Enumeration of the bases of the bicircular matroid on a complete bipartite graph

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Abstract

We enumerate the bases of the bicircular matroid on $K_{m,n}$. The structure of bases of the bicircular matroid in relation to the bases of the cycle matroid is explored. The techniques herein may enable the enumeration of the bases of bicircular matroids on larger classes of graphs; indeed one of the motivations for this work is to show the extendibility of the techniques recently used to enumerate the bases of the bicircular matroid on K_n.

1 Introduction

A *matroid* M is an ordered pair $(E, \mathcal{I})$ of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:

- $\emptyset \in \mathcal{I}$.
- If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $(I_1 \cup e) \in \mathcal{I}$.

M is called a ***matroid on*** E if it is the matroid[1] $(E, \mathcal{I})$. The elements of E are called *edges*, with the set of edges of a matroid M denoted by $E(M)$. The *order of* E, denoted by $|E|$, is the number of edges in the matroid. The members of $\mathcal{I}$ are subsets of $E(M)$ called the *independent sets* of M, and the number of independent sets of the matroid is denoted by $|\mathcal{I}|$. A subset of E that is not in $\mathcal{I}$ is called *dependent.* A maximal independent set of M is called a *basis* of M, denoted by B, with the set of bases of M written $\mathcal{B}(M)$. The number of bases is denoted by $|\mathcal{B}(M)|$. All bases of M are equicardinal, with their cardinality, or *rank*, denoted by $r(M)$. A *circuit* $C(M)$ is a minimal dependent set in a matroid M; removal of any edge from C yields an independent set in the matroid.

Let G be a connected graph[2] (loops and parallel edges allowed) with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. The *cycle matroid* of a graph G, denoted by $M(G)$, is the matroid where E is the set of edges of the graph and $\mathcal{I}$ is the set of sets of edges with no cycle. A basis of $M(G)$ is a spanning tree of G; in particular the bases have cardinality equal to $n-1$. We denote the set of bases of the cycle matroid by $\mathcal{B}_M(G)$.

The *bicircular matroid* of G is the matroid $B(G)$ defined on E whose circuits are the subgraphs which are subdivisions of one of the graphs: (i) two loops on the same vertex, (ii) two loops joined by an edge, (iii) three edges joining the same pair of vertices. Here and elsewhere we identify a

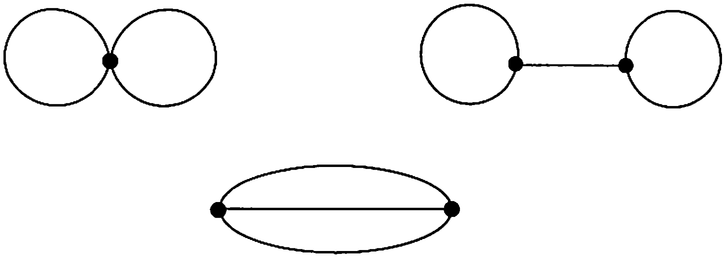

Figure 1: Bicycles of G

subgraph and its collection of edges. The circuits of $B(G)$ are called the *bicycles* of G. They are the connected subgraphs of G containing exactly

[1] A general reference for matroid theory is [5].
[2] A general reference for graph theory is [7] or [8].

two independent cycles [3]. A set of edges is independent in $B(G)$ provided that each connected component contains at most one cycle of G. The rank of a set X of edges is $\rho(X) = n(X) - t(X)$ where $n(X)$ is the number of vertices incident with the edges of X and $t(X)$ is the number of (nontrivial) tree components of X. If G is a tree, then E is an independent set and hence a basis of $B(G)$. If G is a connected graph that is not a tree, then the bases of $B(G)$ are the spanning subgraphs of G each of whose connected components is a unicyclic subgraph of G; in particular the bases have cardinality equal to n. We denote the set of bases of the bicircular matroid by $\mathcal{B}_B(G)$.

A graph is *bipartite* if its vertex set can be partitioned into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B. A *complete bipartite graph* is a bipartite graph in which each vertex of A is joined to each vertex of B by just one edge.

The simple bipartite graph G with m vertices in the first set and n vertices in the second set with the maximal number of bases of $B(G)$ is $K_{m,n}$. $B(K_{m,n})$ gives the greatest number of bases, but the actual number of bases of $B(K_{m,n})$ requires a bit of calculation. Since $K_{2,2}$ is a 4-cycle, $B(K_{2,2})$ has only one basis. The bases of $B(K_{3,3})$ that are Hamiltonian cycles are given in Figure 2. All other bases have a 4-cycle. There are

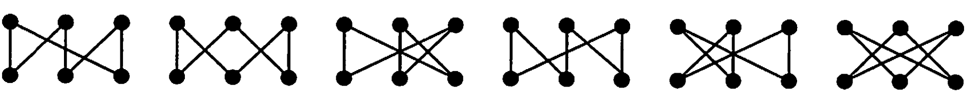

Figure 2: Bases of the bicircular matroid on $K_{3,3}$ that are Hamiltonian cycles.

$\binom{3}{2}\binom{3}{2}$ ways of picking a 4-cycle from $K_{3,3}$. Once the 4-cycle has been chosen there are eight ways to complete the cycle into a basis of $B(K_{3,3})$. For one 4-cycle, these eight completions are shown in Figure 3. Thus there are $6 + 9 \cdot 8 = 78$ bases of $B(K_{3,3})$.

A basis of a bicircular matroid on G can be composed of many components, each of which is a unicyclic subgraph of G. Neudauer, Meyers and Stevens addressed the enumeration of the single-component bases of a bicircular matroid on a general graph G in [4]. In that same article the number of single-component bases and then the total number of bases of the bicircular matroid on the complete graph were found. Here we extend these results to the complete bipartite graph $K_{m,n}$. First we find the precise number of connected bases of the bicircular matroid on $K_{m,n}$ and then in the final section we calculate the total number of bases of the bicircular matroid on $K_{m,n}$. This number includes the connected bases and the

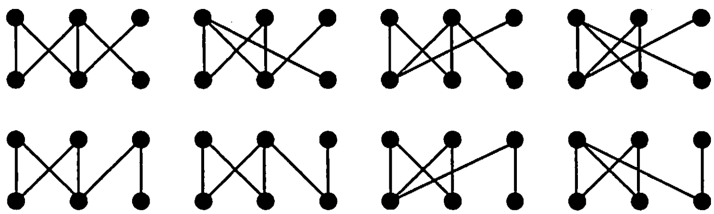

Figure 3: Eight completions to Bases of the bicircular matroid on $K_{3,3}$ that contain a fixed 4-cycle.

bases with more than one component. Tables with these explicit values for $K_{m,n}$ with $m, n \leq 10$ are provided. In our calculations we use results from the theory of partitions and compositions, and include the proofs of more relevant results.[3]

2 Bases of the bicircular matroid on the graph G

For a graph G, a *unicyclic spanning subgraph* is a unicyclic subgraph which includes every vertex of G and is a basis of the bicircular matroid $B(G)$. In fact it is a basis with exactly one connected component. The set of unicyclic spanning subgraph is denoted by $\tau(G)$. Neudauer, Meyers and Stevens [4] prove a number of results about $\tau(G)$, which we restate here, and develop a technique for constructing bases of the bicircular matroid from independent sets of the cycle matroid on G.

Lemma 1 ([4]). *Given an independent set of the cycle matroid on G, to each connected component of the independent set add an edge (not already in the independent set) either joining two vertices of that component or joining it to another component. The resulting set of edges is a basis of the bicircular matroid on G.*

Adding a single edge to a basis of the cycle matroid on G will produce a basis of the bicircular matroid on G. In particular, adding an edge to a spanning tree of G will construct a unicyclic spanning subgraph of G, a basis of the bicircular matroid with exactly one component.

Theorem 2 ([4]). *If G is a simple, connected graph, the following equations relate $|\tau_i(G)|$, the number of unicyclic spanning subgraphs of G with*

[3]A general reference on the theory of partitions is [1].

cycles of length i, and the number of spanning trees of G:

$$|\tau(G)| = \sum_{i=3}^{n} |\tau_i(G)| \tag{1}$$

$$\sum_{i=3}^{n} i|\tau_i(G)| = (|E(G)| - r(M(G)))(|\mathcal{B}_M G|). \tag{2}$$

In particular the number of unicyclic spanning subgraphs is

$$|\tau(G)| = (|E(G)| - r(M(G)))(|\mathcal{B}_M G|) - \sum_{i=3}^{i=n} (i-1)|\tau_i(G)|. \tag{3}$$

A *Hamiltonian cycle* of a graph G is a cycle on G which contains each vertex of G exactly once. The number of Hamiltonian cycles [7] of $K_{n,n}$ is $(n!)^2/(2n)$.

Theorem 3 **([2]).** *The number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.*

3 Enumerating the connected bases of the bicircular matroid on $K_{m,n}$

We approach the problem of enumerating the single component bases of the bicircular matroid on $K_{m,n}$ Instead of starting with a spanning tree and adding an edge to obtain a unicyclic spanning subgraph, we begin with a cycle of length $2i$ and add edges, expanding this to a spanning subgraph. In this section we calculate the exact number of connected bases of $B(K_{m,n})$.

Consider a cycle of length $2i$ in $K_{m,n}$. The induced subgraph on these $2i$ vertices is $K_{i,i}$. There are $(i!)^2/(2i)$ ways that such a cycle can be constructed. Denote the vertices in this cycle by C. Of the $2i$ vertices in this cycle, we choose k vertices to which we will connect the remaining $m + n - 2i$ vertices.

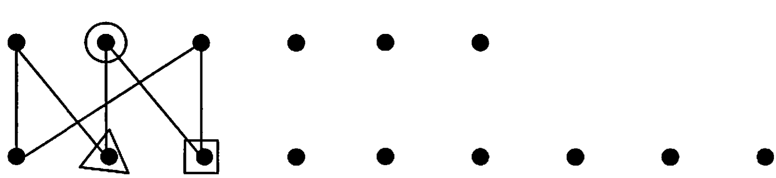

Figure 4: Choose k vertices on the cycle to which we will connect the remaining $m + n - 2i$ vertices.

We associate one vertex from the opposite set, and not on the cycle, to each of these vertices chosen on the cycle

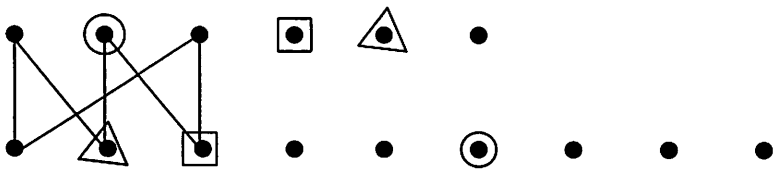

Figure 5: Associate one vertex from the opposite set to each vertex chosen on the cycle.

We then partition the $m+n-2i-k$ remaining vertices into k parts.

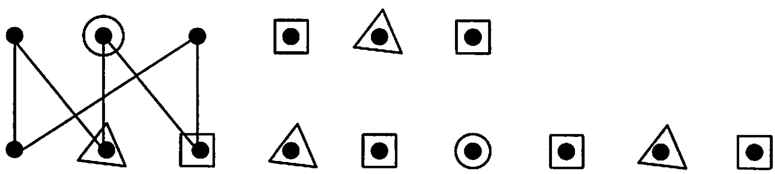

Figure 6: Partition the remaining vertices not on the cycle.

For each part, we construct a spanning tree on the vertices of that part together with the vertex of the cycle to which that set will connect.

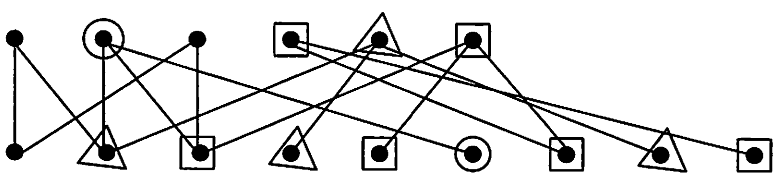

Figure 7: Construct a spanning tree on each partition

We need to partition this set of $m+n-2i-k$ vertices.

Let $D(t,k,m)$ be the array with k columns whose rows are each an ordered partition, or composition, of t consisting of integers at least m. That is each row of $D(t,k,m)$ contains k integers at least m which sum to t. Let us establish some facts about $D(t,k,m)$.

Lemma 4. *$D(t,k,m)$ has exactly $\binom{t-(m-1)k-1}{k-1}$ rows*

Proof. Consider the array $A(t,k)$ which is defined in [4] as the matrix with k columns whose rows are each of the ordered partitions, or compositions, of t consisting of positive integers. This is identical to $D(t,k,1)$. In [4] it was established that this array has $\binom{t-1}{k-1}$ rows. If we subtract exactly $m-1$ from all entries of $D(t,k,m)$ the result will be $A(t-(m-1)k,k) =$

$D(t-(m-1)k,k,1)$. This establishes the the number of rows of $D(t-(m-1)k,k,m)$. □

We will also need to know what the entries of $D(t,k,m)$ are. We shall denote the entry in the i^{th} row and j^{th} column by $d_{i,j}(t,k,m)$. Simple adaptation of Lemma 5 from [4] in the same manner as in the proof of Lemma 4 establishes:

Lemma 5. *Let* $i' = i - \sum_{l=m}^{d_{i,1}(t,k,m)-1} \binom{t-l-(m-1)(k-1)-1}{k-2}$ *to simplify the expression. Then we have*

$$d_{i,j}(t,k,m) = \begin{cases} t & \text{if } k=1 \\ \min\left\{z \mid 0 \le z \le t,\ \sum_{l=m}^{z-(k-1)m} \binom{t-l-(m-1)(k-1)-1}{k-2} \ge i\right\} & \text{if } j=1 \\ d_{i',j-1}\left(t-d_{i,1}(t,k,m),k-1\right) & j>1 \end{cases} \tag{4}$$

We now choose k vertices of the cycle, C, to be adjacent to vertices not contained in the cycle. We choose k_1 from $A \cap C$ and $k_2 = k - k_1$ from $B \cap C$. Clearly $k_1, k_2 \le i$. For each of the k_1 vertices in $A \cap C$ we choose a vertex of $B \setminus C$ that will be part of the spanning tree connected to this vertex. Since this will require k_1 vertices from $B \setminus C$, we also require that $k_1 \le n - i$ and similarly $k_2 \le m - i$.

We now use the rows of the appropriate $D(t, k_j, m)$ to partition the remaining $t = m - i - k_2$ vertices in A and the remaining $t = n - i - k_1$ vertices in B into parts, each associated with one of the k vertices chosen on the cycle. Each such vertex on the cycle now has a vertex in the opposing set associated with it, and possibly additional vertices as determined by the compositions given by $D(t, k_j, m)$. The set associated with a vertex in the cycle is counted l times where l is the cardinality of this set's intersection with the opposing set, A or B, as appropriate. We need to compensate for this overcounting. The number of possible ways of minimally connecting these sets together is the number of spanning trees on the corresponding complete bipartite graph.

Recapping, to enumerate every unicyclic spanning subgraph of $K_{m,n}$ we

1. Choose $2i$ vertices to form a cycle in $K_{m,n}$;

2. Choose $k = k_1 + k_2$ vertices in the cycle to connect to the remaining vertices;

3. Associate one vertex from the opposite set to each of these;
4. Partition the remaining vertices into k parts using $D(t, k_j, m)$; and
5. Construct a spanning tree on the complete bipartite graph induced on each of these k sets.

This gives us the following theorem.

Theorem 6. *The number of unicyclic spanning subgraphs of $K_{m,n}$ is given by*

$$|\tau(K_{m,n})| = \sum_{i=2}^{\min(n,m)} \binom{m}{i}\binom{n}{i}\frac{(i!)^2}{2i} \sum_{k_1=0}^{\min(n-i,i)} \binom{i}{k_1}\binom{n-i}{k_1} k_1! \cdot$$
$$\sum_{k_2=0}^{\min(m-i,i)} \binom{i}{k_2}\binom{m-i}{k_2} k_2! \cdot$$
$$\sum_{j_1=1}^{\binom{n-i+k_2-1}{k_1+k_2-1}} \sum_{j_2=1}^{\binom{m-i+k_1-1}{k_1+k_2-1}} (n-i-k_1)!(m-i-k_2)! \cdot$$
$$\prod_{f_1=1}^{k_1+k_2} \frac{(d_{j_1,f_1}(n-i-k_1, k_1+k_2, 0)+1)^{d_{j_2,f_1}(m-i-k_2,k_1+k_2,0)}}{(d_{j_1,f_1}(n-i-k_1,k_1+k_2,0)+\delta_{f_1 \le k_1})!} \cdot$$
$$\frac{(d_{j_2,f_1}(m-i-k_2,k_1+k_2,0)+1)^{d_{j_1,f_1}(n-i-k_1,k_1+k_2,0)}}{(d_{j_2,f_1}(m-i-k_2,k_1+k_2,0)+\delta_{f_1 > k_1})!}. \qquad (5)$$

We include the special cases where either $n - i = 0$ or $m - i = 0$ by including them in the summation subject to the convention that $k_1 + k_2 > 0$ unless $m = n = i$. We give specific values of $|\tau(K_{m,n})|$ in Table 1

We establish that there are more bases of the bicircular matroid on $K_{m,n}$ than the cycle matroid. For $K_{m,n}$, assuming $m \le n$, Theorem 2 gives

$$\frac{|\mathcal{B}_M(K_{m,n})|(|E(K_{m,n}| - r(M(K_{m,n})))}{2m} \le |\tau(K_{m,n})| \le |\mathcal{B}_M(K_{m,n})|(|E(K_{m,n}| - r(M(K_{m,n}))).$$

Since $|\mathcal{B}_M(K_{m,n})| = m^{n-1}n^{m-1}$, this becomes

$$\frac{m^{n-1}n^{m-1}(mn-(m+n-1)}{2m} \le |\tau(K_{m,n})| \le m^{n-1}n^{m-1}(mn-(m+n-1)).$$

When $m \ge 4$ or $n \ge 6$ the left hand side is larger than the number of spanning trees. In this case there are more unicyclic spanning subgraphs and more bases of the bicircular matroid than spanning trees on $K_{m,n}$. This leaves only a finite number of cases to calculate. For $m+n \leq 6$ there are more spanning trees than bases. For $m=2$, $n=5$ the two sets have equal cardinality. In all other cases there are more bases than spanning trees.

The sequence $\tau(K_{2,n})$ can be found in Neil Sloane's On-line Encyclopedia of Integer Sequences as sequence A001788 [6]. It is listed there as the number of faces in an (n+1)-dimensional hypercube and is also equal to $\sum_{i=1}^{n} i^2 \binom{n}{i}$.

4 Enumerating the bases of the bicircular matroid on $K_{m,n}$

Unlike the bases of the cycle matroid on a graph, the bases of the bicircular matroid need not be connected. A basis of the bicircular matroid may have more than one unicyclic component [3]. We have enumerated the connected bases of the bicircular matroid on $K_{m,n}$. In this section we will find an enumeration of all of the bases of the bicircular matroid on $K_{m,n}$.

We first note that since a cycle in a bipartite graph has at least four vertices, two in each part, a graph with multi-component bases must have a minimum of 8 vertices. Thus $|\mathcal{B}_B(K_{m,n})| = |\tau(K_{m,n})|$ for $m \leq 3$ or $n \leq 3$.

When we have more than one component, each of the p components is a unicyclic spanning subgraph on the induced complete bipartite graph on the vertices of that part. Therefore, we need only partition the vertices and find the number of unicyclic spanning graphs on each set of vertices to determine the number of multi-component bases of the bicircular matroid.

Since each component must contain a minimum of 2 vertices in each part, the maximum number of components is $\min(\lfloor m/2 \rfloor, \lfloor n/2 \rfloor)$. To facilitate manipulating these components we use the array $H(n,p,m)$, defined in [4], as the $s(n,p,m) \times p$ matrix in which each row consists of a partition of n, each of whose parts is not less than m (we will be interested in $m=2$). We denote the element found on the i^{th} row and the j^{th} column by $h_{i,j}(n,p,m)$.

Lemma 7 ([4]).

$$s(n,p,m) = \begin{cases} \sum_{l=m}^{\lfloor n/p \rfloor} s(n-l,p-1,l) & \textit{if } mp \leq n \\ 0 & \textit{otherwise} \end{cases} \tag{6}$$

where

$$s(n,2,m) = \left\lfloor \frac{n+2-2k}{2} \right\rfloor.$$

Lemma 8 ([4]). *Let* $i' = i - \sum_{l=m}^{h_{i,1}(n,p,m)-1} s(n-l, p-1, m)$ *to simplify the expression. Then we get*

$$h_{i,j}(n,k,m) = \begin{cases} \min\{z \mid m \le z \le \lfloor n/p \rfloor,\ \sum_{l=m}^{z} s(n-l,p-1,m) \ge i\} & \text{if } j = 1 \\ h_{i',j-1}\left(n - h_{i,1}(n,p,m), p-1, m\right) & j > 1 \end{cases} \tag{7}$$

To find the number of multi-component bases of the bicircular matroid on $K_{m,n}$, we partition the vertices into p components. For each component we use the rows of $H(v,p,2)$ to enumerate the different sizes of each part. If there are multiple parts of the same size we need to correct for the over-counting caused by the fact that q parts with the same number of vertices can be chosen $q!$ ways indistinguishably. We adopt the same notation as in [4]: For row r of $H(v,p,2)$ we denote the number of times an entry k appears by $x_k(r)$ and define

$$\rho(r) = \prod_{k=2}^{n-2(p-1)} x_k(r)!.$$

Theorem 9. *The number of bases of the bicircular matroid on* $K_{m,n}$, *for* $m \ge 4$ *and* $n \ge 4$ *is given by*

$$|\mathcal{B}_B(K_{m,n})| = \sum_{p=1}^{\min(\lfloor m/2 \rfloor, \lfloor n/2 \rfloor)} \sum_{r_1=1}^{s(m,p,2)} \sum_{r_2=1}^{\binom{n-p-1}{p-1}} \frac{m!n!}{\rho(r_1)} \cdot \prod_{q=1}^{p} \frac{\tau(K_{h_{r_1,q}(m,p,2), d_{r_2,q}(n,p,2)})}{h_{r_1,q}(m,p,2)!\ d_{r_2,q}(n,p,2)!}. \tag{8}$$

We give specific values of $|\mathcal{B}_B(K_{m,n})|$ in Table 2

5 Conclusion

We have enumerated the bases of the bicircular matroid on $K_{m,n}$ and shown that in all but a small number of cases this set is larger than the set of spanning trees. This extends the method of the enumeration of the bases of the bicircular matroid on K_n in [4]. This work grew out of the undergraduate senior thesis project of A.M. Meyers. We suggest that enumerating further families is a possibly publishable project of ideal scope and range for an undergraduate.

We hope that the techniques herein may enable the enumeration of the bases of bicircular matroids on other classes of graphs; indeed one of the motivations for this work was to show the extendibility of the techniques developed in [4]. Families of graphs that could be investigated next are complete multipartite graphs and regular graphs. Strongly regular graphs, in particular, may be sufficiently structured as to make this analysis readily feasible.

Our current focus is to investigate the asymptotic behaviour of $|\mathcal{B}_B(K_{m,n})|$. We finally conjecture that for a fixed number of vertices $B(K_{m,n})$ has the largest number of bases when the parts of $K_{m,n}$ are as equal as possible.

6 Acknowledgements

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$$
\begin{aligned}
|\tau(K_{2,2})| &= 1\\
|\tau(K_{2,3})| &= 6\\
|\tau(K_{2,4})| &= 24\\
|\tau(K_{3,3})| &= 78\\
|\tau(K_{2,5})| &= 80\\
|\tau(K_{3,4})| &= 612\\
|\tau(K_{2,6})| &= 240\\
|\tau(K_{3,5})| &= 3780\\
|\tau(K_{4,4})| &= 8424\\
|\tau(K_{2,7})| &= 672\\
|\tau(K_{3,6})| &= 20250\\
|\tau(K_{4,5})| &= 85920\\
|\tau(K_{2,8})| &= 1792\\
|\tau(K_{3,7})| &= 98658\\
|\tau(K_{4,6})| &= 731520\\
|\tau(K_{5,5})| &= 1359640\\
|\tau(K_{2,9})| &= 4608\\
|\tau(K_{3,8})| &= 449064\\
|\tau(K_{4,7})| &= 5515776\\
|\tau(K_{5,6})| &= 17269200\\
|\tau(K_{2,10})| &= 11520\\
|\tau(K_{3,9})| &= 1942056\\
|\tau(K_{4,8})| &= 38105088\\
|\tau(K_{5,7})| &= 189073500\\
|\tau(K_{6,6})| &= 314452800
\end{aligned}
$$

Table 1: The number of unicyclic spanning subgraphs of $K_{m,n}$ for $4 \le m+n \le 12$ and $2 \le m \le n$.

$$
\begin{aligned}
|\mathcal{B}_B(K_{2,2})| &= 1\\
|\mathcal{B}_B(K_{2,3})| &= 6\\
|\mathcal{B}_B(K_{2,4})| &= 24\\
|\mathcal{B}_B(K_{3,3})| &= 78\\
|\mathcal{B}_B(K_{2,5})| &= 80\\
|\mathcal{B}_B(K_{3,4})| &= 612\\
|\mathcal{B}_B(K_{2,6})| &= 240\\
|\mathcal{B}_B(K_{3,5})| &= 3780\\
|\mathcal{B}_B(K_{4,4})| &= 8442\\
|\mathcal{B}_B(K_{2,7})| &= 672\\
|\mathcal{B}_B(K_{3,6})| &= 20250\\
|\mathcal{B}_B(K_{4,5})| &= 86280\\
|\mathcal{B}_B(K_{2,8})| &= 1792\\
|\mathcal{B}_B(K_{3,7})| &= 98658\\
|\mathcal{B}_B(K_{4,6})| &= 735840\\
|\mathcal{B}_B(K_{5,5})| &= 1371040\\
|\mathcal{B}_B(K_{2,9})| &= 4608\\
|\mathcal{B}_B(K_{3,8})| &= 449064\\
|\mathcal{B}_B(K_{4,7})| &= 5556096\\
|\mathcal{B}_B(K_{5,6})| &= 17476200\\
|\mathcal{B}_B(K_{2,10})| &= 11520\\
|\mathcal{B}_B(K_{3,9})| &= 1942056\\
|\mathcal{B}_B(K_{4,8})| &= 38427648\\
|\mathcal{B}_B(K_{5,7})| &= 191908500\\
|\mathcal{B}_B(K_{6,6})| &= 319899150
\end{aligned}
$$

Table 2: The number of bases of the bicircular matroid on $K_{m,n}$ for $4 \leq m+n \leq 12$ and $2 \leq m \leq n$.