Restricted edge-connectivity and minimum edge-degree

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Abstract

Let G be a connected graph and $S \subset E(G)$. If $G - S$ is disconnected without isolated vertices, then S is called a restricted edge-cut of G. The restricted edge-connectivity $\lambda' = \lambda'(G)$ of G is the minimum cardinality over all restricted edge-cuts of G. A connected graph G is called λ'-connected, if $\lambda'(G)$ exists. For a λ'-connected graph G, Esfahanian and Hakimi have shown, in 1988, that $\lambda'(G) \leq \xi(G)$, where $\xi(G)$ is the minimum edge-degree. A λ'-connected graph G is called λ'-optimal, if $\lambda'(G) = \xi(G)$.

Let G_1 and G_2 be two disjoint λ'-optimal graphs. In this paper we investigate the cartesian product $G_1 \times G_2$ to be λ'-optimal. In addition, we discuss the same question for another operation on G_1 and G_2, and we generalize a recent theorem of J.-M. Xu on non λ'-optimal graphs.

Keywords: Edge-connectivity; Restricted edge-connectivity; λ'-optimal graphs; Products of graphs; Atoms

1. Terminology and introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. For $X \subseteq V(G)$ let $G[X]$ be the subgraph induced by X, $\bar{X} = V(G) - X$, and $(X, \bar{X}) = (X, \bar{X})_G$ the set of edges in G with one end in X and the other in $\bar{X}$. If x is a vertex of a graph G, then $N(x) = N_G(x)$ denotes the set of vertices adjacent to x and $N[x] = N_G[x] = N(x) \cup \{x\}$. More generally, we define $N(X) = N_G(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N_G[X] = N(X) \cup X$ for a subset X of $V(G)$. The vertex v is an end vertex if $d_G(v) = 1$, and an isolated vertex if $d_G(v) = 0$, where $d(x) = d_G(x) = |N(x)|$ is the degree of $x \in V(G)$. We denote by $\delta = \delta(G)$ the minimum degree, by $\Delta = \Delta(G)$ the maximum degree, and by $n = n(G) = |V(G)|$ the order of G. We write C_n for a cycle of length n, K_n for the complete graph of order n, and $K_{n,m}$ for the complete bipartite graph. A star is a complete bipartite graph $K_{1,m}$ with $m \geq 2$, and the unique vertex of degree m is its center.

Let G be a connected graph and $S \subset E(G)$. If $G - S$ is disconnected without isolated vertices, then S is called a *restricted edge-cut* of G. The *restricted edge-connectivity* $\lambda' = \lambda'(G)$ of G is defined as the minimum of $|S|$ over all restricted edge-cuts S of G. A connected graph G is called *λ'-connected*, if $\lambda' = \lambda'(G)$ exists. A restricted edge-cut S of G is a *λ'-cut*, if $|S| = \lambda'$. Obviously, for any λ'-cut S, the graph $G - S$ consists of exactly two components. The concept of restricted edge-connectivity was introduced by Esfahanian and Hakimi [2] in 1988.

For $e = xy \in E(G)$, let $\xi_G(e) = d_G(x) + d_G(y) - 2$ be the *edge-degree* of e, and let $\xi(G) = \min\{\xi_G(e) : e \in E(G)\}$ be the *minimum edge-degree* of G. In 1988, Esfahanian and Hakimi [2] have shown that for every connected graph G of order $n \geq 4$, except a star, $\lambda'(G)$ exists and satisfies the inequality

$$\lambda'(G) \leq \xi(G). \tag{1}$$

A λ'-connected graph G is called *λ'-optimal*, if $\lambda'(G) = \xi(G)$. If $(X, \bar{X})$ is a λ'-cut, then $X \subset V(G)$ is called a *λ'-fragment*. Clearly, if X is a λ'-fragment, then $\bar{X}$ is also a λ'-fragment. Let

$$r(G) = \min\{|X| : X \text{ is a } \lambda'\text{-fragment of } G\}.$$

A λ'-fragment X of G is called a *λ'-atom* of G, if $|X| = r(G)$. Obviously, $2 \leq r(G) \leq \frac{1}{2}|V(G)|$ and $G[X]$ as well as $G[\bar{X}]$ are connected, when X is a λ'-atom. Recently, J.-M. Xu [5] has proved

Theorem 1.1 (Xu [5] 2000) A λ'-connected graph G is λ'-optimal if and only if $r(G) = 2$.

Observation 1.2 If G is a λ'-optimal graph, then

$$\lambda'(G) \le \Delta(G) + \delta(G) - 2 \quad (2)$$
$$\lambda'(G) \ge 2\delta(G) - 2 \quad (3)$$
$$\lambda'(G) \ge \delta(G) \quad (4)$$

Proof. Let $e = xy \in E(G)$ such that $d_G(x) = \delta(G)$.

$$\lambda'(G) = \xi(G) \le \xi_G(e) = d_G(x) + d_G(y) - 2 \le \delta(G) + \Delta(G) - 2,$$

and (2) proved. Because of $\lambda'(G) = \xi(G) \ge \delta(G) + \delta(G) - 2$, also (3) is true. Since G is connected, the last inequality is valid for $\delta(G) = 1$. For $\delta(G) \ge 2$, (4) follows from (3). □

Lemma 1.3 Let G be a λ'-connected graph. If A is a subset of $V(G)$ such that $G[A]$ as well as $G[\bar{A}]$ contain a component with at least two vertices, then $|(A, \bar{A})| \ge \lambda'(G)$.

Proof. Firstly, assume that $G[A]$ is connected. If H is a component of $G[\bar{A}]$ with at least two vertices, then let $B = V(H)$. Since G is connected, we see that $G - V(H) = G[\bar{B}]$ is also connected with $A \subseteq \bar{B}$. Hence $(B, \bar{B})$ is a restricted edge-cut of G, and we conclude that $\lambda'(G) \le |(B, \bar{B})| \le |(A, \bar{A})|$.

If $G[A]$ is not connected, then let $A' \subset A$ be a maximal subset such that $G[A']$ is connected. Since there is no edge from A' to $A - A'$, it follows from the first case that $|(A, \bar{A})| \ge |(A', \bar{A}')| \ge \lambda'(G)$. □

Corollary 1.4 Let G be a λ'-connected graph and $A \neq \emptyset$ a proper subset of $V(G)$. Then $|(A, \bar{A})| \ge \min\{\lambda'(G), \delta(G)\}$. In addition, if $G[A]$ consists of only isolated vertices, then $|(A, \bar{A})| \ge |A|\delta(G)$.

Let G_1 and G_2 be two disjoint graphs of order n with the vertex sets $V(G_1) = \{x_1, x_2, \ldots, x_n\}$ and $V(G_2) = \{y_1, y_2, \ldots, y_n\}$. The graph G consisting of the disjoint union of G_1 and G_2 together with the edges $x_i y_i$ for $i = 1, 2, \ldots, n$, is denoted by $G_1 \odot G_2$. The edges $x_i y_i$ are referred to as *cross edges*.

In this paper we prove that $r(G) \ge \max\{3, \delta(G)\}$ for a non λ'-optimal graph G. This generalizes a corresponding result of J.-M. Xu [5] on regular graphs. If G_1 and G_2 are two disjoint λ'-optimal graphs of order n such that $2 \le \xi(G_i) \le n - 2$ for $i = 1, 2$, then we show that $G_1 \odot G_2$ is also λ'-optimal. Furthermore, we investigate the cartesian product $H_1 \times H_2$ to be λ'-optimal, when H_1 and H_2 are disjoint λ'-optimal graphs. Different examples will show that the presented results are best possible.

2. Non λ'-optimal graphs

Theorem 2.1 Let G be a λ'-connected graph. If G is not λ'-optimal, then

$$r(G) \geq \max\{3, \delta(G)\}.$$

Proof. Let X be a λ'-atom of G. In view of the hypothesis that G is not λ'-optimal, it follows from Theorem 1.1, that $r = r(G) = |X| \geq 3$. Therefore, it remains to show that $r \geq \delta(G)$.

Let $u \in X$ such that $s = d_G(u) = \min_{x \in X} d_G(x)$. Clearly, $G[X]$ is connected, and hence there exists a vertex $v \in X$ which is adjacent to u. Because of $d_G(v) \geq \xi(G) - s + 2$ and since G is not λ'-optimal, we obtain

$$\begin{aligned} \xi(G) &> \lambda'(G) = |(X, \bar{X})| \geq \sum_{x \in X} d_G(x) - r(r-1) \\ &\geq d_G(v) + (r-1)d_G(u) - r(r-1) \\ &\geq \xi(G) - s + 2 + (r-1)s - r(r-1) \end{aligned}$$

This inequality implies

$$(r - (s-1))(r-2) > 0$$

and consequently, because of $r - 2 > 0$, we deduce that $r \geq s = d_G(u) \geq \delta(G)$. □

Corollary 1.4 (J.-M. Xu [5] 2000) Let G be a λ'-connected and k-regular graph. If G is not λ'-optimal, then $r(G) \geq k \geq 3$.

Example 2.3 Let $n \geq 3$ be an integer and let $G = K_n \odot K_n$. Then $\xi(G) = 2n-2$, $\lambda'(G) = r(G) = \delta(G) = n$. Therefore, G is not λ'-optimal and $r(G) = \delta(G)$. This example shows that Theorem 2.1 is best possible.

Using Turán's [4] bound $|E(G)| \leq \frac{1}{4}|V(G)|^2$ for triangle-free graphs G, one can prove the next result, analogously to Theorem 2.1.

Theorem 2.4 Let G be a λ'-connected and triangle-free graph. If G is not λ'-optimal, then

$$r(G) \geq \max\{3, 2\delta(G) - 1\}.$$

Example 2.5 Let $n \geq 3$ be an integer and let $K^i_{n,n-1}$ be a complete bipartite graph with the partite sets A_i and B_i such that $|A_i| = n-1$ and $B_i = \{b^i_1, b^i_2, \ldots, b^i_n\}$ for $i = 1, 2$. Define G as the disjoint union of $K^1_{n,n-1}$ and $K^2_{n,n-1}$ together with the edges $b^1_j b^2_j$ for $j = 1, 2, \ldots, n$. Then $\xi(G) = 2n-2$, $\lambda'(G) = \delta(G) = n$, and $r(G) = 2n-1$. Therefore, G is not λ'-optimal and $r(G) = 2\delta(G) - 1$. This example shows that Theorem 2.4 is best possible.

3. Sufficient conditions for $G_1 \odot G_2$ to be λ'-optimal

If G_1 and G_2 are two disjoint graphs of order n, $G = G_1 \odot G_2$, and $c(G) = \min\{d_{G_1}(u) + d_{G_2}(v) : uv \in E(G) \text{ a cross edge}\}$, then

$$\xi(G) = \min\{\xi(G_1) + 2, \xi(G_2) + 2, c(G)\}. \tag{5}$$

Theorem 3.1 Let G_1 and G_2 be two disjoint λ'-optimal graphs of order $n \geq 4$. If $2 \leq \xi(G_i) \leq n-2$ for $i = 1, 2$, then $G_1 \odot G_2$ is also λ'-optimal.

Proof. Assume that $G = G_1 \odot G_2$ is not λ'-optimal and let X be a λ'-atom of G. Then, $|(X, \bar{X})_G| = \lambda'(G) < \xi(G)$ and, according to Theorem 1.1, we have $r(G) = |X| \geq 3$. Now we investigate different cases.

Case 1. Let $X \subseteq V(G_1)$ or $X \subseteq V(G_2)$, say $X \subseteq V(G_1)$. Since X is a λ'-atom of G, we observe that $G_1[X]$ is connected.

Subcase 1.1. The subgraph $G_1 - X$ contains a component with at least two vertices. Since G_1 is λ'-optimal, Lemma 1.3 and (5) lead to the contradiction

$$\xi(G) > |(X, \bar{X})_G| \geq |(X, \bar{X})_{G_1}| + |X| \geq \lambda'(G_1) + 3 = \xi(G_1) + 3 > \xi(G).$$

Subcase 1.2. The subgraph $G_1 - X$ consists of only isolated vertices or $X = V(G_1)$. Then, in view of the hypothesis $n \geq \xi(G_1) + 2$, we obtain the contradiction

$$\xi(G) > |(X, \bar{X})_G| \geq |(X, \bar{X})_{G_1}| + |X| \geq (n - |X|)\delta(G_1) + |X| \geq n \geq \xi(G).$$

Case 2. Let $X_i = X \cap V(G_i) \neq \emptyset$ for $i = 1, 2$, and, assume without loss of generality, that $|X_1| \geq |X_2| \geq 1$. Consequently, $|X_1| \geq 2$. Since $X = X_1 \cup X_2$ is a λ'-atom of G, we observe that $|X_1| + |X_2| \leq n$ and thus, $|X_2| \leq \frac{n}{2}$.

Subcase 2.1. The subgraphs $G[X_i]$ and $G_i - X_i$ contain a component with at least two vertices for $i = 1, 2$. Then, because of $\xi(G_2) \geq 2$, Lemma 1.3 yields the contradiction

$$\begin{aligned} \xi(G) &> |(X, \bar{X})_G| \geq |(X_1, \bar{X}_1)_{G_1}| + |(X_2, \bar{X}_2)_{G_2}| \\ &\geq \lambda'(G_1) + \lambda'(G_2) = \xi(G_1) + \xi(G_2) \geq \xi(G_1) + 2 \geq \xi(G). \end{aligned}$$

Subcase 2.2. The subgraphs $G[X_1]$ and $G_1 - X_1$ contain a component with at least two vertices. If $G_2 - X_2$ consists of isolated vertices, then, Lemma 1.3 yields the contradiction

$$\begin{aligned}\xi(G) &> |(X,\bar{X})_G| \geq |(X_1,\bar{X}_1)_{G_1}| + |(X_2,\bar{X}_2)_{G_2}| \\ &\geq \lambda'(G_1) + (n - |X_2|)\delta(G_2) \geq \xi(G_1) + 2 \geq \xi(G).\end{aligned}$$

If $G[X_2]$ consists of isolated vertices, then, Lemma 1.3 yields the contradiction

$$\begin{aligned}\xi(G) &> |(X,\bar{X})_G| \geq |(X_1,\bar{X}_1)_{G_1}| + |(X_2,\bar{X}_2)_{G_2}| + |X_1| - |X_2| \\ &\geq \lambda'(G_1) + |X_2|\delta(G_2) + |X_1| - |X_2| \geq \xi(G_1) + 2 \geq \xi(G).\end{aligned}$$

Subcase 2.3. The subgraphs $G[X_2]$ and $G_2 - X_2$ contain a component with at least two vertices. If $G_1 - X_1$ consists of isolated vertices, then, Lemma 1.3 implies the contradiction

$$\begin{aligned}\xi(G) &> |(X,\bar{X})_G| \geq |(X_2,\bar{X}_2)_{G_2}| + |(X_1,\bar{X}_1)_{G_1}| + |X_1| - |X_2| \\ &\geq \lambda'(G_2) + (n - |X_1|)\delta(G_1) + |X_1| - |X_2| \geq \xi(G_2) + 2 \geq \xi(G).\end{aligned}$$

If $G[X_1]$ consists of isolated vertices, then, Lemma 1.3 implies the contradiction

$$\begin{aligned}\xi(G) &> |(X,\bar{X})_G| \geq |(X_2,\bar{X}_2)_{G_2}| + |(X_1,\bar{X}_1)_{G_1}| \\ &\geq \lambda'(G_2) + |X_1|\delta(G_1) \geq \xi(G_2) + 2 \geq \xi(G).\end{aligned}$$

Subcase 2.4. The subgraphs $G[X_1]$ or $G_1 - X_1$ consist of isolated vertices, and the subgraphs $G[X_2]$ or $G_2 - X_2$ consist of isolated vertices.

If $G[X_1]$ contains only isolated vertices, then, with respect to the hypothesis that X is a λ'-atom of G, we conclude that $|X_2| = |X_1|$ and that $G[X_2]$ is connected. Hence, all the vertices of $G_2 - X_2$ are isolated. Consequently, we obtain the contradiction

$$\begin{aligned}\xi(G) &> |(X,\bar{X})_G| \geq |(X_1,\bar{X}_1)_{G_1}| + |(X_2,\bar{X}_2)_{G_2}| \\ &\geq |X_1|\delta(G_1) + (n - |X_2|)\delta(G_2) \geq |X_1| + n - |X_2| = n \geq \xi(G).\end{aligned}$$

If all the vertices of $G_1 - X_1$ and $G_2 - X_2$ are isolated, then

$$\begin{aligned}\xi(G) &> |(X,\bar{X})_G| \geq |(X_1,\bar{X}_1)_{G_1}| + |(X_2,\bar{X}_2)_{G_2}| \\ &\geq (n - |X_1|)\delta(G_1) + (n - |X_2|)\delta(G_2) \\ &\geq 2n - |X_1| - |X_2| \geq n \geq \xi(G),\end{aligned}$$

a contradiction.

Finally, we assume that $G_1 - X_1$ as well as $G[X_2]$ consist of isolated vertices. Then

$$\begin{aligned}\xi(G) &> |(X,\bar{X})_G| \geq |(X_1,\bar{X}_1)_{G_1}| + |(X_2,\bar{X}_2)_{G_2}| + |X_1| - |X_2| \\ &\geq (n-|X_1|)\delta(G_1) + |X_2|\delta(G_2) + |X_1| - |X_2| \\ &\geq n - |X_1| + |X_2| + |X_1| - |X_2| = n \geq \xi(G),\end{aligned}$$

a contradiction. Since we have discussed all possible cases, the proof is complete. □

Example 3.2 Let H be a connected vertex-transitive graph of order $4k-1$ and degree $2k$ for $k \geq 2$. In view of Xu [5] (Theorem 6), H is λ'-optimal with $\lambda'(H) = \xi(H) = 4k-2 = n(H)-1$. If we define G by $H \odot H$, then $\xi(G) = 4k = n(H)+1$, however $\lambda'(G) \leq n(H) < \xi(G)$. Thus, G is not λ'-optimal. These examples show that Theorem 3.1 is best possible.

The *k-cube* Q_k is defined as $Q_0 = K_1$ and $Q_k = Q_{k-1} \odot Q_{k-1}$ for $k \geq 1$.

Corollary 3.3 (Esfahanian [1] 1989) The k-cube Q_k is λ'-optimal for $k \geq 2$.

Proof. Obviously, Q_2 is λ'-optimal. Because of $\xi(Q_p) = 2(p-1) \leq 2^p - 2 = n(Q_p) - 2$, it follows recursively from Theorem 3.1 that the $(p+1)$-cube is λ'-optimal. □

Theorem 3.4 Let G_1 and G_2 be two disjoint λ'-optimal graphs of order $n \geq 4$. If $\xi(G_i) \geq n-1$ for $i = 1,2$, then every λ'-atom of $G = G_1 \odot G_2$ consists of the endpoints of a cross edge (that means that G is λ'-optimal) or of $V(G_1)$ or $V(G_2)$.

Proof. Firstly, we observe that $\delta(G_i) + \Delta(G_i) - 2 \geq \xi(G_i) \geq n-1$ and thus, $\delta(G_i) \geq n - \Delta(G_i) + 1 \geq 2$ for $i = 1,2$.
If we assume that there exists a λ'-atom X of G with $2 < |X| < n$, then by the definition of G, we see that $n \geq |(X,\bar{X})_G|$. Now we distinguish the same cases as in the proof of Theorem 3.1.

Case 1. Let $X \subset V(G_1)$ or $X \subset V(G_2)$, say $X \subset V(G_1)$. Since X is a λ'-atom of G, it follows that $G_1[X]$ is connected.

Subcase 1.1. The subgraph $G_1 - X$ contains a component with at least two vertices. Since G_1 is λ'-optimal, Lemma 1.3 and (5) lead to the contradiction

$$\begin{aligned}n &\geq |(X,\bar{X})_G| \geq |(X,\bar{X})_{G_1}| + |X| \\ &\geq \lambda'(G_1) + 3 = \xi(G_1) + 3 > n.\end{aligned}$$

Subcase 1.2. The subgraph $G_1 - X$ consists of only isolated vertices. Then, because of $\delta(G_1) \geq 2$, we obtain the contradiction

$$n \geq |(X, \bar{X})_G| \geq |(X, \bar{X})_{G_1}| + |X| \geq (n - |X|)\delta(G_1) + |X| > n.$$

Case 2. Let $X_i = X \cap V(G_i) \neq \emptyset$ for $i = 1, 2$, and, assume without loss of generality, that $|X_1| \geq |X_2| \geq 1$. Similarly to Case 1 in this proof and Case 2 in the proof of Theorem 3.1, one can show that this is impossible.

Therefore, every λ'-atom X of G consists of two vertices or of $V(G_1)$ or $V(G_2)$. Analogously to Case 1, one can prove that $|X| = 2$ and $X \subset V(G_1)$ or $X \subset V(G_2)$ as well as $|X| = n$ with $X_i \neq \emptyset$ for $i = 1, 2$ is not possible. Consequently, $X = V(G_1)$, or $X = V(G_2)$, or X consists of two endpoints of a cross edge. □

4. The cartesian product

The *cartesian product* $G = G_1 \times G_2$ of two disjoint graphs G_1 and G_2 consists of the vertices of the cartesian product $V(G_1) \times V(G_2)$, and two vertices (a, u) and (b, v) are adjacent in $G = G_1 \times G_2$, whenever $a = b$ and u adjacent to v in G_2 or $u = v$ and a adjacent to b in G_1. Therefore,

$$d_G((a, u)) = d_{G_1}(a) + d_{G_2}(u),$$

and if $ab \in E(G_1)$ or $uv \in E(G_2)$, then

$$\xi_G((a, u), (b, u)) = \xi_{G_1}(ab) + 2d_{G_2}(u),$$

$$\xi_G((a, u), (a, v)) = \xi_{G_2}(uv) + 2d_{G_1}(a),$$

respectively, and consequently

$$\xi(G) = \min\{\xi(G_1) + 2\delta(G_2), \xi(G_2) + 2\delta(G_1)\}. \tag{6}$$

Theorem 4.1 Let G_1 and G_2 be two disjoint λ'-optimal graphs and $G = G_1 \times G_2$. Then G is λ'-optimal or the λ'-atoms of G have the form $\{y\} \times V(G_2)$ for a vertex $y \in V(G_1)$ with $d_{G_1}(y) = \delta(G_1)$ or $V(G_1) \times \{v\}$ for a vertex $v \in V(G_2)$ with $d_{G_2}(v) = \delta(G_2)$.

Proof. Let $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $\delta(G_1) = \delta_1$, and $\delta(G_2) = \delta_2$. Since G_1 and G_2 are λ'-connected, we note that $n_i \geq 4$ and $\delta_i \geq 1$ for $i = 1, 2$. Assume that G is not λ'-optimal, and let X be a λ'-atom of G. Then, $|(X, \bar{X})_G| = \lambda'(G) < \xi(G)$ and, according to Theorem 1.1, we have $r(G) = |X| \geq 3$. Furthermore, let A be the projection of X on $V(G_1)$, and let B be the projection of X on $V(G_2)$.

In the first two cases we assume that $A \neq V(G_1)$ and $B \neq V(G_2)$.

Case 1. Let $X = \{x\} \times B$ or $X = A \times \{u\}$, say $X = \{x\} \times B$. Since X is a λ'-atom of G, we observe that $G_2[B]$ is connected.

Subcase 1.1. The subgraph $G_2 - B$ contains a component with at least two vertices. Since G_2 is λ'-optimal, Lemma 1.3 and (6) lead to the contradiction

$$\begin{aligned} \xi(G) &> |(X,\bar{X})_G| \geq |(B,\bar{B})_{G_2}| + |X| d_{G_1}(x) \\ &\geq \lambda'(G_2) + |X|\delta_1 \geq \xi(G_2) + 3\delta_1 > \xi(G). \end{aligned}$$

Subcase 1.2. The subgraph $G_2 - B$ contains only isolated vertices. Since $3 \leq |B| = |X| \leq n_2 - 1$, the inequality (2) yields the contradiction

$$\begin{aligned} \xi(G) &> |(X,\bar{X})_G| \geq (n_2 - |B|)\delta_2 + |B|\delta_1 \\ &\geq (n_2 - |B|)\delta_2 + |B| - 2 + 2\delta_1 \\ &= \delta_2 + (n_2 - |B| - 1)\delta_2 + |B| - 2 + 2\delta_1 \\ &\geq \delta_2 + n_2 - 3 + 2\delta_1 \geq \delta_2 + \Delta(G_2) - 2 + 2\delta_1 \\ &\geq \xi(G_2) + 2\delta_1 \geq \xi(G). \end{aligned}$$

Now define $a = |A|$ and $b = |B|$ and let, without loss of generality, $X = A_1 \cup A_2 \cup \ldots \cup A_b$ with $A_k = \{(x_i, u_k) \in X\}$ for $1 \leq k \leq b$ and $X = B_1 \cup B_2 \cup \ldots \cup B_a$ with $B_j = \{(x_j, u_i) \in X\}$ for $1 \leq j \leq a$. Clearly, $A_k \neq \emptyset$ for $1 \leq k \leq b$ and $B_j \neq \emptyset$ for $1 \leq j \leq a$. Since X is connected, there ist at least one edge $(x,u)(x,v)$ and at least one edge $(x,u)(y,u)$ in $G[X]$.

Case 2. Let $2 \leq a \leq n_1 - 1$ and $2 \leq b \leq n_2 - 1$.

Subcase 2.1. There exists an index r such that, without loss of generality, $G_1[A_r]$ and $G_1[\bar{A}_r]$ contain an edge. Applying Corollary 1.4 and (4), we see that $|(B_j, \bar{B}_j)| \geq \delta_2$ for $1 \leq j \leq a$ and thus, Lemma 1.3 implies the contradiction

$$\xi(G) > |(X,\bar{X})_G| \geq \lambda'(G_1) + a\delta_2 \geq \xi(G_1) + 2\delta_2 \geq \xi(G).$$

Subcase 2.2. The subgraphs $G_1[A_k]$ or $G_1[\bar{A}_k]$ consist of isolated vertices for $1 \leq k \leq b$ and the subgraphs $G_2[B_j]$ or $G_2[\bar{B}_j]$ consist of isolated vertices for $1 \leq j \leq a$. In this case, in particular, all vertices of the subgraphs $G_1 - A$ and $G_2 - B$ are isolated. Then, with respect to Corollary 1.4 and (2), we obtain the contradiction

$$\begin{aligned} \xi(G) &> |(X,\bar{X})_G| \geq (n_1 - a)\delta_1 + a\delta_2 \\ &\geq (n_1 - a)\delta_1 + a - 2 + 2\delta_2 = \delta_1 + (n_1 - a - 1)\delta_1 + a - 2 + 2\delta_2 \\ &\geq \delta_1 + n_1 - 3 + 2\delta_2 \geq \delta_1 + \Delta(G_1) - 2 + 2\delta_2 \\ &\geq \xi(G_1) + 2\delta_2 \geq \xi(G). \end{aligned}$$

Consequently, we deduce that $|A| = n_1$ or $|B| = n_2$, say $|A| = n_1$.

Firstly, we investigate the case $|B| \leq n_2 - 1$. According to (4), $\lambda'(G_2) \geq \delta_2$, and hence, on the one hand, Corollary 1.4 implies

$$|(X, \bar{X})_G| \geq n_1 \delta_2.$$

On the other hand, if we choose $Y = V(G_1) \times \{v\}$ with $d_{G_2}(v) = \delta_2$, then $|Y| \leq |X|$ and $|(Y, \bar{Y})_G| = n_1 \delta_2$. Hence, X is only a λ'-atom, if X has the form $X = V(G_1) \times \{u\}$ with $d_{G_2}(u) = \delta_2$.

Finally, we discuss the case $|A| = n_1$ and $|B| = n_2$. We assume, without loss of generality, that $n_2 \geq n_1$. Because of $2|X| \leq n_1 n_2$, we observe that $A_k = V(G_1) \times u_k$ for only $\lfloor \frac{n_2}{2} \rfloor$ such sets and $B_j = x_j \times V(G_2)$ for only $\lfloor \frac{n_1}{2} \rfloor$ such sets. According to (4), $\lambda'(G_i) \geq \delta_i$ for $i = 1, 2$. Therefore, Corollary 1.4 and (2) yield in the case $\delta_1 \geq 2$ the contradiction

$$\begin{aligned}
\xi(G) &> |(X, \bar{X})_G| \geq \frac{n_1}{2}\delta_2 + \frac{n_2}{2}\delta_1 \\
&= \frac{n_1}{2}\delta_2 + (\frac{n_2}{2} - 1)\delta_1 + \delta_1 \geq \frac{n_1}{2}\delta_2 + n_2 - 2 + \delta_1 \\
&\geq \delta_1 + n_1 - 2 + 2\delta_2 \geq \xi(G_1) + 2\delta_2 \geq \xi(G).
\end{aligned}$$

In the case $\delta_1 = 1$ and $\delta_2 \geq 3$, we obtain

$$\begin{aligned}
\xi(G) &> |(X, \bar{X})_G| \geq \frac{n_1}{2}\delta_2 + \frac{n_2}{2}\delta_1 \\
&= (\frac{n_1}{2} - 2)\delta_2 + 2\delta_2 + \frac{n_2}{2} \geq \frac{3n_1}{2} - 6 + 2\delta_2 + \frac{n_2}{2} \\
&= \delta_1 + n_1 - 7 + 2\delta_2 + \frac{n_1}{2} + \frac{n_2}{2} \geq \delta_1 + \Delta(G_1) - 2 + 2\delta_2 \\
&\geq \xi(G_1) + 2\delta_2 \geq \xi(G),
\end{aligned}$$

a contradiction. If $\delta_1 = 1$ and $\delta_2 = 2$, we conclude

$$\begin{aligned}
\xi(G) &> |(X, \bar{X})_G| \geq \frac{n_1}{2}\delta_2 + \frac{n_2}{2}\delta_1 \\
&= (\frac{n_1}{2} - 1)\delta_2 + \delta_2 + \frac{n_2}{2} \geq n_1 - 2 + \delta_2 + 2 \\
&= \delta_1 + n_1 - 3 + 2\delta_2 \geq \delta_1 + \Delta(G_1) - 2 + 2\delta_2 \\
&\geq \xi(G_1) + 2\delta_2 \geq \xi(G),
\end{aligned}$$

a contradiction. In the remaining case $\delta_1 = \delta_2 = 1$, we see that

$$|(X, \bar{X})_G| \geq \frac{n_1}{2}\delta_2 + \frac{n_2}{2}\delta_1 \geq n_1 = n_1 \delta_2.$$

However, if we choose, as above, $Y = V(G_1) \times \{v\}$ with $d_{G_2}(v) = \delta_2$, then $|(Y, \bar{Y})_G| = n_1\delta_2$. Consequently, because of $|Y| < |X|$, the set X is not a λ'-atom. □

The *k-dimensional toroidal mesh* $C(d_1, d_2, \ldots, d_k)$, studied by Ishigami [3], can be represented as the cartesian product $C_{d_1} \times C_{d_2} \times \ldots \times C_{d_k}$, where C_{d_i} are cycles of length d_i for $i = 1, 2, \ldots, k$.

Corollary 4.2 (Xu [5] 2000) Let $C(d_1, d_2, \ldots, d_k)$ be the k-dimensional toroidal mesh. Then, $C(d_1, d_2, \ldots, d_k)$ is λ'-optimal if $d_i \geq 4$ for each $i = 1, 2, \ldots, k$.

Proof. Obviously, the cycle C_{d_1} is λ'-optimal. Now assume that $H_r = C(d_1, d_2, \ldots, d_r)$ is λ'-optimal for $1 \leq r < k$. Then, by Theorem 4.1, $G = C(d_1, d_2, \ldots, d_{r+1}) = H_r \times C_{d_{r+1}}$ is λ'-optimal, or the λ'-atoms X of G have the form $V(H_r) \times \{u\}$ or $\{x\} \times V(C_{d_{r+1}})$. In both of the last two cases, we find that $|(X, \bar{X})_G| \geq 8r$. However, $\lambda'(G) \leq \xi(G) \leq 4(r+1)-2 = 4r+2$, and consequently, the graph G is λ'-optimal. Inductively, we obtain the desired result. □

Remark 4.3 With help of Theorem 4.1 it is possible to give a further short proof of Corollary 3.3.

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