No maximal partial spread of size 115 in $PG(3,11)$

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Abstract

It is proved that there is no maximal partial spread of size 115 in PG(3,11).

1 Introduction

A *maximal partial spread* in $PG(3,q)$ is a set S of mutually skew lines such that any line of $PG(3,q)$ intersects at least one of the lines of S. Maximal partial spreads were first studied by Dale Mesner in 1967 [15]. He observed that if you pick a line ℓ_1 in $PG(3,q)$, and then a second line ℓ_2 skew to the first line, and then a third line ℓ_3 skew to these two lines, and so on, then this process either terminates before a certain bound, or can be continued until you get a spread. A spread is a set of q^2+1 mutually skew lines in $PG(3,q)$ which covers all points of $PG(3,q)$.

Aiden A. Bruen extended Mesners result. He showed in 1971 that for a maximal partial spread S in $PG(3,q)$

$$q+\sqrt{q}<|S|<q^2+1-\sqrt{q}.$$

He also constructed maximal partial spreads of the sizes q^2-q+1 and q^2-q+2 in $PG(3,q)$.

There have been several attempts to improve these results, see [2]. The best upper bound for maximal partial spreads in

$PG(3,q)$ is now given by Aart Blockhuis. It follows from his results on blocking sets [1] that for a maximal strictly partial spread S in $PG(3,p)$, p prime,

$$|S| < p^2 + 1 - \frac{p+1}{2}.$$

In [13] we showed that this bound cannot be improved in general. A maximal partial spread in $PG(3,q)$ for $q = 7$ of size

$$45 = q^2 - q + 3 = q^2 - \frac{q+1}{2}$$

was introduced.

In the finite projective spaces $PG(3,q)$, q prime, the next cases to settle are when q=11 and the possible sizes 113, 114 and 115. The aim of this paper is to prove the following theorem:

Theorem: There is no maximal partial spread of the size 115 in $PG(3,11)$.

Let us also mention that Glynn proved in 1981 that no maximal partial spread in $PG(3,q)$ has a size smaller than $2q$ [6] and that several maximal partial spreads of size smaller than $q^2 - q + 1$ have been constructed, see [8], [9], [10], [11] and [12].

2 Preliminaries

For a general introduction to this subject see e.g. [14].

The *deficiency* δ of a maximal partial spread in $PG(3,q)$ of size n is the integer $\delta = q^2 + 1 - n$.

It was proved in [7] that to any partial spread in $PG(3,q)$ corresponds a 2-weight code C over the alphabet $GF(q)$ and of word length $\delta(q+1)$. The words of C either have weight δq or weight $\delta(q+1)$. This fact was used in [7] to give an upper bound for the possible sizes of a maximal partial spread.

Blockhuis, Brower and Wilbrink translated this coding point of view into geometry [3]. We will use their translation but not their terminilogy. Instead of holes and non holes, rich and poor planes, we will talk about white and black points respectively white and black planes.

With this terminology we adopt the following results from [7]:

To any partial spread S of $PG(3,q)$ of deficiency δ corresponds a coloring of the points and the planes of $PG(3,q)$ in the colors white and black such that:

(i) Any white plane contains $\delta + q$ white points;

(ii) Any black plane contains δ white points;

and dually

(iii) Any white point is contained in $q + \delta$ white planes;

(iv) Any black point is contained in δ white planes.

By simple counting arguments, see [7], it is easy to prove

(v) The number of white points is $\delta(q+1)$;

(vi) The number of white planes is $\delta(q+1)$.

A line with exactly α white points will be called *a line of weight* α or an α*-line*. In [7] it was proved that

(vii) Any α-line is contained in exacly α white planes;

(viii) The weight α of a line either equals $q+1$ or is less then or equal to δ;

(ix) The partial spread S is maximal if and only if there is no $q+1$-line.

Let x_ν denote the number of $(\nu+1)$-lines through a white point. We will use the following two equations proved in [7]:

$$x_1 + 2x_2 + 3x_3 + \dots + qx_q = \delta(q+1) - 1; \tag{1}$$

$$2x_1 + 6x_2 + 12x_3 + \dots + q(q+1)x_q = (\delta+q)(\delta+q-1). \tag{2}$$

If x_ν denotes the number of $(\nu+1)$-lines in a white plane then the same equations will be true.

A line through two points P and Q will be denoted by PQ. A plane π is said to be a *plane of the line* l if π is one of the $q+1$ planes that contain the line l.

Lemma 2.1: If an α-line and a β-line pass through the same black point P then $\alpha + \beta \leq \delta + 1$ where δ is the deficiency of the maximal partial spread.

Proof: There are, by (iv), δ white planes that contain the point P. There are by (vii) α white planes of the α-line and β white planes of the β-line. As there are at most one common white plane of the α-line and the β-line, the lemma follows.

Lemma 2.2: Assume that $\delta = (q+3)/2$ is the deficiency of a maximal partial spread and let π be a white plane of a $(\delta - 1)$-line l. Through any white point W of π, $W \notin l$, there is one and only one 2-line l_W of π such that l and l_W intersect at a black point.

Proof: Let B be a black point of l. There is by (iv) one and only one white plane π_B not containing ℓ and intersecting l at the point B. The plane π_B intersects, by (vii), the plane π in at least two white points. There are $(q+1)/2$ black points of l and $q+1$ white points of $\pi \setminus l$. Hence the lemma is proved when we have showed that any white point of $\pi \setminus l$ is contained in at most one white plane not containing the plane π and intersecting l at a black point.

Assume that W is a white point of $\pi \setminus l$ contained in two white planes π_1 and π_2 not containing ℓ and intersecting l at black points. The planes π_1 and π_2 must meet each of the black planes of l at a white point. There are $(q+1)/2$ black planes of the line l and each of these planes contains exactly one white point not on the line l. The intersection line of π_1 and π_2 must contain all these $(q+1)/2$ white points. As the white point W is a point of this intersection line, the weight of this line will be at least $(q+1)/2+1$. As this number equals δ, we get by (viii) and (ix) that this intersection line must be a δ-line. We conclude, by (vii), that there are exactly $q+1-\delta$ black planes of the intersection line of π_1 and π_2. At least one of these black planes must meet a white point of the line l. This black plane will have more than δ white points.

The following lemma is well known:

Lemma 2.3: The set of white points of a white plane forms a blocking set of the plane.

Proof: If there is a 0-line of the plane, then that line, by (vii), cannot be contained in any white plane.

Lemma 2.4: Every δ-line intersects every white plane at a white point.

Proof: In $PG(3,q)$ any line intersect any plane. Assume that the intersection point of a δ-line with a white plane π is a black point P.

Any plane, white or black, of the δ-line intersects the white plane at a line in π through P. By the previous lemma any line through P in π contains at least one white point. Consequently, any plane of the δ-line will contain at least $\delta + 1$ white points. This implies that there will be no black planes of the δ-line.

3 The proof

A maximal partial spread of size 115 in $PG(3,q)$, $q = 11$, has the deficiency $\delta = 7$. We show the non existence of such a maximal partial spread, by showing that there cannot be any set of $\delta(q+1) = 84$ white points satisfying the conditions of the previous section.

The proof is divided into several steps. In the first of these steps we show, that if there was such a set of white points, then there will be no 5-lines. In the next step we show that there will be no 4-lines. This will imply that there are either at least four 6-lines, two 7-lines and one 6-lines or three 7-lines. Each of these possibilities will then be excluded.

Step 1: There is no 5-line.

Proof: Assume that ℓ is a 5-line and let π be one of the black planes of ℓ. We consider a white point P_1 of $\pi \setminus \ell$.

To any black point Q of ℓ there is, by (vii) of Section 2, at least one white plane containing the line P_1Q. As the line ℓ contains seven black points, there will be at least seven white planes $\pi_1, \pi_2, ..., \pi_7$, containing the point P_1 and meeting only black points of ℓ.

Let π' be another black plane of ℓ with the two white points P_1' and P_2' of $\pi' \setminus \ell$. Again by (vii) of Section 2, each of the white planes $\pi_1, \pi_2, ..., \pi_7$ meets at least one of the points P_1' and P_2'. It follows that at least four of these white planes will meet the same white point. We may assume that the white planes π_1, π_2, π_3 and π_4 meet the point P_1'.

We now consider the line P_1P_1'. Let π'' be any of the other five black planes of the 5-line ℓ. The intersection line of π'' and each of the white planes π_1, π_2, π_3 and π_4 all contain a white point. As there are only two white points of $\pi'' \setminus \ell$, the only possibility is, that the white point on the intersection line of the planes π_1, π_2, π_3 and π_4 and the plane π'', is the intersection point of the line P_1P_1' with the plane π''. As the line P_1P_1' meets seven black planes of ℓ, we conclude that P_1P_1' must contain at least seven white points. By (viii) of Section 2, as $\delta = 7$, we get that the only possibility is that P_1P_1' is a 7-line.

We now consider the black planes of the 7-line P_1P_1'. There are five such black planes. As at least four of the seven white planes of P_1P_1', the planes π_1, π_2, π_3 and π_4, meet black points of ℓ, there will be at least two black planes of P_1P_1' that meet white points of ℓ. These black planes will contain more than seven white points, which is impossible.

Step 2: There is no 4-line.

Proof. Assume there is a 4-line ℓ. Let Π denote the set of white planes that does not contain any white point of ℓ.

Let π be any of the black planes of ℓ and let P be a white point of $\pi \setminus \ell$. We first prove

(i) There is no line throught P contained in four of the planes of Π.

To prove this, assume that ℓ' is a line contained in four of the planes of Π.

We first exclude the possibility that ℓ' is contained in one of the black planes of ℓ. If this had been the case, then ℓ' would intersect ℓ at a black point. By (vii) of Section 2, ℓ' has at least four white points. As a black plane has seven white points, and as ℓ and ℓ' are both contained in the black plane π, we get a contradiction.

Any of the four planes of Π that contain ℓ' must meet the black planes of ℓ at white points not on the line ℓ. As for any black plane of ℓ, there are only three such white points, the only possibility (compare the proof of *Step 1*) is that ℓ' meets all the eight black planes of ℓ at white points. Hence, ℓ' will have at least eight white points, which is impossible by (ix).

Now we prove

(ii) Let π, π' and π'' be three black planes of ℓ and assume that the three points P, P' and P'', where $P \in \pi \setminus \ell$, $P' \in \pi' \setminus \ell$ and $P'' \in \pi'' \setminus \ell$, are collinear.

If the line PP' is contained in three planes of Π then any line through P and a white point of π' will intersect π'' at a white point.

To prove this we first note that any line through P and a black point of ℓ is, by (vii) of section 2, contained in at least one white plane of Π. Hence there are at least eight white planes of Π that contain the point P. We denote these planes by $\pi_1, \pi_2, ..., \pi_8$, and assume that π_1, π_2 and π_3 are the planes containing the line PP'.

Denote the three white points of $\pi' \setminus \ell$ by P', Q' and R'. According to (i) above, we may assume that π_4, π_5 and π_6 contain the point Q' and π_7 and π_8 the point R'.

Let P'', Q'' and R'' denote the three white points of $\pi'' \setminus \ell$.

From (i) we get that none of the planes $\pi_4, \pi_5, ..., \pi_8$ contains the point P''. Two of the planes π_4, π_5 and π_6 hence meet both either the point R'' or the point Q''. Without loss of generality we may assume that π_4 and π_5 contain the point Q''.

The intersection line of π_4 and π_5 will hence contain the points P, Q' and Q''. As π_6 contains the line PQ', this plane will also contain the point Q''.

By (i), none of the planes π_7 and π_8 will contain the point Q''. Hence these two planes both contain the point R''. The intersection line of π_7 and π_8 therefore contain the three white points P, Q'' and R''. (ii) is proved.

Let W denote the set of those white points of the black planes of the 4-line ℓ that are not a point of the line ℓ.

Next we prove

(iii) If a line ℓ' is contained in three of the planes of Π, then ℓ' meets at most three of the white points of W.

To prove this, assume that there is a line ℓ' which meets four of the points of W. Denote these points by P_1, P_2, P_3 and P_4, and the planes containing these points by π_1, π_2, π_3 and π_4, $P_i \in \pi_i$ for $i = 1, 2, 3$ and 4.

Denote the white points of $\pi_i \setminus \ell$ by P_i, P_i' and P_i'' for $i = 1, 2, 3$ and 4.

We first show that the intersection of W with these four planes are 12 white points, contained in the same white plane π'.

From (ii) we get that through any of the white points P_i, $i = 1, 2, 3$ and 4, there are three lines that intersect the planes π_1, π_2, π_3 and π_4 at white points. Hence the three such lines through P_1 are intersected by the three such lines through P_4.

Consequently these lines are lines of the same plane π'.

To the set of 12 points P_i, P_i' and P_i'', $i = 1, 2, 3$ and 4, we add a point Q. Q is the intersection point of the plane π' with the line ℓ. From what is proved above we deduce that these 13 points constitute, together with the lines between these points, a projective plane of order 3 in PG(3,11). Now *(iii)* is proved.

(iv) Let P be a white point of a black plane π of the 4-line ℓ, $P \in \pi \setminus \ell$. There are eight lines through P, each containing three planes of Π, and four lines through P, each containing at least two planes of Π.

We prove this by showing that there are only three lines through P that meet only one further point of the set of white points W.

Let $\pi_1, \pi_2, ..., \pi_8$ denote a set of eight white planes from Π that contain the point P. Assume there is a another black plane π' of the line ℓ with a white point $P' \in W \cap \pi'$ such that the line PP' is contained in three of the planes $\pi_1, \pi_2, ..., \pi_8$ and such that PP' does not meet any further white point of W.

Without loss of generality we may assume that the planes π_1, π_2 and π_3 contain the line PP'. These three planes then meet the remaining six black planes of the 4-line ℓ at distinct white points. Let π_i be any of the planes $\pi_4, \pi_5, ..., \pi_8$. If π_i meets the plane π_1 at only one of these white points, then π_i will meet either π_2 or π_3 at at least four white points. This contradicts *(iii)*.

Most trivial counting arguments now shows *(iv)*.

We now have arrived to the final part of the proof of *step 2*.

Consider a white point P of $\pi \cap W$, where π is a black plane of the 4-line ℓ. There are eight white planes $\pi_1, \pi_2, ..., \pi_8$ of Π containing the point P, and meeting distinct black points of the 4-line ℓ.

We will consider the intersection lines of the planes $\pi_1, \pi_2, ..., \pi_8$ with a plane π' of ℓ, $\pi' \neq \pi$.

With the same arguments as were used above, we may without loss of generality assume that π_1, π_2 and π_3 all contains a point P_1' of π', the planes π_4, π_5 and π_6 a point P_2' of the same plane π' and the planes π_7 and π_8 the point P_3'. (These three points don't have to be white points.) Further, for $i = 1, 2, 3$ and $j = 4, 5, 6$ we denote by P_{ij}' the intersection point of the planes π_i, π_j and π'.

By counting pairs of intersections of planes, we get that the 12 intersections lines of the planes $\pi_1, \pi_2, ..., \pi_8$ described in *(iv)* are the only, up to equivalence, possible intersection lines of these planes . Hence the two planes π_7 and π_8 meet each of the planes π_1, π_2 and π_3 at their intersection line with one of the planes π_4, π_5 and π_6. Without loss of generality we may assume that π_7 meets the points $P_3', P_{14}', P_{25}', P_{36}'$, and that π_8 meets the points $P_3', P_{15}', P_{26}', P_{34}'$.

We will make some "affine calculations" in π'. To get to that possibility we delete the line $P_1'P_2'$ from π'.

We may assume that the intersection lines of π' with π_1, π_2 and π_3 are the lines

$$l_a^v = \{(a, y) \mid y \in GF(11)\}$$

and the intersection lines of π' with π_4, π_5 and π_6 are the lines

$$l_b^h = \{(x, b) \mid x \in GF(11)\}.$$

We have to consider two cases.

Case 1: $P_3' \notin P_1'P_2'$. Without loss of generality we may assume that P_3' has the coordinates (0,0) and that π_1 intersects π' at the line ℓ_1^v

As the plane π_7 meets the points (0,0) and (1,1), we get that the intersection line of π_7 with the affine plane π' consists of the points

$$\{(x, x) \mid x \in GF(11)\}.$$

Consequently, as this line contains the point on the intersection of π_2 with π_5 π_3 with π_6, we get that for the intersection lines $\ell_a^v, \ell_b^v, \ell_c^h$ and ℓ_d^h of these planes with π', $a = c$ and $b = d$.

Further, as the points (0,0), P'_{15}, P'_{26} and P'_{34} are collinear we get that

$$\frac{1-0}{a-0} = \frac{a-0}{b-0} = \frac{b-0}{1-0}$$

which implies that $a^3 = 1$. $a = 1$ is the only element $a \in GF(11)$ satisfying this equation.

Case 2: $P'_3 \in P'_1P'_2$. Without loss of generality we may assume that the intersection lines of π' with π_1, π_2 and π_3 are the lines ℓ_0^v, ℓ_1^v and ℓ_a^v and the intersection lines of π' with π_4, π_5 and π_6 are the lines ℓ_0^h, ℓ_1^h and ℓ_b^h.

As in the case 1, we conclude that $a = b$ and that the points $(1,0), (a,1)$ and $(0,a)$ are collinear and parallell to the line through (0,0) and (1,1). This implies that

$$\frac{1-0}{a-1} = 1, \ \frac{a-1}{0-a} = 1.$$

The first of these two equalities implies that $a = 2$ and the second that $2a = 1$.

Step 3: Any white plane contains either four 6-lines, two 7-lines and one 6-line or three 7-lines.

Proof: We consider the two equations (1) and (2) of Section 2. From *step 1* and *step 2* we may assume that $x_3 = 0$ and $x_4 = 0$. Thus we get the following two equations:

$$x_1 + 2x_2 + 5x_5 + 6x_6 = 83$$

$$2x_1 + 6x_2 + 30x_5 + 42x_6 = 306.$$

From these two equations follows that

$$15x_5 + 24x_6 = 57 + x_1 \geq 57.$$

As there are only 18 white points on a white plane, the only possible values of the integers x_5 and x_6 are those stated above.

Step 4: No white plane contains four 6-lines.

Proof: Assume that there are four 6-lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 of a white plane. As the number of white points in the plane is 18, all these four 6-lines cannot pass through the same point of the plane. Hence three of the lines form a triangle. Assume ℓ_1, ℓ_2 and ℓ_3 form a triangle.

Let P_1 be a white point of ℓ_1 and assume $P_1 \notin \ell_i$ for $i = 2, 3, 4$. By Lemma 2.2 of Section 2, there is only one 2-line through P_1 which does not meet ℓ_3. There are three white points of ℓ_2 that do not belong to any of the lines ℓ_1, ℓ_3 and ℓ_4. Hence, at least two of the lines through P_1 and a white point of ℓ_2 must meet a white point of ℓ_3. As there is only one intersection point of ℓ_3 and ℓ_4, we get that at least one of these two lines meets ℓ_3 at a white point W, $W \notin \ell_4$. By Lemma 2.1, this line ℓ meets ℓ_4 at a white point. As ℓ meets the lines ℓ_i, $i = 1, 2, 3$ and 4, at distinct white points, and as there are no more white points in the plane than those of the lines $\ell_i, i = 1, 2, 3, 4$, we get that ℓ must be a 4-line.

Step 5: No white plane contains two 7-lines and one 6-line.

Proof: We first assume that the two 7-lines and the 6-line form a triangle. Denote by P the only white point of the plane that is not a point of any of these lines.

Let P' and P'' be two white points of the 6-line, and assume that neither P' nor P'' are points of a 7-line.

By Lemma 2.1 of section 2, both lines PP' and PP'' meet the 7-lines at distinct white points. Hence, one of these two lines meets the two 7-lines at a white point, which is not the intersection point of the two lines. That line must be a 4-line.

Assume now, that the two 7-lines and the 6-line meet at a point P. By Lemma 2.1 of Section 2, P must be a white point.

Let ℓ_1 and ℓ_2 denote the two 7-lines and let ℓ_3 denote the 6-line. Let $P_{i1}, P_{i2}, ..., P_{i5}$ denote the black points of the line ℓ_i for $i = 1, 2$. As the plane we consider is a white plane, any line contains a white point, see Lemma 2.3. Further, all white points are points of the lines ℓ_1, ℓ_2 and ℓ_3. Hence, any line $P_{1i}P_{2j}$, $1 \leq i \leq 5$ and $1 \leq j \leq 5$, meets the line ℓ_3 at a white point. We now show that this is impossible, by making calculations in the affine plane that we get by letting ℓ_1 be a line at infinity.

Without loss of generality we may assume the following:

(i) The black points of ℓ_2 are the points $P_{2i} = (y_i, 0)$, $i = 1, 2, 3, 4, 5$ where $y_i \in GF(11)$.

(ii) The five white points of $\ell_3 \setminus \{P\}$ are the points $P_{3i} = (x_i, 1)$, $i = 1, 2, 3, 4, 5$ where $x_i \in GF(11)$.

We may assume that the parrallell class of lines passing through the point P_{11} consists of the lines

$$l_a^v = \{(a, y) \mid y \in GF(11)\}.$$

Thus, as the lines $P_{11}P_{2i}$ for $i = 1, 2, 3, 4, 5$ all meet white points of ℓ_3, we may also without loss of generality assume that $x_i = y_i$ for $i = 1, 2, 3, 4, 5$.

The lines $P_{2i}P_{3j}$,$1 \leq i \leq 5$ and $1 \leq j \leq 5$, only meet black points of ℓ_1, and hence these 25 lines belong to five parallell classes of lines. By considering one of these parallell classes, we find, after a suitable numbering of the elements $x_1, x_2, ..., x_5$, that

$$\frac{x_2 - x_1}{1 - 0} = \frac{x_3 - x_2}{1 - 0} = ...$$

This implies that there is an element t of $GF(11)$ such that

$$x_2 = x_1 + t, \quad x_3 = x_2 + t, \quad ....$$

from which we get that $x_1 = x_1 + kt$ for some integer $k \leq 5$. As $GF(11)$ is a field, this is impossible.

Now the statement in step 5 is proved.

As a consequence of what is proved above we get the following statements:

(A) There are only 1-lines, 2-lines, 3-lines and 7-lines.

(B) Any white plane contains three 7-lines.

Further, by the use of the equations (1) and (2) of Section 2 we get, compare the proof of step 3, that

(C) Through any white point there are three 7-lines.

We are now ready for the final step.

Step 6: To any three 7-lines of a white plane there are another three 7-lines such that these six 7-lines constitute a tetrahedreon.

Proof: Let ℓ_1, ℓ_2 and ℓ_3 be three 7-lines of a white plane π. As π only contains 18 white points, there is no point of π that is contained in all of these three lines. Hence the three lines ℓ_1, ℓ_2 and ℓ_3 constitute a triangle. The intersection point $P_{i,j}$, $1 \leq i < j \leq 3$ of the lines ℓ_i and ℓ_j must be a white point, see Lemma 2.1 of Section 2.

Let ℓ denote the third 7-line through P_{12}, $\ell \notin \pi$, see (C) above. We first show

(i) All 7-lines through white points of ℓ will intersect the line ℓ_3.

Let P be a white point of ℓ and assume $P \notin \pi$. Assume that there is a 7-line $\ell' \neq \ell$ through P that does not meet the line ℓ_3. By Lemma 2.4 of Section 2, ℓ' meets a white point of π. Hence ℓ' either intersects ℓ_1 or ℓ_2. Without loss of generality we may assume that ℓ' intersects ℓ_1.

Let π' denote the plane containing the lines ℓ and ℓ_1 and let

π'' denote the plane containing the lines ℓ and ℓ_2. As there are three 7-lines in each white plane π' and π'', and as ℓ contains seven white points, there must be at least four white points P' of ℓ such that the two 7-lines from P', distinct from ℓ, intersect the line ℓ_3. Let π''' denote the plane containing the line ℓ_3 and one such point P'. By Lemma 2.4, the line ℓ', described in the previous paragraph, intersects π''' at a white point. This white point must be on a 7-line $\ell''' \in \pi'''$ through P'. The intersection point of ℓ' and ℓ''' must, as $\ell' \in \pi'$, be contained in the plane π'. As $P' \in \pi'$ we get that the line $\ell''' \in \pi'$. As π' only contains three 7-lines, we get a contradiction, and the only possibility is that the line ℓ' contains the point P_{13}.

Now (i) is proved.

(ii) There is a point P of ℓ, $P \notin \pi$, such that the two 7-lines ℓ' and ℓ'', distinct from ℓ, through P intersects ℓ_3 at the points P_{13} and P_{23} respectively.

There are fourteen 7-lines, distinct from ℓ, through the white points of ℓ. All these lines intersect, by (i), the line ℓ_3. Further from (C), there are fourteen 7-lines that intersect the 7-line ℓ_3. Hence all 7-lines intersecting ℓ_3 must intersect the line ℓ. We thus conclude that the third 7-line ℓ', distinct from ℓ_1 and ℓ_3, through P_{13} must intersect ℓ at a white point P of ℓ.

A third 7-line ℓ'', distinct from ℓ and ℓ', through P, intersects by (i) the line ℓ_3. If we use (i) with (ℓ_1, ℓ_2, ℓ_3) permutated to (ℓ_1, ℓ_3, ℓ_2) we may conclude that ℓ'' intersects ℓ_2.

Now (ii) is proved and thereby also step 6.

Now to the final step of the proof of the theorem.

By (B) above, if there is a maximal partial spread of size 115 in $PG(3, 11)$, then there must be a 7-line ℓ. By (vii) of Section 2, there are seven white planes of this 7-line ℓ. From step 6 we deduce, that the white planes of this 7-line ℓ are paired together two and two in order to produce distinct tetrahedreons. Finally, the integer seven is not divisible by the integer two.

References

[1] A. Blockhuis, On the size of a blocking set in PG(2,p), Combinatorica 14(1994)111-114.

[2] A. Blockhuis and K. Metsch, On the size of a maximal partial spread, Designs, Codes and Cryptography 3(1993)187-191.

[3] A. Blockhuis, A.E. Brouwer and H.A. Wilbrink, Hedens bound on maximal partial spreads, Discrete Math. 74(1989)335-339.

[4] A.A. Bruen, Partial spreads and replaceable nets, Canad. J. Math. 23(1971)381-392.

[5] A.A. Bruen and J. Thas, Partial spreads, Packings and Hermitian manifolds in PG(3,q), Math. Z. 151(1976)207-214.

[6] D.G. Glynn, A lower bound for maximal partial spreads in $PG(3,q)$, Ars Combin. 13(1982)39-40.

[7] O. Heden, Maximal partial spreads and two-weight codes, Discrete Math. 62(1986)277-293.

[8] O. Heden, A greedy search for maximal partial spreads in $PG(3,7)$, Ars Combin. 32(1991)253-255.

[9] O. Heden, Maximal partial spreads and the modular n-queen problem, Discrete Math. 120(1993)75-91.

[10] O. Heden, Maximal partial spreads and the modular n-queen problem II, Discrete Math. 142(1995)97-106.

[11] O. Heden, Maximal partial spreads in $PG(3,5)$, submitted to Ars Combinatoria.

[12] O. Heden, Maximal partial spreads and the modular n-queen problem III, submitted to Discrete Math.,

[13] O. Heden, A maximal partial spread of size 45 in $PG(3,7)$, to appear in Ars Combin.

[14] J. W. P. Hirschfeld, Finite Projective Spaces of Three dimensions, Clarendon Press, Oxford, 1985.

[15] D. Mesner, Sets of disjoint lines in $PG(3,q)$, Canad. J. Math. 19(1967)273-280.

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