THE NICHE CATEGORY OF SPARSE GRAPHS

STEVE BOWSER AND CHARLES CABLE

ABSTRACT. The niche graph of a digraph D is the undirected graph defined on the same vertex set in which two vertices are adjacent if they share either a common in-neighbor or a common out-neighbor in D. A hierarchy of graphs exists, depending on the condition of being the niche graph of a digraph having, respectively, no cycles, no cycles of length two, no loops, or loops. Our goal is to classify in this hierarchy all graphs of order $n \geq 3$ having a generated subgraph isomorphic to the discrete graph on $n-2$ vertices.

1. INTRODUCTION AND PRELIMINARIES

A survey of the beginnings of the study of niche graphs can be found in [8]. Some more recent work has distinguished categories of niche graphs and attempted to identify the niche graph categories of the graphs belonging to various classes. (See, for example, [1],[3],[10]). The categories we employ in this paper are the ones found in the literature, with the addition of one suggested by the complete description of the niche graphs of all tournaments in [5]. They are, moreover, the same ones used in [4], in which the niche category of each graph in a class, called the dense graphs, is given. The purpose of the present work is to give the niche graph category of each sparse graph.

Notation.

- *If x and y are vertices, the undirected edge between x and y will be denoted $[x,y]$ and the arc from x to y will be denoted $x \rightarrow y$.*
- *The set of in-neighbors of a vertex x in a digraph D will be denoted $in_D(x)$. Similarly, the set of out-neighbors will be denoted $out_D(x)$. In case there is no danger of ambiguity, the subscript will be suppressed.*
- *The discrete graph of order n will be denoted I_n.*
- *The set of non-negative integers will be denoted $\mathbb{N}$.*
- *The cardinality of a finite set A will be denoted $|A|$.*

Date: December 11, 2000.
Key words and phrases. Niche graph.

Definition 1. *Given a digraph $D = (V, A)$, the* **niche graph** *of D is an (undirected) graph $G = (V, E)$ such that $[x, y] \in E$ if and only if either $in(x) \cap in(y)$ or $out(x) \cap out(y)$ is nonempty. We will also say that D is a* **niche digraph** *of G. A niche digraph for a given graph will be called* **minimal** *if its arc set cannot be reduced without changing the niche graph.*

Definition 2. *Among niche graphs, we identify the categories listed below. (It is shown in [4] that these categories are nested and distinct.) A graph G is:*

- *an* **acyclic** *niche graph if it has an acyclic niche digraph.*
- *an* **asymmetric** *niche graph if it has an asymmetric niche digraph (i.e., one without loops or cycles of length 2).*
- *a* **cyclic** *niche graph if it has a niche digraph without loops.*
- *a* **loop** *niche graph if it has an arbitrary niche digraph (possibly containing loops).*

Definition 3. *We will call a graph of order $n \geq 3$* **sparse** *if it contains a generated subgraph isomorphic to I_{n-2} and* **dense** *if it contains a generated subgraph isomorphic to K_{n-2}.*

Example. *For $j \in \mathbb{N}$, any subgraph of $K_{2,j}$ with at least 3 vertices is sparse. Indeed, if $\{r, s\}$ is the two element set in the partition of the vertices defining $K_{2,j}$, then another characterization of sparse graphs is those which are subgraphs with at least 3 vertices of $K_{2,j} + [r, s]$, for some $j \in \mathbb{N}$.*

The goal is to determine the niche graph categories to which each sparse graph belongs. We begin by observing that with the exception of I_3 and I_4, all sparse graphs of order less than 5 are also dense, so their niche categories are given in [4]. For the record, all are acyclic niche graphs except $K_{1,3}$, which is a loop niche graph but not a cyclic niche graph, and K_3 and C_4, both of which are cyclic niche graphs but not asymmetric niche graphs.

Thus, our attention turns to sparse graphs of order at least 5. To make reference to these graphs easier, we single out the following representation.

We say that a graph G of order n *satisfies condition* ★ if:

(1) $n \geq 5$,

(2) $V(G) = W \cup \{r, s\}$, where $\langle W \rangle \cong I_{n-2}$ and $deg(r) = maxdeg(G)$.

The symbols r and s will be reserved throughout the rest of the paper for the two vertices of G outside of W and r is reserved for a (distinguished) vertex of G of maximum degree. Our determination of niche category of graphs satisfying condition ★ will be split into the case in which r is not adjacent to s in G, undertaken in Section 2, and the case in which r is adjacent to s, covered in Section 3. We close this introductory section with a result that holds in either case.

Lemma 1. *Let $i = |W \cap nbr_G(r)|$. If $i > 4$, then G is not a cyclic niche graph and if $i > 3$, then G is not an asymmetric niche graph.*

Proof. Let D be an cyclic niche digraph for a graph G satisfying ★. The subdigraph induced by the set of arcs incident to either r or s must be a subdigraph of the one shown in Figure 1 (where we do not assume that the vertices a, b, c, and d are distinct). Since $nbr_G(r) \subseteq \{d, c\} \cup in(a) \cup out(b)$ and since neither $in(a)$ nor $out(b)$ can contain more than one vertex of W, the first result follows. If D is asymmetric, then there is no loss of generality in assuming that D does not contain the arc $s \rightarrow r$. With this change to Figure 1, the vertex c is no longer a potential member of $nbr_G(r)$, so the second result follows.

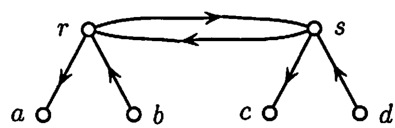


FIGURE 1

□

2. Vertices r and s not adjacent in the niche graph

Throughout this section we assume that G is a graph satisfying ★ and that vertices r and s are not adjacent.

Lemma 2. *If $deg_G(r) > 4$, then G is not a loop niche graph.*

Proof. Suppose that D is a minimal loop niche digraph for a graph G satisfying ★ in which r and s are not adjacent. Since r can have at most one in-neighbor in W and at most one out-neighbor in W and since the same is true of s, D must be a subdigraph of the digraph in Figure 2. Here the vertices in the set $\{a, b, c, d, w, x, y, z\}$ are not assumed to be distinct and vertices which are isolated in D are not depicted.

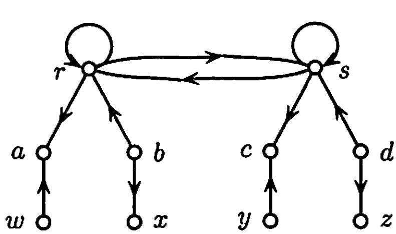


FIGURE 2

If D contains a loop at either r or s, then it can contain neither $r \rightarrow s$ nor $s \rightarrow r$, since either would produce the forbidden edge $[r, s]$ in G. On the other hand, if D contains neither $r \to s$ or $s \to r$, then clearly $deg_G(r) \leq 4$, so we may assume that neither r nor s has a loop. In this case, $nbr_G(r) \subseteq \{c, d, w, x\}$. □

The following short-hand will simplify reference to the sparse graphs under consideration in this section.

Definition 4. *Each graph G satisfying condition ★ in which r and s are not adjacent can be represented by a 4-tuple (i, j, k, m) where $i = deg_G(r)$, $j = deg_G(s)$, $k = |nbr_G(r) \cap nbr_G(s)|$, and $m =$ the number of isolated vertices in W. (Although we will not require the fact, we note that this representation is unique for graphs satisfying the above conditions.)*

Using Lemma 2 and the assumptions of this section, we need consider only graphs represented by (i, j, k, m) for which $0 \leq k \leq j \leq i \leq 4$. Moreover, Lemma 1 says that none of these graphs with $i = 4$ are asymmetric niche graphs.

Remark 1. *If G is niche graph which is respectively acyclic, asymmetric, cyclic, or loop, then so is $G \cup I_m$ (for any positive integer m). Indeed, if D is a niche digraph for G, then $D \cup I_m$ is a niche digraph for $G \cup I_m$.*

Lemma 3. *If $deg_G(r) = 4$, then G is a cyclic niche graph unless G is $(4, 1, 1, m)$ or $(4, 0, 0, m)$, for some non-negative integer m. The latter two forms of sparse graphs are loop niche graphs but not cyclic niche graphs.*

Proof. For the first sentence of the lemma statement, it suffices, by Remark 1, to consider the case in which the number of isolated vertices in W is zero. Of the graphs under consideration, there are a total of 15 sparse graphs of the form $(4, j, k, 0)$, since $0 \leq k \leq j \leq 4$. Table 1 provides cyclic niche graphs for each of these graphs except $(4, 1, 1, 0)$ and $(4, 0, 0, 0)$, for which it provides loop niche graphs. The $4-$ *tuple* $(4, 0, 0, m)$ represents the graph $K_{1,4} \cup I_m$ and in [3] it is proved that $K_{1,4}$ cannot be made into a cyclic niche graph by the addition of any finite number of isolated vertices. Thus, graphs of the form $(4, 0, 0, m)$ are not cyclic niche graphs. We will use this result to show that the same is true of graphs of the form $(4, 1, 1, m)$.

Let $m \geq 0$ and let G be the graph $(4, 1, 1, m)$ with $nbr_G(r) = \{a, b, c, d\}$ and $nbr_G(s) = \{a\}$. Suppose that G is the niche graph of the minimal cyclic digraph D. We may assume, without loss of generality that a and s have a common in-neighbor, x, in D. Now a must have another in-neighbor, y, since otherwise $D - \{x \to a\}$ would be a cyclic niche digraph for $(4, 0, 0, m)$. Similarly, s must have another in-neighbor, z. If $y = z$, then, since x can have no additional out-neighbors and a can have no additional in-neighbors, removing $x \to a$ from D would not alter its niche graph, contradicting the assumption that D is minimal. So y and z are distinct. It follows that $[x, y]$ and $[x, z]$ are edges in G and, since D has no loops, x must be r, y is a neighbor in G other than a, say b, and z is either a or c. Thus, we may assume that one of the two digraphs depicted in Figure 3 is a subdigraph of D.

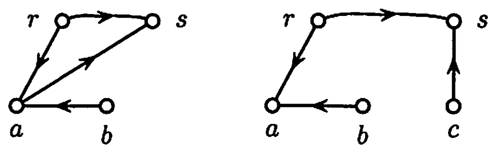


FIGURE 3

Since r may have no additional out-neighbors and neither a nor s may have additional in-neighbors, the two additional edges in G at r must arise from distinct common in-neighbors, which are therefore adjacent in G. Since neither of the latter may be r, the only candidates are s and a. It is routine to check that this is impossible without introducing either loops in D or unwanted edges in G. Thus the supposed cyclic digraph D cannot exist. □

Lemma 4. *If* $deg_G(r) \leq 3$, *then* G *is an acyclic niche graph with the following exceptions:* $(2,2,2,1)$ *and* $(3,1,0,0)$ *are cyclic but not asymmetric niche graphs; for each* $m \in \mathbb{N}$, $(3,0,0,m)$ *is a loop niche graph but not a cyclic niche graph and* $(3,3,3,m)$ *is an asymmetric niche graph but not an acyclic one.*

Proof. It is shown in [6] that the graphs $(2,2,2,m)$ are acyclic niche graphs for $m > 1$, and in [4] that $(2,2,2,0)$ is cyclic, so Remark 1 implies that $(2,2,2,1)$ is also cyclic. The graphs $(3,1,0,m)$ are acyclic for $m > 0$ as is seen by considering the digraph depicted below for $(3,1,0,1)$ and referring to Remark 1.

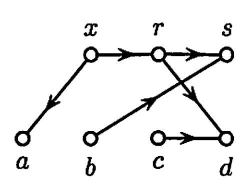


FIGURE 4. Niche digraph for $(3,1,0,1)$

For the remainder of the positive statements in the lemma, it suffices, again by Remark 1, to let $m = 0$. There are three such graphs (satisfying ★) with $i = 2$. Again one may refer to [6] to see that $(2,2,1,0)$ and $(2,2,0,0)$ are acyclic niche graphs. Table 1 provides the required digraphs for $(2,1,0,0)$ and for the remaining cases with $i > 2$.

We address the negative statements regarding each of the four exceptions one at a time. The graph $(2,2,2,0)$ is C_4, which is, as already mentioned, not asymmetric. A short argument now extends this conclusion to $(2,2,2,1)$. Suppose that D is an asymmetric niche digraph for $G = (2,2,2,1)$, that a and b are the (shared) neighbors of r and s in G, and that x is the isolated vertex. There is no loss in assuming that x is a common in-neighbor of vertices of an edge of G, since otherwise the removal of x would provide an asymmetric niche digraph for C_4. So, assume that D contains the arcs $x \to r$ and $x \to a$. Neither r nor a can have another in-neighbor, so a and s must have a common out-neighbor, which must be b, since D is asymmetric. Now it is impossible to produce the edge $[r,b]$ in the niche graph of D without contradicting the assumption that D is asymmetric.

It is shown in [3] that $(3,0,0,m)$ is not a cyclic niche graph for any natural number m, and this fact can be used to demonstrate that $(3,1,0,0)$ is not asymmetric. Suppose, on the contrary, that the neighbors of r in $(3,1,0,0)$ are a,b,c, that the neighbor of s is d, and that D is a minimal asymmetric niche digraph. Without loss of generality, assume that vertex x is a common in-neighbor of s and d. Both s and d must have an additional in-neighbor in D, since otherwise the removal of one of the arcs $x \to s$ or $x \to d$ would produce a cyclic niche digraph for $(3,0,0,m)$ for some m. These additional in-neighbors, which must be distinct by the same argument used in Lemma 3, are both adjacent to x in the niche graph, so x must be r and the additional in-neighbors may be taken to be a and b. Thus the arcs depicted in Figure 5 are forced.

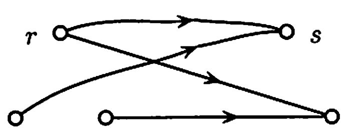


FIGURE 5

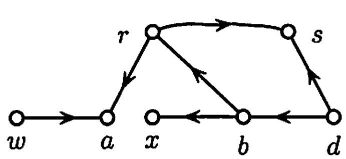
No additional out-neighbors of r are allowed and neither s nor d can have additional in-neighbors, so r and c must have a common in-neighbor in D, yet none can exist without either introducing unwanted edges in the niche graph or violating the asymmetric condition on D.

Finally, suppose that $m \geq 0$ and that D is a minimal acyclic niche digraph for $G = K_{2,3} \cup I_m = (3,3,3,m)$. Ignoring vertices isolated in D, it must be a subdigraph of that depicted in Figure 6 with the arc $s \to r$ removed (where we do not assume that the vertices other than r and s are distinct). It follows that $nbr_G(r) \subseteq \{d,w,x\}$ and $nbr_G(s) \subseteq \{a,y,z\}$, but for G, these four sets must be equal and each contain 3 distinct vertices. To complete the proof, it suffices to show that $a \notin nbr_G(r)$. There are three cases to consider.

(1) $a = d$. In this case , y must be x and z must be w, but this produces a cycle $a \to w \to a$, contradicting the assumption that D is acyclic.
(2) $a = w$. This produces a forbidden loop at a.
(3) $a = x$. In this case, $b = w$. If $y = w$, then a and c are either equal or adjacent, but neither is allowed, so $y = d$ and $z = w$. Now d is a common in-neighbor of b and c, so b and c must be equal. This too is impossible since D now contains the cycle $r \to s \to b \to r$.

□

Remark 2. *Table 1 gives digraphs for each of the sparse niche graphs satisfying condition ★ in which r and s are not adjacent and $m = 0$. In the interest of saving space, we have selected between two formats for these descriptions. In most cases, the last column lists alterations to the digraph depicted in Figure 6. In each such digraph, the vertices are distinct unless an alteration states otherwise. The arc $s \to r$ is denoted by the more compact form (s, r) in the Table. For example, one obtains a niche digraph for $(3, 2, 0, 0)$ by identifying the vertices b and z in Figure 6 and removing the arc $s \to r$ and the vertices c and y. The result is the digraph below.*



In three cases the required digraph cannot be described in the above manner, so a "correspondence" of the digraph is provided. For example "r:r,a; b:r,x" is the correspondence of the digraph with arc set $\{r \to r, r \to a, b \to r, b \to x\}$.

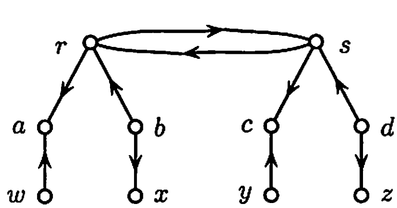


FIGURE 6

We close this section with a summary.

Theorem 1. *If G is a sparse graph satisfying ★ and r is not adjacent to s, then:*

(1) *G is not a (loop) niche graph for $deg(r) > 4$.*
(2) *G is a cyclic niche graph but not an asymmetric niche graph if $deg(r) = 4$, with the exception of those represented by $(4, 0, 0, m)$ or $(4, 1, 1, m)$ ($m \in \mathbb{N}$), which are loop niche graphs but not cyclic niche graphs.*
(3) *G is an acyclic niche graph if $deg(r) \leq 3$ with the exception of:*
  (a) *$(3, 3, 3, m)$, which is asymmetric*
  (b) *$(2, 2, 2, 1)$ and $(3, 1, 0, 0)$, both of which are cyclic niche graphs but not asymmetric niche graphs.*
  (c) *$(3, 0, 0, m)$, which is a loop niche graph but not a cyclic niche graph.*

i	j	k	*D*
4	4	4	a=d, b=c, x=y, w=z
4	4	3	a=d, b=c, x=y
4	4	2	a=d, b=c
4	4	1	a=d
4	4	0	(no change)
4	3	3	a=d, c=b, x=y, -z
4	3	2	a=d, b=c, -z
4	3	1	a=d, -z
4	3	0	-z
4	2	2	a=d, b=c, -y, -z
4	2	1	a=d, -y, -z
4	2	0	-y, -z
4	1	1	r:r,a; s:c; b:r,x; c:a; x:c
4	1	0	a=b, -y, -z
4	0	0	r:r,a; b:r,x; w:a

i	j	k	*D*
3	3	3	-(s,r), a=x, b=c, d=y, -w, -z
3	3	2	-(s,r), a=x, c=z, -w, -y
3	3	1	-(s,r), a=d, y=b, -x
3	3	0	-(s,r), b=z, c=w
3	2	2	-(s,r), a=x, b=z, -c, -w, -y
3	2	1	-(s,r), a=d, c=x, -y, -z
3	2	0	-(s,r), b=z, -c, -y
3	1	1	-(s,r), a=x, -c, -w, -y, -z
3	1	0	a=b, -y, -d, -z
3	0	0	r:r,a; b:r,x
2	1	0	r:b; a:r,b; b:s,c

TABLE 1. Niche digraphs for r not adjacent to s

3. Vertices r and s are adjacent in the niche graph

Throughout this section we assume that G is a graph satisfying ★ and that vertices r and s are adjacent.

Notation. *The following abbreviations will be useful:*

$\overline{nbr}(r) = nbr_G(r) - \{s\}$ $\quad\quad$ $\overline{nbr}(s) = nbr_G(s) - \{r\}$

$\overline{deg}(r) = |\overline{nbr}(r)|$ $\quad\quad$ $\overline{deg}(s) = |\overline{nbr}(s)|$

$\overline{(i,j,k,m)}$ *describes* G *where*

$$i = \overline{deg}(r) \qquad j = \overline{deg}(s)$$
$$k = |nbr_G(r) \cap nbr_G(s)| \qquad m = \textit{the number of isolated vertices in } W.$$

It is clear again that each G under consideration in this section is described by a unique barred $4-$*tuple*. The next lemma gives the main reduction result in this case.

Lemma 5. *If* G *satisfies the assumptions of this section and* $\overline{deg}(r) > 6$, *then* G *is not a (loop) niche graph.*

Proof. Let G satisfy the assumptions of this section and suppose D is a minimal digraph with niche graph G. If isolated vertices are ignored, then D must be a subdigraph of the digraph in Figure 2, where it is not assumed that the vertices a, b, c, d, w, x, y, z are distinct. It follows $nbr_G(r) \subseteq \{s, a, b, c, d, w, x\}$ and that $\overline{deg}(r) \leq 6$. □

Lemma 6. *If* $i \geq 3$, *then* $\overline{(i,j,0,m)}$ *lies in the same niche graph categories as* $\overline{(i,j,0,0)}$.

Proof. Suppose $i \geq 3$ and that D is a minimal niche digraph of $G = \overline{(i,j,0,m)}$. Suppose that $\{a,b,c\} \subseteq nbr_G(r)$ and that x is an isolated vertex of G. Since D is minimal, x cannot have exactly one out-neighbor (the removal of that arc would not change the niche graph). The proof will be completed by showing that if x has more than one out-neighbor, then $i = 3$ and $j = 1$, and that, in this case, it is possible to replace D by a new niche digraph D' for G of the same category as D and in which every vertex which is isolated in G is also isolated in D'. Of course, multiple out-neighbors of x are adjacent in the niche graph, so we consider possibilities:

(1) $x \to r$ and $x \to s$. Since x is isolated in G, r can have no additional in-neighbors, the edges $[r,a]$ and $[r,b]$ must arise from distinct common out-neighbors. These out-neighbors are adjacent in G and neither is equal to r, so one must be s, which means that r and x are adjacent, a contradiction.

(2) $x \to r$ and $x \to a$. Again, r can have no additional in-neighbors, so $[r,b]$ and $[r,c]$ must arise from distinct common out-neighbors. As before, one must be s and the other, say d, is adjacent to s. In this case, the edge $[r,s]$ must also arise from a common out-neighbor. This vertex cannot be s but is adjacent to d, an impossibility.

(3) $x \to s$ and $x \to d$ (where d is a neighbor of s in G). The edge $[r,s]$ must arise from a common out-neighbor. By the first case above (with all arrows reversed), this out-neighbor cannot be an isolated vertex from W, so it must be adjacent in G to either r or s. In the

second case, D has the arcs $r \to e$ and $s \to e$, where e is a second neighbor of s in G. Since s can have no additional in-neighbors and no additional out-neighbors, it is impossible to form the edge $[s,e]$. Therefore, we are left with the first case and may assume that D has the arcs $s \to a$ and $r \to a$. If both $[r,b]$ and $[r,c]$ arise from (necessarily distinct) common in-neighbors, then these in-neighbors are adjacent in G, so one must be s, which in turn produces a forbidden edge between a and another vertex of W. So, without loss of generality, the edge $[r,b]$ must arise from a common out-neighbor and the only candidate is r. These observations force the arcs shown in figure 7.

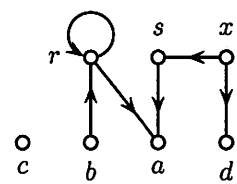


FIGURE 7

The vertex r can have no additional in-neighbors and no additional out-neighbors, so the only way to achieve the edge $[r,c]$ is by the addition of the arc $b \to c$. Now the in-neighbors of r can have no additional out-neighbors and the out-neighbors of r can have no additional in-neighbors. The same statement holds for s, which itself can have no additional neighbors in the minimal digraph D. It follows that G must be $\overline{(3,1,0,m)}$ for some $m \geq 1$ and that D is a loop digraph which is not a cyclic digraph and in which x is the only vertex isolated in G and not isolated in D. If one removes the arcs $x \to s$ and $x \to d$ and adds $d \to s$ and $d \to d$, one arrives at the (minimal) digraph D' whose existence was predicted earlier.

Thus, if $i \geq 3$, we may assume that the vertices which are isolated in $\overline{(i,j,0,m)}$ are isolated in a minimal niche digraph (of whatever category). These isolated vertices may be removed to produce a niche digraph of the same category for $\overline{(i,j,0,0)}$.

□

Lemma 7. *If $i \geq 4$, then $\overline{(i,j,k,m)}$ is not a cyclic niche graphs unless $i = 4$ and k is either 4 or 2.*

Proof. Let G be represented by $\overline{(i,j,k,m)}$ with $i \geq 4$ and suppose that D is a minimal cyclic niche digraph for G. If vertices isolated in D are ignored, then D must be a subdigraph of the digraph depicted in Figure 6, where, as usual, we do not assume that the vertices other than r and s are distinct. Since $|\overline{nbr}(r)| \geq 4$, it follows that

$$\overline{nbr}(r) = \{w, x, c, d\} \text{ and } \overline{nbr}(s) \subseteq \{a, b, y, z\},$$

that the vertices w, x, c, and d are distinct, and that D must contain both of the arcs $r \longrightarrow s$ and $s \longrightarrow r$. We may assume, without loss of generality, that the edge $[r, s]$ in G arises from the identification $a = c$, which implies $w = y$, so $k \geq 2$. It remains to note that, since D has no loops, $x \neq b$ and $z \neq d$, and that $(b = d) \iff (x = z)$, thus if $k > 2$, then $k = 4$. □

The last three results of this section refer to Table 2 which provides in the rightmost column the niche graph category of the graph described by (i, j, k, m). (This will be addressed in Lemma 9.) Note that as a result of our notational conventions, $0 \leq k \leq j \leq i$.

<table>
<tr><th></th><th>i</th><th>j</th><th>k</th><th>G</th></tr>
<tr><td>1</td><td>6</td><td>5,6</td><td>< 4</td><td rowspan="6">not loop</td></tr>
<tr><td>2</td><td>6</td><td>< 5</td><td>< 2</td></tr>
<tr><td>3</td><td>5</td><td>4,5</td><td>1</td></tr>
<tr><td>4</td><td>5</td><td></td><td>0</td></tr>
<tr><td>5</td><td>4</td><td></td><td>0</td></tr>
<tr><td>6</td><td>3</td><td>3</td><td>0</td></tr>
<tr><td>7</td><td>3</td><td>< 3</td><td>0</td><td>loop, not cyclic</td></tr>
<tr><td>8</td><td>4</td><td>4</td><td>4</td><td rowspan="3">cyclic, not asymmetric</td></tr>
<tr><td>9</td><td>4</td><td></td><td>2</td></tr>
<tr><td>10</td><td>3</td><td>1,3</td><td>j</td></tr>
</table>

TABLE 2. Niche graph categories for exceptions, r adjacent to s

Lemma 8. *If G is one of the graphs on line 10 in Table 2 , then any vertex which is isolated in G is also isolated in any minimal asymmetric niche digraph for G.*

Proof. Let D be a minimal asymmetric niche digraph for such a graph G, let a, b, c be neighbors of r, and let x be an isolated vertex of G. Since D is minimal, x cannot have exactly one out-neighbor. It suffices, then to show that x cannot have two out-neighbors. There are three cases to be eliminated.

(1) $x \longrightarrow r$ and $x \longrightarrow s$. The argument in this case is the same as that given in the corresponding case in the proof of Lemma 6

(2) $x \longrightarrow r$ and $x \longrightarrow a$. The edges $[r, b]$ and $[r, c]$ require distinct out-neighbors in D which are adjacent in G, so one must be s. WLOG, D has the arcs $r \longrightarrow s$ and $b \longrightarrow s$. The edge $[r, s]$ requires a common out-neighbor which is adjacent to s and therefore also adjacent to r in G. This out-neighbor can be neither a nor b, since D is asymmetric, so D must contain the arcs $r \longrightarrow c$ and $s \longrightarrow c$. This

leaves no vertex to serve as a common out-neighbor for the edge $[r, c]$.

(3) $x \to s$ and $x \to a$. Since s can have no additional in-neighbors, the edge $[r, s]$ arises from a common out-neighbor, say u, which cannot be a. Similarly, $[r, a]$ must arise from a common out-neighbor, say v. If $v \neq u$, then v and u are adjacent in G and so one of them must be either r or s, but this is impossible in the asymmetric digraph D. So D has the arcs, $r \to u, a \to u$, and $s \to u$, which means that the arcs $x \to s$ and $x \to a$ could be removed without affecting the niche graph, contradicting minimality.

□

Lemma 9. *Let G be the graph represented by $\overline{(i,j,k,m)}$. Table 2 gives the niche category of those graphs G satisfying the stated conditions on the parameters.*

Proof. The positive statements in Table 2 are demonstrated by digraphs given in Table 3, so the rest of the proof is devoted to the negative statements in Table 2. Lines 8 and 9 follow from Lemma 1. Lemma 6 reduces lines 4-7 to the case m=0, i.e. no isolated vertices. These graphs are investigated in [10] where the listed conclusions are proved. Lemma 8 reduces line 10 to the case m=0. These two graphs are both dense (as well as sparse) and it is proved in [4] that these graphs are not asymmetric.

Lines 1-3 follow by counting in Figure 2. Suppose D has a niche graph of the form $\overline{(6,j,k,m)}$. Referring to Figure 2, $nbr_G(r) = \{s, a, b, c, d, w, x\}$ (and these must all be distinct) and $nbr_G(s) \subseteq \{r, a, b, c, d, y, z\}$. In order to achieve $k < 4$, it will be necessary to eliminate at least one of the vertices a, b, c, d from $nbr_G(s)$. Since the arcs $r \to s$ and $r \to a$ are required for the neighbors of r, a cannot be eliminated from $nbr_G(s)$. Similarly, it is not possible to eliminate b. To eliminate c (or d) from $nbr_G(s)$, it is necessary to remove the loop $s \to s$ from Figure 2, but this would eliminate both c and d from $nbr_G(s)$. Thus, if $j > 4$ in G, then $k \geq 4$, this proves line 1. On the other hand, if $k < 2$, then at least one of a, b must be eliminated from $nbr_G(s)$, and, as already pointed out, this is impossible without affecting $nbr_G(r)$. Thus line 2 is proved. Suppose now that G is of the form $\overline{(5,j,1,m)}$. Then, the cardinality of $\{a, b, c, d\}$ is either 3 or 4. In the first case, two of these three must be eliminated from the five vertices in $\{a, b, c, d, y, z\}$ (to achieve $k = 1$). Similarly, in the second case, three vertices must be eliminated from $\{a, b, c, d, y, z\}$. Thus, in either case, $j \leq 3$ and line 3 is proved. □

Remark 3. *Table 3 is analogous to Table 1. For $i > 4$, it describes digraphs by alterations to the one depicted in Figure 2. For $i \leq 4$, it provides the correspondence for the suitable digraph.*

Theorem 2. *Let G be a sparse graph satisfying ★ in which r and s are adjacent. Unless G is one of the graphs covered in Lemma 9, its niche category is determined by $i = \overline{deg}_G(r)$ as follows: G is an acyclic niche graph if $i \leq 3$, G is a loop niche graph but not a cyclic niche graph if $4 \leq i \leq 6$, and G is not a loop niche graph for $i > 6$.*

Proof. For $i < 3$, the graphs $\overline{(i, j, k, 0)}$ are, with two exceptions, dense and it is shown in [4] that these are acyclic niche graphs. The exceptions are $\overline{(2,1,0,0)}$ and $\overline{(2,2,0,0)}$, both of which are shown in [9] to be acyclic niche graphs. By Remark 1, the result for $i < 3$ and arbitrary m follows. In light of Lemmas 5, 7, and 9, it remains to provide acyclic niche digraphs for the graphs with $i = 3$ which are not covered in lines 6 or 7 in Table 2, cyclic niche digraphs for those described in lines 8, 9, and 10, and loop niche digraphs for those described in line 7 and for those with $4 \leq i \leq 6$ not covered in Table 2. Table 3 does this for the case $m = 0$. As already mentioned, this is sufficient. Finally, the statement for $i > 6$ is a restatement of Lemma 5. □

References

[1] C. A. Anderson, *Loop and cyclic niche graphs*, Linear Alg. Applic. **217** (1995), 5–13.

[2] Stephen Bowser and Charles Cable, *Cliques and niche graphs*, Congressus Numeratium **76** (1990), 151–156.

[3] Stephen Bowser and Charles Cable, *Cyclic and loop niche numbers of novas and wheels*, Congressus Numeratium **103** (1994), 215–218.

[4] Stephen Bowser and Charles Cable, *The Niche Category of Dense Graphs*, Ars Combinatoria, to appear.

[5] Stephen Bowser, Charles Cable and Richard Lundgren, *Niche Graphs and Mixed Pair Graphs of Tournaments*, to appear in J. Graph Theory.

[6] Charles Cable, Kathryn F. Jones, J. R. Lundgren, and Suzanne Seager, *Niche Graphs*, Discrete Appl. Math. **23** (1989), 231–241.

[7] W. V. Gehrlein and P. C. Fishburn, *The smallest graphs with niche number three*, Computers & Mathematics with Applications **27**(1994), 53–57.

[8] J. R. Lundgren, *Food webs, competition graphs, competition-common enemy graphs, and niche graphs*, in F. S. Roberts, editor, *Applications of Combinatorics and Graph Theory to the Biological and Social Sciences*, Springer-Verlag, 1989. In *Volumes in Mathematics and Its Applications, Vol. 17*, 221–244.

[9] Suzanne Seager, *Niche Graph Properties of Trees*, Congressus Numeratium **81** (1991), 149–156.

[10] Suzanne Seager, *Relaxations of niche graphs*, Congressus Numeratium **96** (1993), 205–214.

Department of Mathematics,Allegheny College, Meadville, Pennsylvania 16335

E-mail address, S. Bowser: sbowser@allegheny.edu

E-mail address, C. Cable: ccable@allegheny.edu

i	j	k	D
6	6	6	w=z,x=y
6	6	5	w=z
6	6	4	
6	5	5	w=z , -y
6	5	4	-z
6	4	4	-y , -z
6	4	3	-(s,s), x=y
6	4	2	-(s,s)
6	3	3	-(s,s) , x=y, -z
6	3	2	-(s,s) , -y
6	2	2	-(s,s) , -y , -z
5	5	5	w=z , -x , -y
5	5	4	-x , -z
5	5	3	b=c
5	5	2	-(r,s)
5	4	4	-x , -y, -z
5	4	3	-(s,s), b=c, x=y
5	4	2	-(s,s), b=c
5	3	3	a=b , -y , -z
5	3	2	-(s,s), -x, -z
5	3	1	-(s,s), a=b
5	2	2	-(s,s), a=b,w=z, -y
5	2	1	-(s,s) , a=b , -y
5	1	1	-(s,s) , a=b , -y , -z
4	4	4	r:s,a; s:r,a; b:r,s,x; w:a
4	4	3	r:r,s,a; s:r,c; b:r; d:d,s; y:c
4	4	2	r:s,a; s:r,a; b:r,x; w:a; d:s,z
4	4	1	r:r,a; s:r,s,b; b:r,x; d:s,z; w:a; y:b
4	3	3	r:r,s,a; s:r,c; b:r; d:d,s
4	3	2	r:s,a; s:r,a; b:r,x; w:a; d:s
4	3	1	r:r,a; s:r,c; a:s,z; b:r,x; y:c
4	2	2	r:s,a; s:r,a; a:r,x; d:s; w:a
4	2	1	r:r,a; s:r,c; a:s,z; b:r,x
4	1	1	r:r,a; s:r,c; b:r,x
3	3	3	r:s,a; s:r,a; b:r,s; w:a
3	3	2	r:a; s:r,a; b:r,s,x; w:a
3	3	1	r:s,a; s:a; b:r,x; d:s,b; w:a
3	2	2	r:s,a; s:a; b:r,a; d:s
3	2	1	r:a; s:r,a; b:r,x; w:a
3	2	0	r:r,a; s:r,c; a:s,z; w:a; y:c
3	1	1	r:s,a; s:r,a; a:r,x; d:s
3	1	0	r:r,a; s:r,c; a:s,z; w:a
3	0	0	r:r,a; s:r,c; w:a

TABLE 3. Niche digraphs for r adjacent to s