MAXIMUM COMPLEMENTARY P_3-PACKINGS OF K_v

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Abstract

Motivated by the work of Granville, Moisiadis and Rees, we consider in this paper complementary P_3-packings of K_v. We prove that a maximum complementary P_3-packing of K_v (with $\lfloor \frac{v}{4} \lfloor \frac{2(v-1)}{3} \rfloor \rfloor P_3$s) exists for all integers $v \geq 4$, except for $v = 9$ and possibly for $v \in \{24, 27, 30, 33, 36, 39, 42, 57\}$.

1 Introduction

Let K_v be the complete graph on v vertices, and P_3 a path of length 3. A P_3-packing of K_v is defined to be a family, F, of edge-disjoint P_3s in K_v. The graph spanned by the edges which do not occur in any P_3 of F is referred to as the leave of the P_3-packing.

A complementary P_3-packing of K_v is a P_3-packing of K_v, F, with the property that upon taking the complement of each path in F one obtains a second P_3-packing of $K_v, \overline{F}$. Here the complement of the path $abcd$ is the path $bdac$, that is, $\overline{F} = \{bdac : abcd \in F\}$. In the particular case where the leaves of both F and $\overline{F}$ are empty, a complementary P_3-packing of K_v is nothing else than a complementary P_3-decomposition of K_v, which was first investigated by Granville, Moisiadis and Rees [2]. They proved that

Theorem 1.1 *There exists a complementary P_3-decomposition of K_v if and only if $v \equiv 1 (mod\ 3)$.*

If F is a complementary P_3-packing of K_v, then the set $D = \{\{a, b, c, d\} : abcd \in F\}$ forms a standard $(v, 4, 2)$ packing. Therefore, from Schönheim [5] we have

$$|F| \leq \lfloor \frac{v}{4} \lfloor \frac{2(v-1)}{3} \rfloor \rfloor = \psi(v, 4, 2)$$

where $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \le x$.

In what follows, a complementary P_3-packing of K_v with $\psi(v,4,2)$ P_3s will be called maximum. It is clear that Theorem 1.1 implies that a maximum complementary P_3-packing of K_v exists when $v \equiv 1 \pmod 3$. The purpose of this paper is to treat the cases where $v \equiv 0$ or $2 \pmod 3$. We will prove the following.

Theorem 1.2 *For all integers $v \ge 4$, a maximum complementary P_3-packing of K_v exists, except for $v = 9$ and in the set of possible exceptions $v \in \{24, 27, 30, 36, 39, 42, 57\}$.*

Throughout the remainder of this paper, a familiarity with the definitions and notations for group divisible designs (GDDs) and transversal designs (TDs)is presumed.

2 Recursive Constructions

In order to describe our recursive constructions, we need the notion of a holey self-orthogonal Latin square, which we define below.

Let S be a finite set and $H = \{S_1, S_2, \ldots, S_n\}$ be a partition of S. A holey Latin square having partition H is a $|S| \times |S|$ array L, indexed by S, satisfying the following properties:

(1) every cell of L either contains an element of S or is empty,

(2) every element of S occurs at most once in any row or column of L,

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \le i \le n$ (these subarrays are referred to as holes),

(4) element $s \in S$ occurs in a row or column t if and only if $(s,t) \in (S \times S) \setminus \cup_{1 \le i \le n} (S_i \times S_i)$.

The order of L is $|S|$. The type of L is the multiset $T = \{|S_i| : 1 \le i \le n\}$ and will be denoted by an "exponential" notation.

Two holey Latin squares on symbol set S and hole set H, say L_1 and L_2, are said to be orthogonal if their superposition yields every ordered pair in $(S \times S) \setminus \cup_{1 \le i \le n} (S_i \times S_i)$. A holey Latin square is said to be self-orthogonal if it is orthogonal to its transpose. A self-orthogonal holey Latin square of type T will be denoted by HSOLS (T).

The following existence result for HSOLSs is very useful to our problem under consideration, which is taken from [6], [7] (also see [8]).

Lemma 2.1 *Both an HSOLS $(2^n 1^1)$ and an HSOLS (2^n) exist when $n \ge 4$; and an HSOLS $(2^n u^1)$ exists if $n \ge 1 + u$ and $u \ge 3$.*

We now give the recursive constructions used in the proof of our main theorem.

Lemma 2.2 *Let $n \geq 4$ be an integer. If a maximum complementary P_3-packing K_{12} exists, then so does a maximum complementary P_3-packing of K_{12n}. Furthermore, a maximum complementary P_3-packing of K_{12n+6} exists if a maximum complementary P_3-packing of K_6 exists.*

Proof.
For given n, we have an HSOLS (2^n) and an HSOLS $(2^n 1^1)$ from Lemma 2.1. We may assume that $A = (a_{ij})_{2n}$ is an HSOLS (2^n) on symbol set $S = \{1, 2, \ldots, 2n\}$, and that its hole sets are $H_j = \{j, j+1\}, j = 1, 3, \ldots, 2n-1$. Write $B = (b_{ij})_{2n}$ for the transpose of A.

Now we take the vertex set of K_{12n} to be $S \times Z_6$. Consider the following P_3s in K_{12n}:

$$(i,2)(a_{ij},3)(b_{ij},0)(j,2) \bmod (-,6)$$

$$(a_{ij},0)(i,2)(j,2)(b_{ij},3) \bmod (-,6)$$

where $\{i,j\}$ runs over all pairs from $(S \times S) \setminus \cup_{1 \leq t \leq n} (H_t \times H_t)$ satisfying $1 \leq i < j \leq 2n$. Note that since A and $B = A^T$ are orthogonal, each pair of distinct vertices not in the $H_j \times Z_6 (j = 2t-1, 1 \leq t \leq n)$ occurs in exactly one of the above P_3s and one of their complements. We then put a copy of a maximum complementary P_3-packing of K_{12} on each vertex set $H_j \times Z_6 (j = 2t-1, 1 \leq t \leq n)$ to obtain the required P_3-packing of K_{12n}. To see that this construction satisfies the number of paths required by the definition of maximality, we observe that the number of paths constructed is $(6)(2)(2n^2 - n)$, then subtracting the number of paths $(12n)$ based on the holes $H_j \times Z_6$, and adding the $21n$ paths of the maximum packings of K_12s we get the maximum required number of paths by the bound:

$$12(4n^2 - n) - 12n + 21n = 24n^2 - 3n = \psi(12n, 4, 2)$$

For a maximum complementary of P_3-packing of K_{12n+6}, we make use of an HSOLS $(2^n 1^1)$. The procedure is similar to the above. □

Lemma 2.3 *Let $n \geq 4$ be integer. Then*

(1) a maximum complementary P_3-packing of K_{12n+3} exists;

(2) a maximum complementary P_3-packing of K_{12n+21} exists if a maximum complementary P_3-packing of K_{21} exists.

Proof.
First, we construct a maximum complementary P_3-packing, F, of K_{15} such that the leave of F and its complement contain a common triangle. This is done by the following P_3s, by taking the vertex set of K_{15} as Z_{15}:

9	0	11	1	1	14	6	8	2	4	7	13	11	7	3	6
0	2	5	6	1	13	4	5	2	14	5	10	4	6	10	14
12	0	3	4	1	10	2	12	2	13	11	3	4	10	9	6
8	0	4	12	3	1	6	7	7	2	8	4	5	9	7	8
6	0	5	13	7	1	12	10	3	5	8	9	10	7	5	11
0	13	8	10	11	4	1	5	3	9	4	14	8	12	11	6
0	1	2	3	12	6	2	9	8	3	13	10	7	12	9	13
0	10	11	9	14	8	11	2	5	12	3	14	11	14	12	13
1	9	14	13												

where the triangle is $\{0,7,14\}$.

Now for conclusion (1), the construction is similar to that in the proof of Theorem 2.2. Here we take a set T of 3 infinite points and then replace each set $(H_j \times Z_6) \cup T (j = 2t-1, 1 \leq t \leq n)$ by a maximum complementary P_3-packing of K_{15} constructed above where the triangle is based on T.

For conclusion (2), we start with an HSOLS $(2^n 3^1)$, $A = (a_{ij})_{2n+3}$, on symbol set $S = \{1, 2, \ldots, 2n+3\}$. Suppose that the hole set is $\{H_j = \{j, j+1\} : j = 2t-1, 1 \leq t \leq n\} \cup \{\{2n+1, 2n+2, 2n+3\}\}$. Then the construction can be made as follows.

(1) A set of three infinite points and then replace $(H_j \times Z_6) \cup T (j = 2t-1, 1 \leq t \leq n)$ by a maximum complementary P_3-packing of K_{15} as above and replace $(\{2n+1, 2n+2, 2n+3\}) \times Z_6 \cup T$ by a maximum complementary P_3-packing of K_{21}.

(2) Take the following paths

$$(i,2)(a_{ij},3)(b_{ij},0)(j,2) \bmod (-,6)$$

$$(a_{ij},0)(i,2)(j,2)(b_{ij},3) \bmod (-,6)$$

where $\{i,j\}$ ranges over all pairs of distinct points of S satisfying $1 \leq i < j \leq 2n+3$ which are not in the same hole, and where $(b_{ij})_{2n+3}$ is the transpose of A.

To see that this construction satisfies the number of paths required by the definition of maximality, we observe that the number of paths constructed by step (2) is $(6)(2)(\frac{(2n+3)(2n+2)}{2})$, then subtracting the number of paths $(12n+36)$ based on the $holes \times Z_6$, adding the $33n$ paths of the maximum packings of K_{15}s, and adding the 68 paths of the maximum packing of K_{21} we get the maximum required number of paths by the bound:

$$12[2n^2+5n+3] - (12n+36) + 33n + 68 = 24n^2 + 81n + 68 = \psi(12n+21, 4, 2)$$

It is readily checked that the above two steps produce a maximum complementary P_3-packing of K_{12n+21}, where the vertex set of K_{12n+21} is $(S \times Z_6) \cup T$. □

3 The Proof of Theorem 1.2

Before giving a proof of Theorem 1.2, we require the direct construction of some P_3-packings of K_v with small values of v. We assume that the reader is familiar with the notion of a standard (v, k, λ) packing.

Lemma 3.1 *If $v \in \{6, 8, 11, 12, 17, 18, 21\}$, then there exists a maximum complementary P_3-packing of K_v.*

Proof.
For all stated values of v, we take the vertex set of K_v as Z_v. Then the following paths form the desired P_3-packing.

v=6

5 0 3 1	2 0 1 4	0 4 2 3	3 5 1 2

v=8

1 2 3 5	3 1 7 4	2 0 3 4	0 7 3 6
2 4 1 6	1 5 6 7	5 2 6 0	4 0 5 7

v=11

1 4 2 3	8 1 10 9	2 7 10 8	6 8 3 0
1 2 5 6	1 0 8 7	2 9 7 0	4 10 5 7
10 3 1 5	3 6 2 10	9 3 4 7	9 4 6 10
1 6 0 4	2 0 5 4	3 5 9 0	5 8 9 6

v=12

0 3 2 1	1 0 4 8	5 9 1 10	4 9 2 11
0 9 3 6	2 0 5 7	1 7 11 6	2 10 6 8
0 11 9 10	11 1 3 7	3 10 8 2	3 4 10 7
4 7 0 10	4 1 8 9	11 4 2 6	11 3 8 5
0 8 11 5	10 5 1 6	9 7 2 5	3 5 6 4
9 6 7 8			

v=17

12	2	15	7	15	13	3	2	3	11	12	14	11	7	0	15
1	10	12	9	12	7	10	16	12	16	6	5	6	1	15	11
13	8	3	15	0	13	12	4	9	16	7	13	8	6	9	7
10	13	11	8	8	5	16	3	16	11	6	2	14	8	0	6
4	8	7	2	10	6	15	5	3	4	0	10	7	4	11	5
12	0	9	5	8	9	11	0	2	0	14	6	13	4	1	14
1	3	6	7	4	15	16	14	16	8	10	14	5	7	14	13
0	3	7	1	4	14	15	9	9	13	16	1	13	2	11	10
11	14	3	12	9	3	5	2	0	5	10	15	9	10	2	14
5	4	1	11	3	10	4	6	4	13	6	12	15	8	12	1
1	8	2	4	0	1	2	16								

v=18

17	13	6	4	8	13	10	3	16	12	0	15	5	2	9	3	2	14	0	6
4	17	3	12	10	11	2	13	15	4	14	9	9	11	16	3	4	3	6	10
14	7	10	8	11	17	5	10	16	5	6	2	1	12	9	10	5	15	10	0
17	14	10	2	1	8	2	4	13	11	7	12	6	1	10	12	0	4	12	13
6	7	13	9	2	15	11	4	15	7	16	10	15	6	11	1	11	14	6	16
2	1	9	8	5	1	16	13	1	13	5	14	4	7	5	11	15	13	3	8
3	11	12	14	17	2	0	13	7	8	4	5	16	14	8	12	5	9	17	12
9	15	14	13	16	9	4	10	8	5	12	6	9	0	8	11	8	17	0	11
15	12	2	7	6	9	7	0	0	16	4	1	7	3	2	16	1	3	15	17
7	17	1	14	0	5	3	14	15	8	6	17	3	0	1	7				

v=21

1	5	3	2	0	4	17	2	3	0	8	13	6	18	5	11	11	7	19	15
13	9	2	1	2	5	16	15	17	3	8	16	5	6	16	19	17	15	7	14
10	1	3	7	6	2	8	14	9	3	10	18	5	7	9	0	8	11	13	14
5	15	4	1	2	20	6	10	9	15	3	20	7	10	14	5	20	8	17	18
1	8	18	4	16	2	0	7	12	15	18	3	11	20	5	8	9	10	17	16
15	1	6	17	2	7	20	18	3	12	0	19	18	9	5	13	11	15	13	10
7	1	11	12	12	2	15	8	5	4	13	17	5	10	20	12	17	0	11	10
8	10	0	1	2	11	17	9	18	7	6	4	0	5	12	14	16	18	11	14
9	1	14	19	10	2	18	19	4	19	6	0	9	8	6	15	15	0	13	18
11	19	1	20	13	2	19	12	13	20	4	7	0	20	9	6	16	13	19	20
12	1	17	20	4	3	14	20	8	4	10	19	10	12	6	14	0	14	15	20
14	16	1	13	17	5	19	3	4	12	9	11	6	13	12	17	18	14	17	19
16	0	18	1	6	3	13	7	4	9	16	12	9	19	8	7	16	7	12	8
2	4	11	3	3	16	11	6	4	16	10	15								

Lemma 3.2 *There exists a maximum complementary P_3-packing of K_{45}.*

Proof.
The packing is obtained from a TD $(4, 11)$ in the following two steps:

(1) replace each block $\{a, b, c, d\}$ of a TD $(4, 11)$ by two P_3s: *abcd* and *bdac*;

(2) add an infinite point to each group of the TD, and then construct a maximum complementary P_3-packing of K_{12}.

□

Lemma 3.3 *There does not exist a maximum complementary P_3-packing of K_9.*

Proof.
The conclusion follows from the nonexistence of a $(9, 4, 2)$ packing with $\lfloor \frac{9}{4} \lfloor \frac{16}{3} \rfloor \rfloor$ blocks, which was proved by Hartman [3]. □

Now we give the proof of Theorem 1.2, which is restated below.

Theorem 3.4 *For all integers $v \geq 4$, there exists a maximum complementary P_3-packing of K_v, except for $v = 9$ and possibly for $v \in \{24, 27, 30, 33, 36, 39, 42, 57\}$.*

Proof.
The theorem is true for $v \equiv 1 (\mathrm{mod}\ 3)$ by Theorem 1.1. The theorem is also true for $v \in \{8, 11, 17\}$ by Lemma 3.1. For $v \equiv 2(\mathrm{mod}\ 3)$ and $v \notin \{8, 11, 17\}$, it was shown by Brouwer [1] that a $(v, 4, 1)$-packing with $\lfloor \frac{v}{4} \lfloor \frac{v-1}{3} \rfloor \rfloor$ block exists. Replacing each block $\{a, b, c, d\}$ in such a packing by two P_3s: *abcd* and *bdac* gives a maximum complementary P_3-packing of K_v. Now for the case $v \equiv 0(\mathrm{mod}\ 3)$, the result follows from Lemmas 3.1-3.2 when $v \in \{6, 12, 18, 21, 45\}$. A maximum complementary P_3-packing of K_{15} was given in the proof of Lemma 2.3. The remaining values of $v \equiv 0(\mathrm{mod}\ 3)$ are all covered by Lemmas 2.2 and 2.3.

This completes the proof. □

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