$. In proper context, confusion rarely arises from also calling this graph a Venn diagram. In the rest of this paper the notations $\mathcal{F}$ and $V(\mathcal{F})$ are freely interchanged when it is not important that the Venn diagram is a graph. The Venn diagram $V(\mathcal{F})$ may have multiple edges. The graph $V(\mathcal{F})$ is a planar graph. Note that the Venn diagram depends upon its drawing in the plane. The graph dual of $V(\mathcal{F})$ will be called the *Venn graph,* denoted by $D(\mathcal{F})$.

Quite often we will consider the Venn graph $D(\mathcal{F})$ to be superimposed on the Venn diagram in the plane, and we will use descriptive statements such as "the curve C of $\mathcal{F}$ intersects or crosses the edge e of $D(\mathcal{F})$". (These descriptive statements can be rigorously stated).

Several interesting properties of Venn diagrams and Venn graphs were derived in [2]. Here we simply state those properties we need as remarks and refer the reader to [2] for proofs.

Remark 1 The Venn graph $D(\mathcal{F})$ of a Venn diagram $V(\mathcal{F})$ is a planar, spanning subgraph of the $|\mathcal{F}|$-hypercube.

Remark 2 No two edges in a face in a Venn diagram belong to the same curve.

Remark 3 A Venn graph $D(\mathcal{F})$ is simple and 2-connected, but the deletion of any pair of adjacent vertices does not disconnect the graph.

Remark 4 If $\mathcal{F}$ forms a simple Venn diagram, then each face of $D(\mathcal{F})$ is a quadrilateral, and hence $D(\mathcal{F})$ is a maximal bipartite planar graph.

Remark 5 A simple Venn graph $D(\mathcal{F})$ has connectivity 3.

A Venn diagram with n curves is said to be *symmetric* if rotations through $360/n$ degrees map the family of curves into itself, so that the diagram is not changed by the rotation.

Recently, considerable attention has been given to symmetric Venn diagrams. The concept was introduced by Henderson [13]. The simple Venn diagrams with one, two, or three circles can obviously be represented as symmetric Venn diagrams. Henderson provided two examples of non-simple symmetric Venn diagrams with five curves, using five pentagons and five triangles. He stated in [13] that a diagram with seven curves had been found, but later he could not locate it.

2 Case $p = 7$.

Grünbaum conjectured in [6] that symmetrical 7-Venn diagrams do not exist. He later disproved his own conjecture by giving an example of a simple, symmetric Venn diagram of seven curves in [8]. He then conjectured that symmetric p-Venn diagrams exist for all prime numbers p. Shortly thereafter several others, Edwards [4] and [5], Ruskey [14], and Savage and Winkler (the last authors have never published their example, but it appeared in [14]), gave additional examples of such simple diagrams with seven curves. Bultena *et al.* [1] published a so called "necklace" type, non-simple 7-Venn diagram with the minimum number (21) of vertices, but it seems unlikely that the "necklace" procedure will yield results for larger values of p. Recently it was shown in [11] that there are symmetrical 7-Venn diagrams for all possible size vertex sets.

In this note we construct another symmetrical, non-simple, highly attractive, 7-Venn diagram, having 21 vertices, the minimum possible. In [11] it is shown that the possible numbers of vertices $|V|$ for a symmetric 7-Venn diagram are $|V| = 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105, 112, 119$, and 126. In the same paper a new approach is introduced. It starts with the defining of a graph with special properties and with a labeling of its edges. This graph is a modification of the Venn graph as defined above, but we will still refer to it as the *Venn graph of a Venn diagram*. This definition fits our purpose better.

Definition 1 *A 2-connected, planar, spanning, labeled subgraph G^* of the n-hypercube together with an edge numbering $\mathcal{N}$ of the edges of G^* is called a* Venn graph of a Venn diagram *iff*

1. *To each edge e of G^* is assigned a unique number between 1 and n (called the* edge number*) corresponding to the coordinate where the descriptions of the two end-vertices of e differ;*
2. *Any two faces of G^* share at most one edge with a given edge number; and*
3. *An edge number that appears in a face of G^* appears exactly twice in that face.*

In a case when each face of G^* is a 4-face and the graph G^* is 3-connected, G^* is the Venn graph of a simple Venn diagram.

Since the hypercube is a bipartite graph, each cycle is an even cycle, so each face of any 2-connected, spanning subgraph has an even number of edges. Two edges that are adjacent in the cube cannot share the same edge number, since that would correspond to changing one coordinate and then reversing the change, thus resulting in parallel edges. Furthermore, any curve of a Venn diagram $\mathcal{F}$ is not self-intersecting, thus two faces of the Venn graph $D(\mathcal{F})$ cannot share the same edge number twice.

Construction 2 *If G^* is the Venn graph of an n-Venn diagram and $\mathcal{N}$ is an edge numbering with the above properties, then a planar diagram (multigraph) $\mathcal{F}$ can be constructed in the following way.*

Step 1. *Create a graph $D^*(G^*)$ by placing a vertex x_F in each face F of G^* and joining it to each edge of F by a simple Jordan arc, such that*

- **(i)** *The arcs inside in a face F meet only at the new vertex x_F, and*
- **(ii)** *In every edge of G^* the arcs (exactly two of them) meet in the same point.*

Step 2. *Assign to each simple Jordan arc the edge number of the edge it meets. This identifies each simple closed Jordan curve in the Venn diagram.*

Step 3. *Remove the Venn graph G^* to obtain $\mathcal{F}$.*

It is easy to see that the diagram $\mathcal{F}$ thus obtained is a Venn diagram. It is also clear that if $D(\mathcal{F})$ is a Venn graph (in the usual sense) of a Venn diagram $\mathcal{F}$, that is, it is a subgraph of the n-hypercube, then using the method described above starting with $D(\mathcal{F}) = G^*$ the obtained multigraph

$\mathcal{F}^*$ is graph-isomorphic to the Venn diagram $\mathcal{F}$.

Suppose that a symmetrical 7-Venn diagram has been constructed by rotating one simple Jordan curve through an angle of 360/7 degrees seven times about an appropriate center. We observe that labeling the curves 1 – 7 clockwise, say, induces a unique labeling of each region by a binary 7-tuple having 0 or 1 in the ith coordinate ($i = 1, 2, \ldots, 7$) according to whether the region is outside or inside curve i. Letting $\mathbf{a} =< a_1, a_2, ..., a_7 >$, where a_i is 0 or 1, we define a *shift* s of $\mathbf{a}$ by $s(\mathbf{a}) =< a_7, a_1, a_2, ..., a_6 >$; a *rotation* is a repeated shift.

Definition 3 *Let $\mathcal{B}$ be a set of binary p-tuples. A binary p-tuple* $\mathbf{a}$ *(not necessarily in $\mathcal{B}$) is called* independent *from $\mathcal{B}$ iff* $\mathbf{a}$ *cannot be obtained by a shift or a rotation of any other element of $\mathcal{B}$. A set of binary p-tuples $\mathcal{B}$ is called independent iff every element* $\mathbf{a}$ *of $\mathcal{B}$ is independent from $\mathcal{B}$. The* weight $w(\mathbf{a})$ *of a binary p-tuple* $\mathbf{a}$ *is the number of* 1*'s in the tuple.*

Definition 4 *A* generator $\mathcal{G}$ *is a maximal independent set of binary p-tuples with*

$$1 \leq w(\mathbf{a}) \leq p - 1.$$

If $p = 7$, then in a generator $\mathcal{G}$ there are exactly

$$\frac{\binom{7}{k}}{7}$$

elements with weight k, for each $1 \leq k \leq 6$, and thus, there are

$$\sum_{k=1}^{6} \frac{\binom{7}{k}}{7} = 18$$

elements.

Now we choose a set of binary 7-tuples, a generator. It is not hard to check that the set of 7-tuples in the following table satisfies the conditions of a generator; each is numbered for use in Figure 1.

Next we construct a portion G of the Venn graph of the symmetric 7-Venn diagram, where G is a planar subgraph of the 7-hypercube, together with $\mathcal{S} = \{s_1, s_2, \ldots, s_7\}$, a previously chosen 7-element sequence of *transformations,* (shifts and/or rotations) with the following properties:

weight 1	#	weight 2	#	weight 3	#
1000000	1	1010000	2	1010010	5
		0010010	3	0011010	6
		1000001	4	0101100	7
				1001001	8
				1000011	9
weight 4	#	weight 5	#	weight 6	#
0011110	10	0111110	15	1111101	18
0101110	11	1101101	16		
0101101	12	0111101	17		
1001101	13				
1010011	14				

Table 1: The generator

1. The set $\mathcal{G}$ of descriptions of the vertex set G is the generator set shown in the table;
2. $\{s(\mathrm{a}) \mid \mathrm{a} \in \mathcal{G}, s \in \mathcal{S}\} \cup \{<0,0,\ldots,0>, <1,1,\ldots,1>\}$ is a set of descriptions of a spanning, 2-connected, planar subgraph of the 7-hypercube;
3. $\cup_{s \in \mathcal{S}} s(\mathcal{G})$ is a pairwise disjoint union;
4. If $\mathrm{a} \in \mathcal{G}$, then for every $1 \leq j \leq 7$, $s_j(\mathrm{a}) = s_1 \circ s_1 \ldots \circ s_1(\mathrm{a})$, where s_1 is a shift applied j times, $1 \leq j \leq 7$; and
5. $s_7(\mathcal{G}) = \mathcal{G}$.

The chosen transformations $\mathcal{S} = \{s_1, s_2, \ldots, s_7\}$ rotate this graph repeatedly through 360/7 degrees, creating a planar, spanning subgraph of the 7-hypercube.

Using the descriptions of the vertices of the graph created above, an edge number is assigned to each edge following the process described in Definition 1. (In Figure 1 we show this only for the portion of the graph and not for the whole graph.) It is not hard to check that we have created the Venn graph of a symmetric 7-Venn diagram.

Following the process described in Construction 2, a non-simple, symmetric 7-Venn diagram is created with 21 vertices. Figure 2 shows the entire Venn graph without labels (with the understanding that the dashed edges in the exterior are all incident with the single vertex $<0,0,\ldots,0>$). Figure 3 illustrates the rotational character of the construction.

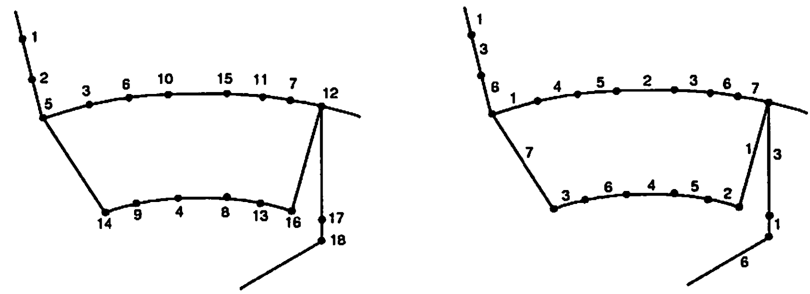

Figure 1: A portion of the Venn graph with the vertex labeling is given in the first figure. The numbers correspond to the numbers in the table. In the second figure the edges are labeled. The successive rotation of this portion through an angle of 360/7 degrees creates the Venn graph of a symmetric 7-Venn diagram.

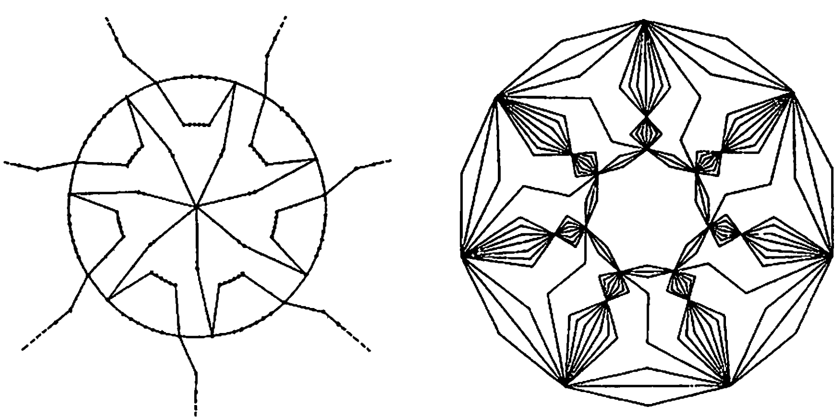

Figure 2: The first figure is the Venn graph. The second is the non-simple, symmetric 7-Venn diagram created from this Venn graph. This Venn diagram has a minimum vertex set, 21 vertices.

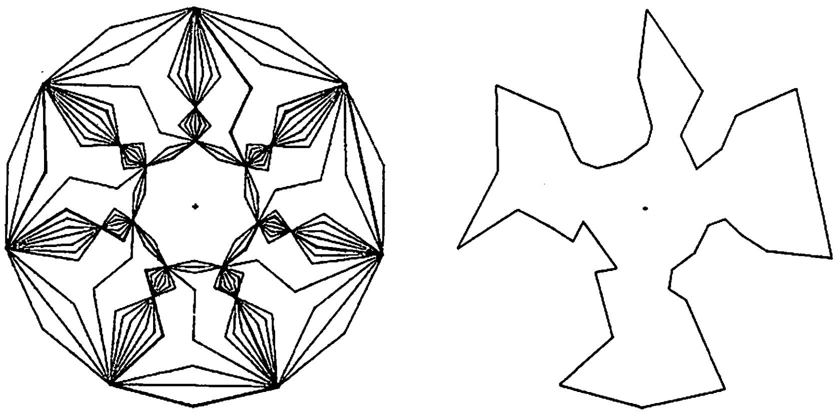

Figure 3: The non-simple, symmetric 7-Venn diagram, and one of the Jordan curves. The rotation of this curve through 360/7 degrees seven times about the center creates the non-simple 7- Venn diagram.

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