The Automorphism Groups of Certain Tetravalent Metacirculant Graphs

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Abstract. Recently, in connection with the classification problem for non-Cayley tetravalent metacirculant graphs, three families of special tetravalent metacirculant graphs, denoted by Φ_1, Φ_2 and Φ_3, have been defined [11]. It has also been shown [11] that any non-Cayley tetravalent metacirculant graph is isomorphic to a union of disjoint copies of a graph in one of the families Φ_1, Φ_2 or Φ_3. A natural question raised from the result is whether all graphs in these families are non-Cayley. In this paper we determine the automorphism groups of all graphs in the family Φ_2. As a corollary, we show that every graph in Φ_2 is a connected non-Cayley tetravalent metacirculant graph.

1. Introduction

Non-Cayley vertex-transitive graphs have attracted considerable attention in the last few years [1, 4–12]. There are two reasons for this. Firstly, the number of such graphs seems to be small in comparison with the number of Cayley graphs (see statistics for vertex-transitive graphs of small orders in [7]). Secondly, it is conjectured (by C. Thomassen [3] and others [14]) that there are only finitely many connected vertex-transitive non-hamiltonian graphs and all such graphs are non-Cayley. At present, only four nontrivial connected vertex-transitive non-hamiltonian graphs are known to exist. These graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from these by replacing each vertex by a triangle. All these four graphs are non-Cayley.

Metacirculant graphs were introduced in [1] as a logical generalization of the Petersen graph for the primary reason of providing a class of vertex-transitive graphs in which there might be some new non-hamiltonian connected vertex-transitive graphs. This class includes many non-Cayley graphs [1]. So a natural problem is to classify non-Cayley graphs in this class. For cubic non-Cayley metacirculants, this problem has been solved in [10]. Recently, in connection with the classification problem for non-Cayley tetravalent metacirculant graphs, three families of special tetravalent metacirculant graphs, denoted by Φ_1, Φ_2 and Φ_3, have been introduced [11]. It has also been proved [11] that any non-Cayley tetravalent metacirculant graph is isomorphic to a union of disjoint copies of a graph in one of the families Φ_1, Φ_2 or Φ_3. A question raised from this result is whether all graphs in these families Φ_1, Φ_2 and Φ_3 are non-Cayley.

In this paper, we determine the automorphism groups of all graphs in the family Φ_2. As a corollary, we show that every graph in Φ_2 is a connected non-Cayley tetravalent metacirculant graph. So the results obtained in this paper are significant for the classification problem for non-Cayley

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tetravalent metacirculant graphs. Besides this, they are interesting themselves by providing an infinite family of connected tetravalent non-Cayley vertex-transitive graphs whose automorphism groups have a rather simple structure.

2. Preliminaries

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. Unless otherwise indicated, our graph-theoretic terminology will follow [2], and our group-theoretic terminology will follow [13]. If G is a graph, then $V(G)$, $E(G)$ and $\mathrm{Aut}(G)$ will denote its vertex-set, its edge-set, and its full automorphism group, respectively. A graph G is said to be *vertex-transitive* if the action of $\mathrm{Aut}(G)$ on $V(G)$ is transitive.

For a group Γ and a subset $S \subset \Gamma$ such that $1_\Gamma \notin S$ and $S^{-1} = S$, the *Cayley graph* on Γ relative to S, $\mathrm{Cay}(\Gamma, S)$, is defined as follows. The vertex-set of $\mathrm{Cay}(\Gamma, S)$ is Γ, and two vertices $f, g \in \Gamma$ are adjacent in $\mathrm{Cay}(\Gamma, S)$ if and only if $fg^{-1} \in S$. It is known that a graph G is isomorphic to a Cayley graph if and only if $\mathrm{Aut}(G)$ contains a regular subgroup. In particular, Cayley graphs are vertex-transitive. However, there are vertex-transitive graphs which are not Cayley graphs, the smallest example being the well-known Petersen graph. Such a graph is called a *non-Cayley* vertex-transitive graph.

A *circulant graph* is a Cayley graph on a cyclic group. It is also usually defined in the following additive form. Let n be a positive integer. We will write Z_n for the ring of integers modulo n and Z_n^* for the multiplicative group of units in Z_n. Further, let S be a subset of Z_n satisfying $0 \notin S = -S$. Then we define the circulant graph $C = C(n, S)$ to be the graph with vertex-set $V(G) = \{v_j : j \in Z_n\}$ and edge-set $E(G) = \{v_j v_h : j, h \in Z_n \text{ and } (h - j) \in S\}$, where subscripts are always reduced modulo n. The subset S of a circulant graph $C(n, S)$ is called its symbol.

Let m and n be two positive integers, $\alpha \in Z_n^*$, $\mu = \lfloor m/2 \rfloor$ and $S_0, S_1, \dots, S_\mu$ be subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$,

(2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$,

(3) if m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the metacirculant graph $G = MC(m, n, \alpha, S_0, \dots, S_\mu)$ to be the graph with vertex-set

$$V(G) = \{v_j^i \ : \ i \in Z_m,\ j \in Z_n\},$$

and edge-set

$$E(G) = \{v_j^i v_h^{i+r} \ : \ 0 \leq r \leq \mu;\ i \in Z_m;\ h, j \in Z_n \text{ and } (h - j) \in \alpha^i S_r\},$$

where superscripts and subscripts are always reduced modulo m and modulo n, respectively. It is easy to verify that the permutations ρ and τ on $V(G)$ with $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$ are automorphisms of G and that

$< \rho, \tau >$ is a transitive subgroup of Aut(G). Thus, metacirculant graphs are vertex-transitive.

The definitions of the three families Φ_1, Φ_2 and Φ_3 of special tetravalent metacirculant graphs were given in [11]. The role of these families for the classification problem for non-Cayley tetravalent metacirculant graphs were explained in the introduction. For the purpose of this paper we recall here only the definition of the family Φ_2.

The family Φ_2 is defined to consist of all tetravalent metacirculant graphs $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$, where $m, n, \alpha, S_0, S_1, \dots, S_\mu$ satisfy the following conditions:

(Φ_{21}) $m > 2$ and $m = rs$ with $r \geq 2$ even and s odd,

(Φ_{22}) n is a positive integer not divisible by 4,

(Φ_{23}) $\alpha^r \equiv -1 \pmod{n}$ but $\alpha^t \not\equiv -1 \pmod{n}$ for any $1 \leq t \leq r-1$,

(Φ_{24}) $S_0 = \{1, -1\}$,

(Φ_{25}) $S_1 = \{0\}$ and $S_2 = \cdots = S_\mu = \emptyset$.

In order to obtain the results of this paper we shall, apart from the group-theoretic techniques, also be using an algorithm for solving special systems of two congruences. We shall show this algorithm in the next section .

Convention: Unless otherwise indicated, congruences in this paper are of modulo n. So for brevity instead of $P(\alpha) \equiv 0 \pmod{n}$ we simply write $P(\alpha) \equiv 0$. Also, when expressing numbers modulo n we always mean that these numbers are $0, 1, \dots, n-1$.

3. Special Systems of Two Congruences

Let

$$P(\alpha) = a_t\alpha^t + a_{t-1}\alpha^{t-1} + \cdots + a_1\alpha + a_0, \text{ and} \tag{3.1}$$

$$Q(\alpha) = b_l\alpha^l + b_{l-1}\alpha^{l-1} + \cdots + b_1\alpha + b_0, \tag{3.2}$$

where $a_t \neq 0, a_0 \neq 0, b_l \neq 0$ and $b_0 \neq 0$ be two polynomials with integer coefficients $a_0, \dots, a_t, b_0, \dots, b_l$. We consider the following problem: find all solutions of the system

$$\begin{cases} P(\alpha) \equiv 0 \\ Q(\alpha) \equiv 0, \end{cases} \tag{3.3}$$

where n and α are unknowns satisfying the following conditions:

(A) n is a positive integer such that it is not divisible by 4 but $\varphi(n)$ is divisible by 4, where φ is the Euler φ-function;

(B) $\alpha \in Z_n^*$ and has the property that there exists an even integer $r \geq 2$ such that $\alpha^r \equiv -1$ but $\alpha^t \not\equiv -1$ for any $1 \leq t \leq r-1$.

We note that systems of congruences considered here differ from usual systems of congruences by the fact that n, the modulo number, is unknown in our systems. Moreover, n and α must satisfy the two additional conditions (A) and (B). We do not attempt to investigate such a system in detail or to give here its general solution. Instead, we give below an algorithm for solving it.

Remark 1. Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ be a graph of the family Φ_2. Then by Condition Φ_{22} n is a positive integer not divisible by 4. By Φ_{23} α must be of order $2r$ in Z_n^*. It follows that $4 \mid |Z_n^*| = \varphi(n)$ because $r \geq 2$ is even by Φ_{21}. Together with Condition Φ_{23} this implies that n and α satisfy Conditions (A) and (B) of System (3.3).

Remark 2. Let n and α satisfy Conditions (A) and (B) of System (3.3). Since r is even, $\alpha^r + 1 = t_1(\alpha + 1) + 2$ and $\alpha^r + 1 = t_2(\alpha - 1) + 2$ for some appropriate integers t_1 and t_2. It follows that $\gcd(\alpha \pm 1, n) \leq 2$ because $\alpha^r + 1 \equiv 0$. It is clear that if $P(\alpha)$ is a polynomial with integer coefficients, then for all odd values of α either all the respective values of $P(\alpha)$ are odd or all the respective values of $P(\alpha)$ are even. Therefore, since n is not divisible by 4, it is not difficult to see that the following statements are true:

(i) If $P(\alpha) = (\alpha \pm 1)\overline{P}(\alpha)$ and for an odd value of α the value of $\overline{P}(\alpha)$ is even, then $P(\alpha) \equiv 0$ if and only if $\overline{P}(\alpha) \equiv 0$;

(ii) If $P(\alpha) = (\alpha \pm 1)\overline{P}(\alpha)$ and for an odd value of α the value of $\overline{P}(\alpha)$ is odd, then $P(\alpha) \equiv 0$ if and only if $2\overline{P}(\alpha) \equiv 0$;

Similarly, since n is not divisible by 4, $\gcd(2^t, n) \leq 2$. Therefore, the following statements are also true:

(iii) If $P(\alpha) = 2^t\overline{P}(\alpha)$ with $t \geq 1$ and for an odd value of α the value of $\overline{P}(\alpha)$ is even, then $P(\alpha) \equiv 0$ if and only if $\overline{P}(\alpha) \equiv 0$;

(iv) If $P(\alpha) = 2^t\overline{P}(\alpha)$ with $t \geq 1$ and for an odd value of α the value of $\overline{P}(\alpha)$ is odd, then $P(\alpha) \equiv 0$ if and only if $2\overline{P}(\alpha) \equiv 0$.

The above statements will be useful in the compiling Tables 5, 6, 7 and 8 of this paper.

An algorithm for solving System (3.3)

Let $P(\alpha)$ and $Q(\alpha)$ be given by (3.1) and (3.2), respectively. Then the value $\max\{t, l\}$ is called the degree of System (3.3). Without loss of generality we may assume that $t \geq l$.

Step 1. Let $h_1 = \operatorname{lcm}(a_t, b_l)/a_t$, $h_2 = \operatorname{lcm}(a_t, b_l)/b_l$, $h_3 = \operatorname{lcm}(a_0, b_0)/a_0$ and $h_4 = \operatorname{lcm}(a_0, b_0)/b_0$. For $t > l$, set $P_1(\alpha) = h_1 P(\alpha) - h_2\alpha^{t-l}Q(\alpha)$ and $Q_1(\alpha) = Q(\alpha)$. For $t = l$, set $P_1(\alpha) = \big(h_1 P(\alpha) - h_2 Q(\alpha)\big)/\alpha^i$ if $h_1 a_0 = h_2 b_0, h_1 a_1 = h_2 b_1, \dots, h_1 a_{i-1} = h_2 b_{i-1}$ but $h_1 a_i \neq h_2 b_i$ and $Q_1(\alpha) = \big(h_3 P(\alpha) - h_4 Q(\alpha)\big)/\alpha^j$ if $h_3 a_1 = h_4 b_1, h_3 a_2 = h_4 b_2, \dots, h_3 a_{j-1} = h_4 b_{j-1}$ but $h_3 a_j \neq h_4 b_j$. The divisions by α^i and α^j here are possible because $\alpha \in Z_n^*$. Form now the system

$$\begin{cases} P_1(\alpha) \equiv 0 \\ Q_1(\alpha) \equiv 0. \end{cases} \tag{3.4}$$

It is clear that the degree of System (3.4) is less than the degree of System (3.3) and any solution of (3.3) is also a solution of (3.4). In general, the converse is not true. Also, it is clear that System (3.4) and the systems

$$\begin{cases} P_1(\alpha) \equiv 0 \\ -Q_1(\alpha) \equiv 0 \end{cases}, \quad \begin{cases} -P_1(\alpha) \equiv 0 \\ Q_1(\alpha) \equiv 0 \end{cases}, \quad \begin{cases} -P_1(\alpha) \equiv 0 \\ -Q_1(\alpha) \equiv 0 \end{cases}, \quad \begin{cases} \overline{P}_1(\alpha) = Q_1(\alpha) \equiv 0 \\ \overline{Q}_1(\alpha) = P_1(\alpha) \equiv 0 \end{cases}$$

have the same set of solutions. Therefore, without loss of generality we always assume that in (3.4) the degree of $P_1(\alpha)$ is not less than the degree of $Q_1(\alpha)$ and the coefficients of the highest powers of α in $P_1(\alpha)$ and $Q_1(\alpha)$ are positive.

Step 2. Repeat Step 1 for System (3.4) and get a new system

$$\begin{cases} P_2(\alpha) \equiv 0 \\ Q_2(\alpha) \equiv 0 \end{cases}$$

with the degree less than the degree of System (3.4) and the set of solutions containing all solutions of System (3.3). Repeat again Step 1 for successively obtained systems until getting either a system of the type

$$\begin{cases} g \equiv 0 \\ Q_i(\alpha) \equiv 0, \end{cases} \tag{3.5}$$

where g is a number and $Q_i(\alpha)$ is a polynomial, or a linear system

$$\begin{cases} c_1\alpha + c_0 \equiv 0 \\ d_1\alpha + d_0 \equiv 0, \end{cases} \tag{3.6}$$

where $0 < c_1 \leq d_1$. Go now to Step 5 if System (3.5) is obtained; otherwise go to Step 3.

Step 3. Let $e_1 = \gcd(c_1, d_1)$. Then there exist integers x_1 and y_1 such that $e_1 = x_1c_1 + y_1d_1$. The integers e_1, x_1 and y_1 can be found, for example, by extended Euclid's algorithm for computing $\gcd(c_1, d_1)$. It follows that

$$x_1(c_1\alpha + c_0) + y_1(d_1\alpha + d_0) = e_1\alpha + e_0 \equiv 0, \tag{3.7}$$

where $e_0 = x_1c_0 + y_1d_0$. Let $f_1\alpha + f_0 = c_1\alpha + c_0$ if $c_1\alpha + c_0 \neq e_1\alpha + e_0$ and $f_1\alpha + f_0 = d_1\alpha + d_0$ if $c_1\alpha + c_0 = e_1\alpha + e_0$. Form the system

$$\begin{cases} e_1\alpha + e_0 \equiv 0 \\ f_1\alpha + f_0 \equiv 0. \end{cases} \tag{3.8}$$

Step 4. Set $g = z_1 e_0 - f_0$ if $f_1 = e_1 z_1$. Then from (3.8) we have $g \equiv 0$. Thus n must be a divisor of g.

Step 5. List all divisors of g, which satisfy Condition (A). Let these divisors be $n_1, n_2, \ldots, n_k$. For each $n = n_i$, by using $Q_i(\alpha) \equiv 0$ if in Step 2 System (3.5) is obtained or (3.7) if in Step 2 System (3.6) is obtained, we get all values $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iw_i}$ satisfying it. The values of n and α from the obtained pairs (n_i, α_{ij}) form possible solutions of System (3.3).

Step 6. Exclude those pairs (n_i, α_{ij}) for which $P(\alpha_{ij}) \not\equiv 0 \pmod{n_i}$ or $Q(\alpha_{ij}) \not\equiv 0 \pmod{n_i}$.

Step 7. Check for each of the remaining pairs (n_i, α_{ij}) if n_i and α_{ij} satisfy Condition (B) and further exclude those pairs for which n_i and α_{ij} do not satisfy it. The values of n and α from the remaining pairs $(n_i,\ \alpha_{ij})$ are solutions of our system (3.3).

In the next section, in the course of the proof of Theorem 1, we will illustrate how this algorithm works to solve such systems of two congruences.

4. The Automorphism Groups of Graphs in Φ_2

The purpose of this section is to prove the following result.

Theorem 1. *Let $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ be a graph in the family Φ_2, i.e., G satisfies Conditions $\Phi_{21} - \Phi_{25}$. Further, let ρ, τ and π be the permutations on $V(G)$ with $\rho(v_j^i) = v_{j+1}^i$, $\tau(v_j^i) = v_{\alpha j}^{i+1}$ and $\pi(v_j^i) = v_j^{-i}$. Then*

(i) If $r = 2$, then $Aut(G) = < \rho, \tau, \pi >$;

(ii) If $r > 2$, then $Aut(G) = < \rho, \tau >$.

Proof. Let $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ be a tetravalent metacirculant graph in the family Φ_2. An edge in G joining two vertices with the same superscripts is called a horizontal edge; otherwise it is called a vertical edge. A walk in G is called horizontal (resp., vertical) if all its edges are horizontal (resp., vertical). Both horizontal and vertical walks are called homogeneous. A maximal homogeneous subwalk of a walk W is called a segment of W.

By the definition of the family Φ_2, it is clear that the subgraphs G_i induced by G on $V^i = \{v_j^i \ : \ j \in Z_n\}$, $i = 0, 1, \ldots, m-1$, are the only horizontal cycles in G. Each G_i has the form $G_i = v_0^i v_{\alpha^i}^i v_{2\alpha^i}^i \cdots v_{(n-1)\alpha^i}^i v_0^i$. The direction on G_i from v_0^i to $v_{\alpha^i}^i$ is called positive. Also, the cycles $F_j = v_j^0 v_j^1 v_j^2 \cdots v_j^{m-1} v_j^0$, $j = 0, 1, \ldots, n-1$, are the only vertical cycles in G. The direction on F_j from v_j^0 to v_j^1 is called positive. For convenience we consider a horizontal cycle G_i as a walk having two segments: the vertical segment $v_0^i v_0^i$ of length 0 and the horizontal segment $v_0^i v_{\alpha^i}^i v_{2\alpha^i}^i \cdots v_{(n-1)\alpha^i}^i v_0^i$ of length n. Similarly, we consider a vertical cycle F_j as a walk having two segments: the vertical segment $v_j^0 v_j^1 v_j^2 \cdots v_j^{m-1} v_j^0$ of length m and the horizontal segment $v_j^0 v_j^0$ of length 0. A homogeneous path with the specified beginning and terminating vertices is said to have a positive length if the direction from its beginning to its terminating is the positive direction of the homogeneous cycle containing it; otherwise it is said to have a negative length.

Let ρ and τ be the permutations on $V(G)$ with $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Then the subgroup $< \rho, \tau >$ is a transitive subgroup of $\text{Aut}(G)$. We say that two cycles C and D in G are equivalent if there exists an element $\gamma \in < \rho, \tau >$ such that $\gamma(C) = D$. It is clear that this relation is an equivalence on the set of cycles in G and so this set is partitioned into equivalent classes. In short, we will call them classes. It is clear that all horizontal cycles form a class and all vertical cycles form another one. For a subfamily Ω of the family Φ_2 some classes exist in a graph G of Ω without any additional conditions and some classes exist in a graph G of Ω only if some additional conditions are satisfied in this graph. In the former case such a class is called Ω-unconditional and in the latter case it is called Ω-conditional. It is usually clear from the context what Ω we are dealing with. So we will simply talk about unconditional or conditional classes without mentioning Ω. Cycles in unconditional (resp., conditional) classes are also called unconditional (resp., conditional) cycles. In most cases, conditions for the existence of certain cycles in G are expressed by congruences of the type $P(\alpha) \equiv 0$, where $P(\alpha)$ is a polynomial in α with integer coefficients. For brevity a cycle of length ℓ is called an ℓ-cycle, a class of ℓ-cycles is called an ℓ-class, and conditions for the existence of an ℓ-cycle are called ℓ-conditions. We say that two ℓ-conditions are equivalent if they are satisfied or are not satisfied in each graph of the family Φ_2 simultaneously. For our convenience we will identify equivalent ℓ-conditions.

Since the subgroup $< \rho, \tau >$ is transitive on $V(G)$, it is not difficult to see that in each class, except the class of horizontal cycles, there are cycles containing v_0^0 as the beginning vertex of a vertical segment of a positive length of these cycles. Such a cycle is called a representative of the class containing it. We also assume that the orientation on a representative of a class is always such that the positive direction on it coincides with the positive direction on the vertical segment containing v_0^0.

For the class of horizontal cycles of length n, the cycle G_0 is assumed to be the representative of the class and the pair $(0,n)$ is called its type. Further, for the class of vertical cycles of length m, the pair $(m,0)$ is called the type of its representative F_0. Now let C be a representative of a class in G which is different from the class of horizontal cycles, as well as from that of vertical cycles. By going from v_0^0 in the positive direction of C, we can talk about a segment following or preceding another one. Also, the vertex of a segment we encounter first is assumed to be its beginning. We represent the cycle C in the form

$$C = C_0 C_1 C_2 \cdots C_{t-1},$$

where C_0 is the vertical segment containing v_0^0, C_{i+1} is the segment following C_i for each $i = 0, 1, \dots, t-2$ and C_{t-1} is the segment preceding C_0 in C. By the definition of segments it is clear that all C_i with i even are vertical and all C_i with i odd are horizontal segments of C. So the number t of segments in C must have the form $2w$ with $w \geq 2$. Let the length (positive or negative) of a segment C_i be l_i. Then the sequence $(l_0, l_1, l_2, \cdots, l_{t-1})$ is called the segment type, or shortly type, of C. It is clear that $\sum_{i \text{ even}} l_i \equiv 0 \pmod m$, and the type of a representative of a class determines this representative uniquely.

Let P be a homogeneous path in G with the beginning vertex v_j^i. For any automorphism $\gamma \in \mathrm{Aut}(G)$, we assume that the beginning vertex of the path $\gamma(P)$ is $\gamma(v_j^i)$. If P is horizontal (resp., vertical), then it is clear by the definitions of ρ and τ that both $\rho(P)$ and $\tau(P)$ are horizontal (resp., vertical). Therefore, if P is horizontal (resp., vertical), then for any element $\gamma \in < \rho, \tau >$ the path $\gamma(P)$ is also horizontal (resp., vertical). Consider now the directions of P and $\gamma(P)$. By the definition of ρ it is not difficult to see that for any integer j the directions of P and $\rho^j(P)$ are always the same, i.e., both are positive or both are negative. Also, if P is vertical, then for any integer i the directions of P and $\tau^i(P)$ are the same, too. But if P is horizontal, then for some integers k the directions of P and $\tau^k(P)$ may be opposite, i.e., one is positive and another is negative. This happens if and only if $k = xm + y$ with x odd and $0 \leq y < m$. Therefore, if for a horizontal path P the directions of P and $\tau^k(P)$ are opposite, then for any horizontal path P' the directions of P' and $\tau^k(P')$ are also opposite. Since any element $\gamma \in < \rho, \tau >$ can be represented in the form $\gamma = \tau^i \rho^j$ with appropriate integers i and j, it follows from the above observations that for any element $\gamma \in < \rho, \tau >$ and any vertical path P the directions of P and $\gamma(P)$ are always the same, whereas for any element $\gamma \in < \rho, \tau >$, either P and $\gamma(P)$ have the same directions for any horizontal path P or P and $\gamma(P)$ have the opposite directions for any horizontal path P.

Now let C be a representative of a class $\mathbf{C}$ with type $(l_0, l_1, \cdots, l_{t-1})$. Then by the assertions proved in the preceding paragraph it is not difficult to see that a sequence $(l'_0, l'_1, \cdots, l'_{t-1})$ is the type of a representative of the class $\mathbf{C}$ if and only if one of the following are true:

(a) For any $i = 0, 1, \ldots, t-1, l'_i = l_{k+i}$, where k is an even index such that $l_k > 0$ and subscripts are always reduced modulo t;

(b) For any $i = 0, 1, \ldots, t-1, l'_i = (-1)^i l_{k+i}$, where k is an even index such that $l_k > 0$ and subscripts are always reduced modulo t;

(c) For any $i = 0, 1, \ldots, t-1, l'_i = -l_{k-i}$, where k is an even index such that $l_k < 0$ and subscripts are always reduced modulo t;

(d) For any $i = 0, 1, \ldots, t-1, l'_i = (-1)^{i+1} l_{k-i}$, where k is an even index such that $l_k < 0$ and subscripts are always reduced modulo t.

The above assertions are helpful for determining whether two representatives given by their types are representatives of the same class.

Let $\mathbf{C}$ be a class of ℓ-cycles in G different from the class of horizontal cycles, as well as from that of vertical cycles. Further, let $(l_0, l_1, \cdots, l_{t-1})$

be the type of a representative C of $\mathbf{C}$. To C we associate the polynomial

$$P_C(\alpha) = l_1\alpha^{l_0} + l_3\alpha^{l_0+l_2} + l_5\alpha^{l_0+l_2+l_4} + \cdots + l_{t-1}\alpha^{l_0+l_2+l_4+\cdots+l_{t-2}},$$

where exponents of α are always reduced modulo m. Then it is clear that $P_C(\alpha) \equiv 0$ is satisfied in G. Also, by Assertions (a)-(d) above it is not difficult to see that for any two representatives C and D of $\mathbf{C}$, $P_C(\alpha) \equiv 0$ and $P_D(\alpha) \equiv 0$ are equivalent ℓ-conditions.

Now let $P(\alpha)$ be a polynomial in α with integer coefficients. We say that a class $\mathbf{C}$ in G exists under the condition $P(\alpha) \equiv 0$ if there exists a representative C of $\mathbf{C}$ such that $P_C(\alpha) = P(\alpha)$. Also, the class of horizontal cycles and the class of vertical cycles are assumed to exist under the conditions $n = c$ and $m = c$, respectively, where c is a constant positive integer.

Since the group $< \rho, \tau >$ acts transitively on the set of horizontal (resp., vertical) edges of G, for any two horizontal (resp., vertical) edges e_1 and e_2, the number of cycles in a class $\mathbf{C}$ containing e_1 is equal to the number of cycles in $\mathbf{C}$ containing e_2. Therefore, we can talk about the number of cycles in a class $\mathbf{C}$ containing a given horizontal (resp., vertical) edge without specifying this edge. Denote this number by $h(\mathbf{C})$ (resp., $v(\mathbf{C})$). They can be counted by considering how many different cycles of the form $\tau^i\rho^j(C)$ contain a specified horizontal edge and a specified vertical edge, respectively, where $i \in \{0, \ldots, 2m-1\}, j \in \{0, \ldots, n-1\}$ and C is a fixed representative of the class $\mathbf{C}$. Further, we denote by $h_\ell(G)$ and $v_\ell(G)$ the number of ℓ-cycles in G containing a given horizontal edge and a given vertical edge, respectively.

The following proposition is crucial for the proof of Theorem 1.

Proposition 1. *Let G be a graph in the family Φ_2. Then there exists an integer ℓ with $4 \le \ell \le 12$ such that $h_\ell(G) \ne v_\ell(G)$.*

This proposition will be proved by a series of lemmas.

Lemma 1. *Let C be a cycle in G. Then C has length at least 4.*

Proof. Assume that G has a cycle C of length 3. If all edges in C are horizontal, then C must be the subgraph G_i induced by G on some $V^i = \{v_j^i : j \in Z_n\}$. Since G_i is an n-cycle, $n = 3$. But $\alpha^2 \equiv 1 \pmod 3$ for any $\alpha \in Z_3^*$. We get a contradiction to Φ_{23}. If all edges in C are vertical, then C has the form $C = v_j^0 v_j^1 v_j^2 v_j^0$, and therefore $m = 3$. This contradicts the fact that $m > 2$ is even by Φ_{21}. Since $|S_1| = 1$, the case that C has two vertical edges and one horizontal edge also cannot occur. Thus G has no cycles of length 3. ■

Lemma 2. *If G has a 4-cycle, then $h_4(G) \ne v_4(G)$.*

Proof. Let C be a 4-cycle in G. Suppose that C has both horizontal and vertical edges. By the transitive action of $< \rho, \tau >$, without loss of generality we may assume that C has the form $C = v_0^1 v_0^0 v_1^0 v_1^1$, where v_0^1 and v_1^1 are adjacent in G_1. But G_1 is the circulant $C(n, R_1)$ with $R_1 = \{\alpha, -\alpha\}$. Therefore, $\alpha = 1$ or $\alpha = -1$, contradicting Φ_{23}. Thus either all edges in C are horizontal or they are all vertical. If all edges in C are horizontal, then $C = G_i$ for some $i = 0, 1, \cdots, m-1$. Therefore, $n = 4$, contradicting Φ_{22}. So all edges in C are vertical. It follows that $m = 4$ and $h_4(G) = 0,\ v_4(G) = 1$. ■

In order to prove Proposition 1 for the cases $5 \leq \ell \leq 12$ we need know necessary informations about ℓ-cycles existing in a graph of Φ_2. For this purpose in the respective table we list all possible ℓ-conditions for graphs in some subfamily of Φ_2 (Column II). For each of them we indicate the number of ℓ-classes existing under this ℓ-condition (Column III), the types of representatives of the respective ℓ-classes (Column IV) and the numbers of ℓ-cycles containing a given horizontal edge (Column V) or a given vertical edge (Column VI) in the respective ℓ-classes.

Lemma 3. *If G has a 5-cycle, then $h_5(G) \neq v_5(G)$.*

Proof. Let C be a 5-cycle in G. If all edges in C are vertical, then $m = 5$, contradicting Φ_{11}. If all edges in C are horizontal, then $C = G_i$ for some $i = 0, 1, \ldots, m-1$. Therefore, $n = 5$. There is only one such a cycle containing a given horizontal edge.

Suppose now that C contains both horizontal and vertical edges. We note that the number of vertical edges in C must be even and the number of horizontal edges in C must be at least 2. Therefore, C contains exactly 2 vertical and 3 horizontal edges. In Table 1 we give necessary informations about 5-cycles existing in graphs of Φ_2. From the table and the fact that G has no unconditional 5-cycles it is clear that in all possible combinations of 5-conditions that may occur in G, $h_5(G) > v_5(G)$. ■

No (I)	5-conditions (II)	Number of 5-classes (III)	Type of a representative of 5-class C (IV)	h(C) (V)	v(C) (VI)
1	$n = 5$	1	(0,5)	1	0
2	$2\alpha + 1 \equiv 0$	1	(1,2,–1,1)	3	2
3	$2\alpha - 1 \equiv 0$	1	(1,2,–1,–1)	3	2
4	$\alpha + 2 \equiv 0$	1	(1,1,–1,2)	3	2
5	$\alpha - 2 \equiv 0$	1	(1,1,–1,–2)	3	2

Table 1

Lemma 4. *If G has a 6-cycle, then there exists an integer ℓ with $4 \leq \ell \leq 6$ such that $h_\ell(G) \neq v_\ell(G)$.*

Proof. Let G have a 6-cycle. If G also has an ℓ-cycle with $\ell = 4$ or $\ell = 5$, then $h_\ell(G) \neq v_\ell(G)$ by Lemmas 2 and 3. Therefore we assume that G has no ℓ-cycles with $\ell \leq 5$. Let C be a 6-cycle in G. If all edges in C are horizontal, then $C = G_i$ for some $i = 0, 1, \ldots, m-1$. It follows that $n = 6$. Since $\alpha^2 \equiv 1 \pmod 6$ for any $\alpha \in Z_6$, we get a contradiction to Φ_{23}. Thus

C must contain a vertical edge. In Table 2 we give necessary informations about 6-cycles existing in graphs of Φ_2 which have no ℓ-cycles with $\ell \leq 5$. We note that Conditions 2 and 3 cannot satisfy simultaneously. The similar note is true for Conditions 4 and 5. Also, if G satisfies Condition 6 and one of Conditions 2 or 3 (resp., Conditions 4 or 5), then by using the algorithm suggested in Section 3 we can see that G also satisfies Condition 5 or Condition 4 (resp., Condition 3 or Condition 2). With these notes and the fact that G has no unconditional 6-cycles it is not difficult to check that in all possible combinations of 6-conditions that may occur in G, $h_6(G) \neq v_6(G)$. ■

No (I)	6-conditions (II)	Number of 6-classes (III)	Type of a representative of 6-class $\mathbf{C}$ (IV)	h($\mathbf{C}$) (V)	v($\mathbf{C}$) (VI)
1	$m = 6$	1	(6,0)	0	1
2	$3\alpha + 1 \equiv 0$	1	(1,3,-1,1)	4	2
3	$3\alpha - 1 \equiv 0$	1	(1,3,-1,-1)	4	2
4	$\alpha + 3 \equiv 0$	1	(1,1,-1,3)	4	2
5	$\alpha - 3 \equiv 0$	1	(1,1,-1,-3)	4	2
6	$r = 2$	1	(2,1,-2,1)	2	4

Table 2

For a class $\mathbf{C}$ of a graph G in Φ_2 and an integer $\ell \geq 3$, denote

$$d(\mathbf{C}) = h(\mathbf{C}) - v(\mathbf{C}) \quad \text{and} \quad d_\ell(G) = h_\ell(G) - v_\ell(G).$$

Also, for an integer $\ell \geq 3$ and an ℓ-condition $P(\alpha) \equiv 0$ satisfied in G, denote

$$d_{G,\ell}\big(P(\alpha) \equiv 0\big) = d(\mathbf{C}_1) + d(\mathbf{C}_2) + \cdots + d(\mathbf{C}_k),$$

where $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_k$ are all ℓ-classes in G existing under one of the ℓ-conditions equivalent to $P(\alpha) \equiv 0$. Further, set $d_{G,\ell}\big(P(\alpha) \equiv 0\big) = 0$ if $P(\alpha) \equiv 0$ is not satisfied in G. For any integer ℓ with $5 \leq \ell \leq 12$, ℓ-conditions listed in the respective tables of this paper for graphs in Φ_2 clearly have the property that the sets of ℓ-classes existing under them are pairwise disjoint. This property will be used frequently without mention. Therefore,

$$d_\ell(G) = d(\mathbf{C}_1^\ell) + d(\mathbf{C}_2^\ell) + \cdots + d(\mathbf{C}_u^\ell) + \sum_{P(\alpha)\equiv 0} d_{G,\ell}\big(P(\alpha) \equiv 0\big), \quad (4.1)$$

where $\mathbf{C}_1^\ell, \mathbf{C}_2^\ell, \ldots, \mathbf{C}_u^\ell$ are all unconditional ℓ-classes in G and $P(\alpha) \equiv 0$ runs through all ℓ-conditions listed in the respective tables of this paper for graphs in Φ_2.

Lemma 5. *If G has a 7-cycle, then there exists an integer ℓ with $4 \le \ell \le 7$ such that $h_\ell(G) \ne v_\ell(G)$.*

Proof. Let G have a 7-cycle. If G also has a t-cycle with $t \le 6$, then by Lemmas 2, 3 or 4, for some integer ℓ with $4 \le \ell \le 6$, $h_\ell(G) \ne v_\ell(G)$. Therefore, we may assume from now on that G has no t-cycles with $t \le 6$.

Since Condition Φ_{23} holds, α must be of order $2r$ in Z_n. It follows that $4 \mid |Z_n|$ because $r \ge 2$ is even. Let C be a 7-cycle in G. If all edges in C are horizontal, then $C = G_i$ for some $i = 0, 1, \dots, m-1$. Therefore, $n = 7$. We have $|Z_7| = 6$ which is not divisible by 4. This contradicts the fact that $4 \mid |Z_n|$ which we have proved above. If all edges in C are vertical, then $C = v_j^0 v_j^1 v_j^2 \ \dots \ v_j^6 v_j^0$ for some $j = 0, 1, \dots, n-1$, and therefore $m = 7$, contradicting Φ_{21}. Thus C must contain both horizontal and vertical edges. It is clear that the number of vertical edges in C must be even and the number of horizontal edges in C must be at least 2. So C has 2 or 4 vertical edges. In Table 3 we give necessary informations about 7-cycles existing in graphs of Φ_2 which have no t-cycles with $t \le 6$.

Assume that among Conditions 9–16 in Table 3 G satisfies at least two different conditions. Let $P(\alpha) \equiv 0$ and $Q(\alpha) \equiv 0$ be two of them. Then the values n and α of G form a solution of the system

$$\begin{cases} P(\alpha) \equiv 0 \\ Q(\alpha) \equiv 0\,. \end{cases} \tag{4.2}$$

This is just a system we have considered in Section 3. So by using the algorithm suggested there we can solve System (4.2). The reader is invited to verify that for any two different conditions $P(\alpha) \equiv 0$ and $Q(\alpha) \equiv 0$ which are among Conditions 9–16 in Table 3, System (4.2) has no solutions. This means that G satisfies at most one of these conditions. Also, it is clear that G has no unconditional 7-classes. We distinguish two cases.

Case 1: G does not satisfy any of Conditions 9–16 in Table 3. In this case, $d_{G,7}\big(S(\alpha) \equiv 0\big) \ge 0$ for any condition $S(\alpha) \equiv 0$ in Table 3. Since G has a 7-cycle, G satisfies at least one of Conditions 1–8. Let $R(\alpha) \equiv 0$ be one of them. We have $d_{G,7}\big(R(\alpha) \equiv 0\big) = 3 > 0$. Therefore, by (4.1) $d_7(G) \ge d_{G,7}\big(R(\alpha) \equiv 0\big) > 0$ and the lemma is true.

Case 2: G satisfies a condition $P(\alpha) \equiv 0$ which is one of Conditions 9–16 in Table 3. In this case, $d_{G,7}\big(S(\alpha) \equiv 0\big) \ge 0$ for any condition $S(\alpha) \equiv 0$ in Table 3, other than $P(\alpha) \equiv 0$. If G does not satisfy any of Conditions 1–8, then $d_7(G) = d_{G,7}\big(P(\alpha) \equiv 0\big) < 0$. If G satisfies a condition $R(\alpha) \equiv 0$ among Conditions 1–8, then by (4.1) $d_7(G) \ge d_{G,7}\big(P(\alpha) \equiv 0\big) + d_{G,7}\big(R(\alpha) \equiv 0\big) \ge 1 > 0$ and the lemma follows. ∎

In any graph G of the family Φ_2 there is one Φ_2-unconditional 8-class with a representative $C = v_0^0 v_0^1 v_\alpha^1 v_\alpha^0 v_{\alpha+1}^0 v_{\alpha+1}^1 v_1^1 v_1^0 v_0^0$. In the next lemma we deal with the case where G also has a conditional 8-cycle with the number

No	7-conditions	Number of 7-classes	Type of a representative of 7-class $\mathbf{C}$	h($\mathbf{C}$)	v($\mathbf{C}$)
(I)	(II)	(III)	(IV)	(V)	(VI)
1	$4\alpha + 1 \equiv 0$	1	(1,4,–1,1)	5	2
2	$4\alpha - 1 \equiv 0$	1	(1,4,–1,–1)	5	2
3	$3\alpha + 2 \equiv 0$	1	(1,3,–1,2)	5	2
4	$3\alpha - 2 \equiv 0$	1	(1,3,–1,–2)	5	2
5	$2\alpha + 3 \equiv 0$	1	(1,2,–1,3)	5	2
6	$2\alpha - 3 \equiv 0$	1	(1,2,–1,–3)	5	2
7	$\alpha + 4 \equiv 0$	1	(1,1,–1,4)	5	2
8	$\alpha - 4 \equiv 0$	1	(1,1,–1,–4)	5	2
9	$2\alpha^2 + 1 \equiv 0$	1	(2,2,–2,1)	3	4
10	$2\alpha^2 - 1 \equiv 0$	1	(2,2,–2,–1)	3	4
11	$\alpha^2 + 2 \equiv 0$	1	(2,1,–2,2)	3	4
12	$\alpha^2 - 2 \equiv 0$	1	(2,1,–2,–2)	3	4
13	$\alpha^2 + \alpha + 1 \equiv 0$	1	(2,1,–1,1,–1,1)	6	8
14	$\alpha^2 + \alpha - 1 \equiv 0$	1	(2,1,–1,1,–1,–1)	6	8
15	$\alpha^2 - \alpha + 1 \equiv 0$	1	(2,1,–1,–1,–1,1)	6	8
16	$\alpha^2 - \alpha - 1 \equiv 0$	1	(2,1,–1,–1,–1,–1)	6	8

Table 3

of horizontal edges different than the number of vertical edges.

Lemma 6. *If G has an 8-cycle with the number of horizontal edges different than the number of vertical edges, then there exists an integer ℓ with $4 \le \ell \le 8$ such that $h_\ell(G) \ne v_\ell(G)$.*

Proof. Let G satisfy the hypothesis of the lemma. If G has a t-cycle with $t \le 7$, then by Lemmas 1–5 there exists an integer ℓ with $4 \le \ell \le 7$ such that $h_\ell(G) \ne v_\ell(G)$. Therefore we may assume from now on that G has no t-cycles with $t \le 7$.

Let C be an 8-cycle in G with the number of horizontal edges different than the number of vertical edges. If all edges in C are horizontal, then $C = G_i$ for some $i = 0, 1, \dots, m-1$. So $n = 8$, contradicting Φ_{22}. Thus C must contain a vertical edge. In Table 4 we give necessary informations about 8-cycles with the number of horizontal edges different than the number of vertical edges existing in graphs of Φ_2 which have no t-cycles with $t \le 7$. We note that in each class $\mathbf{C}$ of 8-cycles with the number of horizontal edges equal to the number of vertical edges, $h(\mathbf{C}) = v(\mathbf{C})$. Therefore, it is not difficult to see from Table 4 that in any possible combinations of 8-conditions that may occur in G, $h_8(G) \ne v_8(G)$. ∎

We partition the family Φ_2 into two subfamilies Ψ_1 and Ψ_2. The subfamily Ψ_1 consists of all graphs of Φ_2 which have either a t-cycle with

No (I)	8-conditions (II)	Number of 8-classes (III)	Type of a representative of 8-class $\mathbf{C}$ (IV)	h(C) (V)	v(C) (VI)
1	$m = r = 8$	1	(8,0)	0	1
2	$5\alpha + 1 \equiv 0$	1	(1,5,-1,1)	6	2
3	$5\alpha - 1 \equiv 0$	1	(1,5,-1,-1)	6	2
4	$4\alpha + 2 \equiv 0$	1	(1,4,-1,2)	6	2
5	$4\alpha - 2 \equiv 0$	1	(1,4,-1,-2)	6	2
6	$2\alpha + 4 \equiv 0$	1	(1,2,-1,4)	6	2
7	$2\alpha - 4 \equiv 0$	1	(1,2,-1,-4)	6	2
8	$\alpha + 5 \equiv 0$	1	(1,1,-1,5)	6	2
9	$\alpha - 5 \equiv 0$	1	(1,1,-1,-5)	6	2

Table 4

$t \leq 7$ or an 8-cycle with the number of horizontal edges different than the number of vertical edges. The subfamily Ψ_2 is the complement of Ψ_1 in Φ_2. By Lemmas 1–6, Proposition 1 is proved for graphs in Ψ_1.

Now let G be a graph in Ψ_2. By definition, $h_\ell(G) = v_\ell(G) = 0$ for each $\ell = 3, \dots, 7$ and $h_8(G) = v_8(G)$. Thus to prove Proposition 1 for graphs in Ψ_2 we must show that $h_\ell(G) \neq v_\ell(G)$ for some ℓ with $9 \leq \ell \leq 12$.

We consider the numbers $h_{10}(G)$ and $v_{10}(G)$ for a graph G in the subfamily Ψ_2. We need know all Ψ_2-unconditional and all possible Ψ_2-conditional 10-classes in G. It is not difficult to see that in G there are two Ψ_2-unconditional 10-classes $\mathbf{C}_1^{10}$ and $\mathbf{C}_2^{10}$ with representatives

$$C_1^{10} = v_0^0 v_0^1 v_\alpha^1 v_{2\alpha}^1 v_{2\alpha}^0 v_{2\alpha+1}^0 v_{2\alpha+1}^1 v_{\alpha+1}^1 v_1^1 v_1^0 v_0^0, \quad \text{and}$$
$$C_2^{10} = v_0^0 v_0^1 v_\alpha^1 v_\alpha^0 v_{\alpha+1}^0 v_{\alpha+2}^0 v_{\alpha+2}^1 v_2^1 v_2^0 v_1^0 v_0^0,$$

respectively. Further, $h(\mathbf{C}_1^{10}) = h(\mathbf{C}_2^{10}) = 6$ and $v(\mathbf{C}_1^{10}) = v(\mathbf{C}_2^{10}) = 4$. In Tables 5, 6, and 7 we give necessary informations about 10-cycles existing in graphs of the subfamily Ψ_2. For simplicity we combine, where it is possible, several similar 10-conditions into one combination with sign "$\pm$" before certain terms which we call ($\pm$)-terms, if the numbers of 10-classes as well as the numbers of 10-cycles containing a given horizontal edge and the numbers of 10-cycles containing a given vertical edge in respective similar classes existing under these conditions and their equivalents are respectively the same. So the numbers in Columns III, V and VI for each of the combinations in Tables 5, 6, and 7 are respectively the number of 10-classes existing under any condition in the combination and its equivalents, the number of 10-cycles containing a given horizontal edge and the number of 10-cycles containing a given vertical edge in any respective similar class existing under any condition in the combination and its equivalents. Also, we indicate in Column IV of these tables only the types of representatives of 10-classes existing under the condition with sign "+" chosen for each ($\pm$)-

term in combinations and its equivalents. The reader who wants to know the types of representatives of 10-classes of other conditions in combinations is invited to list them himself.

No (I)	10-conditions (II)	Number of 10-classes (III)	Type of a representative of 10-class $\mathbf{C}$ (IV)	h($\mathbf{C}$) (V)	v($\mathbf{C}$) (VI)
1	m = r = 10	1	(10,0)	0	1
2	$m > 4$ & $r = 4$	1	(4,1,–4,1)	2	8

Table 5

We explain how Tables 5, 6 and 7 are compiled. In Table 5 we list 10-conditions for the existence of 10-cycles with 10 or 8 vertical edges. In Table 6 we list 10-conditions for the existence of at least one class of 10-cycles with 6 vertical edges. The remaining 10-conditions under which only classes of 10-cycles with 4 or 2 vertical edges exist are listed in Table 7. In these tables we list conditions in the increasing order of the least numbers of segments in cycles of 10-classes existing under them. For a fixed least number of segments we list conditions for possible lengths of horizontal and vertical segments in cycles of 10-classes existing under these conditions. By this way we can obtain all possible 10-conditions. Now we must identify equivalent 10-conditions and exclude those 10-conditions which result in a contradiction with the definition of graphs in the subfamily Ψ_2. The statements (i)–(iv) in Remark 2 (Section 3) are very helpful for this. Take Condition $\alpha^3 - 2\alpha + 1 \equiv 0$ (in Combination 5 of Table 6) as an example. Since $\alpha^3 - 2\alpha + 1 = (\alpha - 1)(\alpha^2 + \alpha - 1)$, by Statement (ii) in Remark 2, $\alpha^3 - 2\alpha + 1 \equiv 0$ if and only if $2\alpha^2 + 2\alpha - 2 \equiv 0$. But again by Statement (ii), $2\alpha^2 + 2\alpha - 2 \equiv 0$ if and only if $\alpha^3 + 2\alpha^2 - 1 = (\alpha + 1)(\alpha^2 + \alpha - 1) \equiv 0$. This shows that if one of these conditions is satisfied in a graph G of Φ_2, then both of the others are also satisfied in G, i.e., they are equivalent. By our convention these 10-conditions must be identified. So for the condition $\alpha^3 - 2\alpha + 1 \equiv 0$ there exist in G six 10-classes listed as in Table 6. Take, as another example, the condition $3\alpha^2 + 2\alpha - 1 \equiv 0$. This condition allows the 10-class with the representative of type (2,3,–1,2,–1,–1) to exist. But $3\alpha^2 + 2\alpha - 1 = (\alpha + 1)(3\alpha - 1)$. So by Statement (i), $3\alpha^2 + 2\alpha - 1 \equiv 0$ if and only if $3\alpha - 1 \equiv 0$, i.e., if one of these conditions is satisfied in a graph G, then the other is also satisfied in G. Under the last condition there exists the 6-class with the representative of type (1,3,–1,–1). Such a class cannot exist in graphs of the subfamily Ψ_2. Because of this, the condition $3\alpha^2 + 2\alpha - 1 \equiv 0$ is not included in these tables.

If G has $r = 2$, then G has the 6-class with the representative of type (2,1,–2,1). Further, if G has $m = 4, 6$ or 8, then vertical cycles of G are respectively 4-cycles, 6-cycles and 8-cycles with the number of horizontal edges different than the number of vertical edges. So G is a graph in the

No	10-conditions	No of 10-classes	Type of a representative of 10-class $\mathbf{C}$	h($\mathbf{C}$)	v($\mathbf{C}$)
(I)	(II)	(III)	(IV)	(V)	(VI)
1	$3\alpha^3 \pm 1 \equiv 0$	1	(3,3,–3,1)	4	6
2	$\alpha^3 \pm 3 \equiv 0$	1	(3,1,–3,3)	4	6
3	$2\alpha^3 \pm \alpha \pm 1 \equiv 0$	1	(3,2,–2,1,–1,1)	8	12
4	$\alpha^3 + 2\alpha \pm 1 \equiv 0$	2	(3,1,–2,2,–1,1)	8	12
			(1,1,2,1,–2,1,–1,1)	4	6
5	$\alpha^3 - 2\alpha \pm 1 \equiv 0$	6	Under $\alpha^3 - 2\alpha + 1 \equiv 0$:		
			(3,1,–2,–2,–1,1)	8	12
			(1,–1,2,1,–2,–1,–1,1)	4	6
			Under $\alpha^3 + 2\alpha^2 - 1 \equiv 0$:		
			(3,1,–1,2,–2,–1)	8	12
			(2,1,1,1,–1,1,–2,–1)	4	6
			Under $2\alpha^2 + 2\alpha - 2 \equiv 0$:		
			(2,2,–1,2,–1,–2)	12	8
			(1,1,1,2,–1,1,–1,–2)	6	4
6	$\alpha^3 + \alpha \pm 2 \equiv 0$	3	Under $\alpha^3 + \alpha + 2 \equiv 0$:		
			(3,1,–2,1,–1,2)	8	12
			Under $\alpha^2 - \alpha + 2 \equiv 0$:		
			(2,1,–2,1,1,–1,–1,1)	4	6
			(1,–2,1,1,–1,1,–1,2)	12	8
7	$\alpha^3 - \alpha \pm 2 \equiv 0$	1	(3,1,–2,–1,–1,2)	8	12
8	$\pm 2\alpha^3 + \alpha^2 + 1 \equiv 0$	3	Under $2\alpha^3 + \alpha^2 + 1 \equiv 0$:		
			(3,2,–1,1,–2,1)	8	12
			Under $2\alpha^2 - \alpha + 1 \equiv 0$:		
			(2,1,–1,–1,1,1,–2,1)	4	6
			(1,–2,1,2,–1,1–1,1)	12	8
9	$\pm 2\alpha^3 + \alpha^2 - 1 \equiv 0$	1	(3,2,–1,1,–2,–1)	8	12
10	$\pm \alpha^3 + 2\alpha^2 + 1 \equiv 0$	2	(3,1,–1,2,–2,1)	8	12
			(2,1,1,1,–1,1,–2,1)	4	6
11	$\alpha^3 \pm \alpha^2 \pm 2 \equiv 0$	1	(3,1,–1,1,–2,2)	8	12
12	$\alpha^3 + \alpha^2 \pm (\alpha - 1) \equiv 0$	2	(3,1,–1,1,–1,1,–1,–1)	8	12
			(2,1,1,1,–2,1,–1,–1)	8	12
13	$\alpha^3 - \alpha^2 \pm (\alpha + 1) \equiv 0$	2	(3,1,–1,–1,–1,1,–1,1)	8	12
			(2,–1,1,1,–2,1,–1,1)	8	12

Table 6

No (I)	10-conditions (II)	Number of 10-classes (III)	Type of a representative of 10-class $\mathbf{C}$ (IV)	h($\mathbf{C}$) (V)	v($\mathbf{C}$) (VI)
1	$7\alpha \pm 1 \equiv 0$	1	(1,7,–1,1)	8	2
2	$5\alpha \pm 3 \equiv 0$	1	(1,5,–1,3)	8	2
3	$3\alpha \pm 5 \equiv 0$	1	(1,3,–1,5)	8	2
4	$\alpha \pm 7 \equiv 0$	1	(1,1,–1,7)	8	2
5	$5\alpha^2 \pm 1 \equiv 0$	1	(2,5,–2,1)	6	4
6	$4\alpha^2 \pm 2 \equiv 0$	1	(2,4,–2,2)	6	4
7	$3\alpha^2 + 3 \equiv 0$	1	(2,3,–2,3)	6	4
8	$2\alpha^2 \pm 4 \equiv 0$	1	(2,2,–2,4)	6	4
9	$\alpha^2 \pm 5 \equiv 0$	1	(2,1,–2,5)	6	4
10	$4\alpha^2 \pm \alpha \pm 1 \equiv 0$	1	(2,4,–1,1,–1,1)	12	8
11	$\alpha^2 \pm 4\alpha \pm 1 \equiv 0$	3	(2,1,–1,4,–1,1)	12	8
			(1,3,1,1,–1,1,–1,1)	12	8
			(1,2,1,1,–1,2,–1,1)	6	4
12	$\alpha^2 \pm \alpha \pm 4 \equiv 0$	1	(2,1,–1,1,–1,4)	12	8
13	$3\alpha^2 \pm 2\alpha + 1 \equiv 0$	2	(2,3,–1,2,–1,1)	12	8
			(1,1,1,3,–1,1,–1,1)	6	4
14	$3\alpha^2 \pm \alpha \pm 2 \equiv 0$	1	(2,3,–1,1,–1,2)	12	8
15	$2\alpha^2 \pm 3\alpha - 1 \equiv 0$	2	(2,2,–1,3,–1,–1)	12	8
			(1,2,1,2,–1,1,–1,–1)	12	8
16	$\alpha^2 \pm 3\alpha - 2 \equiv 0$	2	(2,1,–1,3,–1,–2)	12	8
			(1,2,1,1,–1,1,–1,–2)	12	8
17	$2\alpha^2 \pm \alpha \pm 3 \equiv 0$	1	(2,2,–1,1,–1,3)	12	8
18	$\alpha^2 \pm 2\alpha + 3 \equiv 0$	2	(2,1,–1,2,–1,3)	12	8
			(1,1,1,1,–1,1,–1,3)	6	4
19	$3\alpha^2 \pm 1 \equiv 0$	1	(1,1,1,3,–1,–1,–1,1)	12	8
20	$\alpha^2 \pm 3 \equiv 0$	1	(1,1,1,1,–1,–1,–1,3)	12	8
21	$\alpha^2 \pm 2\alpha - 1 \equiv 0$	1	(1,3,1,1,–1,–1,–1,–1)	12	8

Table 7

subfamily Ψ_1 in these cases. Thus if G is a graph in the subfamily Ψ_2, then $m \geq 10$ and $r \geq 4$. The following lemmas 7 and 8 deal with the cases where G is in Ψ_2 and has $m = 10$ or $r = 4$.

Lemma 7. *Let G be a graph in the subfamily Ψ_2 and further suppose that G has $m = 10$. Then $h_{10}(G) \neq v_{10}(G)$.*

Proof. Since G is in Ψ_2 and has $m = 10$, it is clear that $r = 10$, i.e., G satisfies Condition 1 but does not satisfy Condition 2 of Table 5. Therefore, $d_{G,10}(m = r = 10) = -1$ and $d_{G,10}(m > 4 \ \& \ r = 4) = 0$. For any 10-condition $P(\alpha) \equiv 0$ in Tables 6 and 7 we have $d_{G,10}(P(\alpha) \equiv 0)$ is even. Further, $d(\mathbf{C}_1^{10}) = d(\mathbf{C}_2^{10}) = 2$. So by (4.1) $d_{10}(G)$ is odd. This means that $d_{10}(G) \neq 0$, i.e., $h_{10}(G) \neq v_{10}(G)$. ■

Lemma 8. *Let G be a graph in the subfamily Ψ_2 and further suppose that G has $r = 4$. Then $h_{10}(G) \neq v_{10}(G)$.*

Proof. Since $r = 4$ and G is in Ψ_2, G satisfies Condition 2 of Table 5. Therefore $d_{G,10}(m > 4 \ \& \ r = 4) = -6$. Suppose that besides the 10-condition $(m > 4 \ \& \ r = 4)$ in Table 5, G also satisfies a 10-condition $Q(\alpha) \equiv 0$ in Table 6. Then the values of n and α of G form a solution of the system

$$\begin{cases} \alpha^4 + 1 \equiv 0 \\ Q(\alpha) \equiv 0. \end{cases} \tag{4.3}$$

This is just a system we have considered in Section 3. So by using the algorithm suggested there we can solve System (4.3). Take, for example, $Q(\alpha) = \alpha^3 - \alpha^2 - 2 \equiv 0$ (in Combination 11 of Table 6). Then System (4.3) has the form

$$\begin{cases} P(\alpha) = \alpha^4 + 1 \equiv 0 \\ Q(\alpha) = \alpha^3 - \alpha^2 - 2 \equiv 0. \end{cases} \tag{4.4}$$

By using the algorithm suggested in Section 3, we successively obtain the following systems

$$\begin{cases} P_1(\alpha) = \alpha^3 + 2\alpha + 1 \equiv 0 \\ Q_1(\alpha) = \alpha^3 - \alpha^2 - 2 \equiv 0\,, \end{cases} \qquad \begin{cases} P_2(\alpha) = \alpha^2 + 2\alpha + 3 \equiv 0 \\ Q_2(\alpha) = 3\alpha^2 - \alpha + 4 \equiv 0\,, \end{cases}$$

$$\begin{cases} c_1\alpha + c_0 = 5\alpha - 11 \equiv 0 \\ d_1\alpha + d_0 = 7\alpha + 5 \equiv 0\,, \end{cases} \qquad \begin{cases} e_1\alpha + e_0 = \alpha - 43 \equiv 0 \\ f_1\alpha + f_0 = 5\alpha - 11 \equiv 0\,. \end{cases}$$

Therefore, $g = 5e_0 - f_0 = -204 \equiv 0$. Step 5 of the algorithm gives us the pairs (102,43), (51,43), (34,9) and (17,9), the values of n and α of which form possible solutions of System (4.4). Continuing the algorithm we find that only $n = 34$, $\alpha = 9$ and $n = 17$, $\alpha = 9$ are solutions of (4.4). But it is easily seen that both of these solutions satisfy the congruence $4\alpha - 2 \equiv 0$, which is an 8-condition in Table 4. It follows that graphs G with these values of n and α must be in Ψ_1, a contradiction. The reader is invited to verify that for any other condition $Q(\alpha) \equiv 0$ in Table 6, all solutions of (4.3) also satisfy some conditions in Tables 1–4, i.e., graphs with values of n and α a solution of System (4.3) must be in Ψ_1. This contradicts the hypothesis that G is in Ψ_2. Thus G does not satisfy any 10-conditions in Table 6. Further, since $r = 4$, it is clear that G also does not satisfy the condition $m = r = 10$ in Table 5. We distinguish several cases.

Case 1: G does not satisfy any 10-conditions in Table 7. In this case, $d_{10}(G) = d(\mathbf{C}_1^{10}) + d(\mathbf{C}_2^{10}) + d_{G,10}(m > 4 \;\&\; r = 4) = 2 + 2 - 6 = -2$. This means that $h_{10}(G) \neq v_{10}(G)$ in this case.

Case 2: G satisfies a 10-condition $R(\alpha) \equiv 0$ in one of Combinations 1–4 or 10–21 of Table 7. In this case, $d_{G,10}(R(\alpha) \equiv 0) \geq 4$. Therefore, $d(\mathbf{C}_1^{10}) + d(\mathbf{C}_2^{10}) + d_{G,10}(m > 4 \;\&\; r = 4) + d_{G,10}(R(\alpha) \equiv 0) \geq 2+2-6+4 = 2 > 0$. Since $d_{G,10}(S(\alpha) \equiv 0) \geq 0$ for any 10-condition $S(\alpha) \equiv 0$ in Table 7, by (4.1) we get in total $d_{10}(G) > 0$. Again, $h_{10}(G) \neq v_{10}(G)$ in this case.

Case 3: G satisfies a 10-condition $R(\alpha) \equiv 0$ in one of Combinations 5–9 of Table 7. Then the values n and α of G form a solution of the system

$$\begin{cases} \alpha^4 + 1 \equiv 0 \\ R(\alpha) \equiv 0 . \end{cases}$$

Using the algorithm suggested in Section 3 we can solve this system and find out that it has no solutions. Thus this case cannot occur.

The proof of Lemma 8 is complete. ■

By Lemmas 7 and 8 we may assume from now on that G is a graph in Ψ_2 and does not satisfy any 10-conditions in Table 5.

Lemma 9. *Let G be a graph in the subfamily Ψ_2 and further suppose that G does not satisfy any 10-conditions in Table 5 but satisfies at least two 10-conditions of Table 6. Then $h_{10}(G) \neq v_{10}(G)$.*

Proof. Let $P(\alpha) \equiv 0$ and $Q(\alpha) \equiv 0$ be two different 10-conditions of Table 6 satisfied in G. Then the values of n and α of the graph G form a solution of the system

$$\begin{cases} P(\alpha) \equiv 0 \\ Q(\alpha) \equiv 0 . \end{cases} \tag{4.5}$$

This is just a system we have considered in Section 3. So by using the algorithm suggested there we can solve System (4.5). By the hypotheses of the lemma, G is a graph in the subfamily Ψ_2 not satisfying any 10-conditions in Table 5. So a solution n and α of System (4.5) is required to have the property that it does not satisfy any conditions in Tables 1– 5. Because of this, we add to the algorithm in Section 3 the following step:

Step 8. Exclude those pairs (n_i, α_{ij}) which satisfy any condition in Tables 1–5. The values of n and α from the remaining pairs (n_i, α_{ij}) are solutions of System (4.5).

Remark 3. Suppose that we are using the algorithm suggested in Section 3 to solve (4.5). If in an intermediate obtained system

$$\begin{cases} P_i(\alpha) \equiv 0 \\ Q_i(\alpha) \equiv 0 , \end{cases}$$

either $P_i(\alpha) \equiv 0$ or $Q_i(\alpha) \equiv 0$ is a condition in Tables 1–5, then we can stop the algorithm and conclude that System (4.5) has no solutions.

Let (n_i, α_{ij}) be a solution of System (4.5). By substituting these values n_i and α_{ij} into each of the 10-conditions of Tables 6 and 7 we can see which 10-conditions are satisfied in a graph G with these values of n and α. Therefore, by using Table 6 or Table 7 we can compute $d_{G,10}(P(\alpha) \equiv 0)$ where $P(\alpha) \equiv 0$ is a 10-condition in Table 6 or Table 7. By (4.1) this implies that we also can compute $d_{10}(G)$. The reader is invited to verify that for any two different 10-conditions $P(\alpha) \equiv 0$ and $Q(\alpha) \equiv 0$ of Table 6 and for any graph G with n and α a solution of (4.5), $d_{10}(G) \neq 0$. Lemma 9 is therefore proved. ■

Thus, by Lemma 9 Proposition 1 is proved for a graph G in Ψ_2 not satisfying any 10-conditions of Table 5 but satisfying at least two 10-conditions of Table 6. Therefore, in the next five lemmas we can restrict to consider only graphs of the subfamily Ψ_2 which do not satisfy any 10-conditions of Table 5 and satisfy at most one 10-condition of Table 6.

Lemma 10. *Let G be a graph in the subfamily Ψ_2 and further suppose that G does not satisfy any 10-conditions in Tables 5 and 6. Then $h_{10}(G) \neq v_{10}(G)$.*

Proof. By the hypotheses of the lemma, among 10-conditions G possibly satisfies only 10-conditions in Table 7. We have $a = d(\mathrm{C}_1^{10}) + d(\mathrm{C}_2^{10}) = 4 > 0$. For any 10-condition $P(\alpha) \equiv 0$ in Table 7, $d_{G,10}(P(\alpha) \equiv 0) \geq 0$. It follows by (4.1) that in total $d_{10}(G) \geq a > 0$ in any possible cases. ■

Lemma 11. *Let G be a graph in the subfamily Ψ_2 and further suppose that G does not satisfy any 10-conditions in Table 5 and among 10-conditions in Table 6 G satisfies one and only one 10-condition which is a condition in Combinations 1, 2, 6 or 8. Then $h_{10}(G) \neq v_{10}(G)$.*

Proof. Let $P(\alpha) \equiv 0$ be a condition in Combinations 1, 2, 6 or 8, which is the only 10-condition in Table 6 satisfied in G. Then $d_{G,10}(P(\alpha) \equiv 0) = -2$. We have $b = d(\mathrm{C}_1^{10}) + d(\mathrm{C}_2^{10}) + d_{G,10}(P(\alpha) \equiv 0) = 2 > 0$. For any 10-condition $Q(\alpha) \equiv 0$ in Table 7, we have $d_{G,10}(Q(\alpha) \equiv 0) \geq 0$. It follows by (4.1) that in total $d_{10}(G) \geq b > 0$ in any possible cases. ■

Lemma 12. *Let G be a graph in the subfamily Ψ_2 and further suppose that G does not satisfy any 10-conditions in Table 5 and among 10-conditions in Table 6 G satisfies one and only one 10-condition which is a condition in Combinations 4, 5, 10, 12 or 13. Then $h_{10}(G) \neq v_{10}(G)$.*

Proof. (I) Assume first that G satisfies the hypotheses of the lemma and $P(\alpha) \equiv 0$ is one of the 10-conditions in Combinations 12 or 13, which is the only 10-condition in Table 6 satisfied in G. Then $d_{G,10}(P(\alpha) \equiv 0) = -8$ and $d_{G,10}(R(\alpha) \equiv 0) = 0$ for any other 10-condition $R(\alpha) \equiv 0$ of Table 6. We consider separately several cases.

Case 1: G satisfies a 10-condition $Q(\alpha) \equiv 0$ in one of Combinations 1–4, 11, 13, 15, 16 or 18 of Table 7. Then $d_{G,10}(Q(\alpha) \equiv 0) \geq 6$. We have $c = d(\mathrm{C}_1^{10}) + d(\mathrm{C}_2^{10}) + d_{G,10}(P(\alpha) \equiv 0) + d_{G,10}(Q(\alpha) \equiv 0) \geq 2+2-8+6 = 2 > 0$. Since $d_{G,10}(S(\alpha) \equiv 0) \geq 0$ for any 10-condition $S(\alpha) \equiv 0$ in Table

7, by (4.1) we have in total $d_{10}(G) \geq c > 0$ and the lemma is true in this case.

Case 2: G satisfies a 10-condition $Q(\alpha) \equiv 0$ in one of Combinations 5–10, 12, 14, 17 or 19–21 of Table 7. Then the values of n and α of G form a solution of the system

$$\begin{cases} P(\alpha) \equiv 0 \\ Q(\alpha) \equiv 0 . \end{cases} \tag{4.6}$$

Since G does not satisfy any 10-conditions in Tables 1–5, we can add Step 8 in the proof of Lemma 9 to the algorithm suggested in Section 3 in order to solve System (4.6). The reader is invited to verify that for any 10-conditions $P(\alpha) \equiv 0$ and $Q(\alpha) \equiv 0$ satisfying our assumptions and for any graph G with n and α a solution of (4.6), $h_{10}(G) \neq v_{10}(G)$, i.e., the lemma is also true in this case.

Case 3: G does not satisfy any 10-conditions in Table 7. In this case, we have $d_{10}(G) = d(\mathbf{C}_1^{10}) + d(\mathbf{C}_2^{10}) + d_{G,10}\big(P(\alpha) \equiv 0\big) = 2+2-8 = -4 \neq 0$ and the lemma is again true.

(II) Assume now that G satisfies the hypotheses of the lemma and $P(\alpha) \equiv 0$ is a 10-condition in one of Combinations 4, 5 or 10, which is the only 10-condition in Table 6 satisfied in G. Then $d_{G,10}\big(P(\alpha) \equiv 0\big) = -6$ and $d_{G,10}\big(R(\alpha) \equiv 0\big) = 0$ for any other 10-condition $R(\alpha) \equiv 0$ of Table 6. By arguments similar to those we have used in Part (I) we can show here that $h_{10}(G) \neq v_{10}(G)$. We leave it to the reader to do in detail. ■

Lemma 13. *Let G be a graph satisfying the following conditions:*

(i) G is in the subfamily Ψ_2;

(ii) G does not satisfy any 10-conditions in Table 5;

(iii) Among 10-conditions in Table 6 G satisfies one and only one 10-condition which is a condition in Combinations 3, 7, 9 or 11;

(iv) G satisfies a 10-condition in Table 7.

Then $h_{10}(G) \neq v_{10}(G)$.

Proof. Let $P(\alpha) \equiv 0$ and $Q(\alpha) \equiv 0$ be the 10-conditions which are satisfied in G and are respectively in Table 6 and Table 7. Then $d_{G,10}\big(P(\alpha) \equiv 0\big) = -4$ and $d_{G,10}\big(Q(\alpha) \equiv 0\big) \geq 2$. We have $d = d(\mathbf{C}_1^{10}) + d(\mathbf{C}_2^{10}) + d_{G,10}\big(P(\alpha) \equiv 0\big) + d_{G,10}\big(Q(\alpha) \equiv 0\big) \geq 2 + 2 - 4 + 2 > 0$. Since $d_{G,10}\big(S(\alpha) \equiv 0\big) \geq 0$ for any 10-condition $S(\alpha) \equiv 0$ in Table 7, by (4.1) we have in total $d_{10}(G) \geq d > 0$. ■

If G is a graph satisfying Conditions (i)–(iii) but not satisfying Condition (iv) of Lemma 13, then it is not difficult to see that $h_{10}(G) = v_{10}(G)$. Nevertheless, the following lemma shows that for such a graph $h_{12}(G) \neq v_{12}(G)$.

Lemma 14. *Let G be a graph satisfying the following conditions:*

(i) G is in the subfamily Ψ_2;

(ii) G does not satisfy any 10-conditions in Table 5;

(iii) Among 10-conditions in Table 6 G satisfies one and only one 10-condition which is a condition in Combinations 3, 7, 9 or 11;

(iv) G does not satisfy any 10-condition in Table 7.

Then $h_{12}(G) \neq v_{12}(G)$.

Proof. Let G be a graph satisfying the hypotheses of the lemma and $P(\alpha) \equiv 0$ be the only 10-condition of Table 6 satisfied in G which is a condition in Combinations 3, 7, 9 or 11. Consider 12-classes in this graph. It is not difficult to see that in G there are four Ψ_2-unconditional 12-classes $\mathbf{C}_1^{12}, \mathbf{C}_2^{12}, \mathbf{C}_3^{12}$ and $\mathbf{C}_4^{12}$ which have the representatives with types (1,1,–1,3,1,–1,–1,–3), (1,2,–1,2,1,–2,–1,–2), (1,3,–1,1,1,–3,–1,–1) and (2,1,–2,1,2,–1,–2,–1), respectively. We have

$$h(\mathbf{C}_1^{12}) = h(\mathbf{C}_2^{12}) = h(\mathbf{C}_3^{12}) = 8,$$
$$v(\mathbf{C}_1^{12}) = v(\mathbf{C}_2^{12}) = v(\mathbf{C}_3^{12}) = 4,$$
$$h(\mathbf{C}_4^{12}) = 4 \text{ and } v(\mathbf{C}_4^{12}) = 8.$$

We note that the number of vertical edges in 12-cycles in G must be even, and therefore it is equal to 12, 10, 8, 6, 4, 2 or 0. We partition the set of 12-classes in G into two collections $\mathbf{P}_G$ and $\mathbf{N}_G$. The collection $\mathbf{P}_G$ consists of all 12-classes $\mathbf{C}$ in G with $d(\mathbf{C}) = h(\mathbf{C}) - v(\mathbf{C}) \geq 0$. The collection $\mathbf{N}_G$ consists of all 12-classes $\mathbf{C}$ in G with $d(\mathbf{C}) < 0$. Further, let

$$d(\mathbf{P}_G) = \sum_{\mathbf{C} \in \mathbf{P}_G} d(\mathbf{C}), \text{ and } d(\mathbf{N}_G) = \sum_{\mathbf{C} \in \mathbf{N}_G} d(\mathbf{C}).$$

Then it is clear that $d_{12}(G) = d(\mathbf{P}_G) + d(\mathbf{N}_G)$. It is not difficult to see that cycles in 12-classes of the collection $\mathbf{P}_G$ have the number of vertical edges not greater than 6, whereas cycles in 12-classes of the collection $\mathbf{N}_G$ have the number of vertical edges greater than 6. It is clear that if G has 12-cycles with 12 vertical edges, then $m = 12$ and all these 12-cycles form one 12-class denoted by $\mathbf{C}_5^{12}$. We have $h(\mathbf{C}_5^{12}) = 0$ and $v(\mathbf{C}_5^{12}) = 1$. Therefore, $\mathbf{C}_5^{12} \in \mathbf{N}_G$ if it exists in G. Since G satisfies the hypotheses of the lemma, it has no 12-cycles with 10 vertical edges. We also have $\mathbf{C}_4^{12} \in \mathbf{N}_G$. Thus, if $m = 12$, then the collection $\mathbf{N}_G$ consists of $\mathbf{C}_4^{12}$, $\mathbf{C}_5^{12}$ and conditional classes of 12-cycles with 8 vertical edges; otherwise, it consists of the class $\mathbf{C}_4^{12}$ and conditional classes of 12-cycles with 8 vertical edges. We distinguish two cases.

Case 1: G has no conditional 12-cycles with 8 vertical edges. If $m = 12$, then $d(\mathbf{N}_G) = d(\mathbf{C}_4^{12}) + d(\mathbf{C}_5^{12}) = -5$. If $m \neq 12$, then $d(\mathbf{N}_G) = d(\mathbf{C}_4^{12}) = -4$. On the other hand, since $\mathbf{C}_1^{12}$, $\mathbf{C}_2^{12}$ and $\mathbf{C}_3^{12}$ are contained in $\mathbf{P}_G$, $d(\mathbf{P}_G) \geq d(\mathbf{C}_1^{12}) + d(\mathbf{C}_2^{12}) + d(\mathbf{C}_3^{12}) = 12$. It follows that $d_{12}(G) = d(\mathbf{P}_G) + d(\mathbf{N}_G) > 0$ and the lemma is true in this case.

Case 2: G has a conditional 12-cycle with 8 vertical edges. For graphs satisfying the hypotheses of Lemma 14, in Table 8 we give necessary informations about 12-cycles existing in them. Here we also adopt conventions similar to those for Tables 5, 6 and 7.

No (I)	12-conditions (II)	Number of 12-classes (III)	Type of a representative of 12-class $\mathbf{C}$ (IV)	h($\mathbf{C}$) (V)	v($\mathbf{C}$) (VI)
1	$3\alpha^4 \pm 1 \equiv 0$	1	(4,3,–4,1)	4	8
2	$\alpha^4 \pm 3 \equiv 0$	1	(4,1,–4,3)	4	8
3	$2\alpha^4 \pm \alpha^3 \pm 1 \equiv 0$	1	(4,2,–1,1,–3,1)	8	16
4	$2\alpha^4 + \alpha^2 \pm 1 \equiv 0$	1	(4,2,–2,1,–2,1)	8	16
5	$2\alpha^4 - \alpha^2 + 1 \equiv 0$	1	(4,2,–2,–1,–2,1)	8	16
6	$2\alpha^4 \pm \alpha \pm 1 \equiv 0$	1	(4,2,–3,1,–1,1)	8	16
7	$\alpha^4 \pm 2\alpha^3 - 1 \equiv 0$	2	(4,1,–1,2,–3,–1)	8	16
			(3,1,1,1,–1,1,–3,–1)	4	8
8	$\alpha^4 + 2\alpha^2 \pm 1 \equiv 0$	2	(4,1,–2,2,–2,1)	8	16
			(2,1,2,1,–2,1,–2,1)	4	8
9	$\alpha^4 - 2\alpha^2 - 1 \equiv 0$	2	(4,1,–2,–2,–2,–1)	8	16
			(2,–1,2,1,–2,–1,–2,–1)	4	8
10	$\alpha^4 + 2\alpha \pm 1 \equiv 0$	2	(4,1,–3,2,–1,1)	8	16
			(1,1,3,1,–3,1,–1,1)	4	8
11	$\alpha^4 - 2\alpha - 1 \equiv 0$	2	(4,1,–3,–2,–1,–1)	8	16
			(1,–1,3,1,–3,–1,–1,–1)	4	8
12	$\alpha^4 \pm \alpha^3 \pm 2 \equiv 0$	1	(4,1,–1,1,–3,2)	8	16
13	$\alpha^4 + \alpha^2 + 2 \equiv 0$	1	(4,1,–2,1,–2,2)	8	16
14	$\alpha^4 - \alpha^2 \pm 2 \equiv 0$	1	(4,1,–2,–1,–2,2)	8	16
15	$\alpha^4 \pm \alpha \pm 2 \equiv 0$	1	(4,1,–3,1,–1,2)	8	16
16	$\alpha^4 \pm \alpha^3 \pm \alpha^2 \pm 1 \equiv 0$	2	(4,1,–1,1,–1,1,–2,1)	8	16
			(3,1;1,1,–2,1,–2,1)	8	16
17	$\alpha^4 + \alpha^3 \pm (\alpha - 1) \equiv 0$	2	(4,1,–1,1,–2,1,–1,–1)	8	16
			(3,1,1,1,–3,1,–1,–1)	8	16
18	$\alpha^4 - \alpha^3 \pm (\alpha + 1) \equiv 0$	2	(4,1,–1,–1,–1,1,–1,1)	8	16
			(3,–1,1,1,–3,1,–1,1)	8	16
19	$\alpha^4 \pm \alpha^2 \pm \alpha \pm 1 \equiv 0$	2	(4,1,–2,1,–1,1,–1,1)	8	16
			(2,1,2,1,–3,1,–1,1)	8	16
20	$\pm 2\alpha^3 + \alpha^2 - 1 \equiv 0$	2	(3,1,–1,1,1,1,–3,–1)	4	8
			(2,2,1,2,–1,–1,–2,–1)	12	12
21	$\alpha^3 - \alpha \pm 2 \equiv 0$	2	(3,1,–3,1,1,–1,–1,1)	4	8
			(1,–2,2,1,–2,1,–1,2)	12	12

Each of the 12-conditions in Combinations 20 and 21 is satisfied in G if and only if it is the only 10-condition satisfied in G.

Table 8

Since in this case G has a conditional 12-cycle with 8 vertical edges, G satisfies a condition in Table 8, say $Q(\alpha) \equiv 0$. Therefore, the values n and α of G form a solution of the system

$$\begin{cases} P(\alpha) \equiv 0 \\ Q(\alpha) \equiv 0. \end{cases} \tag{4.7}$$

By using the algorithm suggested in Section 3 we can solve System (4.7). But a graph G we consider here must satisfy the hypotheses of the lemma. So a solution n and α of System (4.7) is required to have the property that it does not satisfy any conditions in Tables 1–7, other than $P(\alpha) \equiv 0$. Because of this, we add to the algorithm in Section 3 the following step:

Step $8^{\#}$. Exclude those pairs (n_i, α_{ij}) which satisfy any condition in Tables 1– 7, other than $P(\alpha) \equiv 0$. The values of n and α from the remaining pairs (n_i, α_{ij}) are solutions of System (4.7).

Remark 4. Suppose that we are using the algorithm suggested in Section 3 to solve System (4.7). If in an intermediate obtained system

$$\begin{cases} P_i(\alpha) \equiv 0 \\ Q_i(\alpha) \equiv 0, \end{cases}$$

either $P_i(\alpha) \equiv 0$ or $Q_i(\alpha) \equiv 0$ is a condition in Tables 1–7, other than $P(\alpha) \equiv 0$, then we can stop the algorithm and conclude that System (4.7) has no solutions.

Let (n_i, α_{ij}) be a solution of System (4.7). By substituting these values n_i and α_{ij} into each 12-condition $R(\alpha) \equiv 0$ of Table 8 we can see if it is satisfied in a graph G with these values of n and α. It follows that we can compute $d_{G,12}(R(\alpha) \equiv 0)$, and therefore $d(\mathbf{N}_G)$ for this graph because the sets of 12-classes existing under 12-conditions in Table 8 are pairwise disjoint. On the other hand, for any conditions $P(\alpha) \equiv 0$ and $Q(\alpha) \equiv 0$ satisfying our assumption and for any graph G with n and α a solution of (4.7) we can check which 12-conditions of either the type $a\alpha^2 + b\alpha + c \equiv 0$ with $|a| + |b| + |c| = 8$ or the type $f\alpha + g \equiv 0$ with $|f| + |g| = 10$ are satisfied in G. Since 12-classes in G satisfying 12-conditions of these types are in $\mathbf{P}_G$, we can estimate $d(\mathbf{P}_G)$ and find out that in any cases $d(\mathbf{P}_G) > -d(\mathbf{N}_G)$, i.e., $d_{12}(G) > 0$. So the lemma is again true in this case.

This completes the proof of Lemma 14. ■

By Lemmas 1–14 Proposition 1 is proved. ■

Now we complete the proof of Theorem 1. Let $V^i = \{v^i_j : j \in Z_n\}$, G_i be the subgraph induced by G on V^i, $i = 0, \dots, m-1$, and φ be an automorphism of G. By Proposition 1, $h_\ell(G) \neq v_\ell(G)$ for some integer ℓ with $4 \leq \ell \leq 12$. Therefore, φ cannot map horizontal edges to vertical edges. It follows that for any $i \in \{0, 1, \dots, m-1\}$ there exists some $j \in \{0, 1, \dots, m-1\}$ such that $\varphi(V^i) = V^j$ because $G_0, G_1, \dots, G_{m-1}$ are n-cycles on $V^0, V^1, \dots, V^{m-1}$, respectively. Therefore, $\{V^0, V^1, \dots, V^{m-1}\}$ is a complete block system for $\Gamma = \mathrm{Aut}(G)$.

Let ρ, τ and π be the permutations on $V(G)$ with $\rho(v_j^i) = v_{j+1}^i, \tau(v_j^i) = v_{\alpha j}^{i+1}$ and $\pi(v_j^i) = v_j^{-i}$. It is not difficult to see that ρ and τ are automorphisms of G. Further, the subgraph G_i is the circulant graph $C(n, R_i)$ with $R_i = \{\alpha^i, -\alpha^i\}$. Therefore, π is an automorphism of G if and only if $\alpha^i \equiv \pm\alpha^{m-i}$ for each $i \in Z_m$, i.e., if and only if $\alpha^{2i} \equiv \pm\alpha^m \equiv \mp 1$ for each $i \in Z_m$. By Condition Φ_{23} it is clear that the last congruences are possible if and only if $\alpha^2 \equiv -1$, i.e., if and only if G has $r = 2$.

Let Γ_0 be the subgroup of Γ consisting of all automorphisms of G which leave V^0 invariant. For each $\Delta \subseteq \Gamma_0$ denote by $\Delta \,|\, V^0$ the restriction of Δ on V^0. Then

$$\Gamma_0 \,|\, V^0 \leq \mathrm{Aut}(G_0). \tag{4.8}$$

On the other hand, it is clear that ρ and τ^m are elements of Γ_0 and $\mathrm{Aut}(G_0) = <\rho, \tau^m> \,|\, V^0$ because G_0 is an n-cycle. So

$$\mathrm{Aut}(G_0) = <\rho, \tau^m> \,|\, V^0 \leq \Gamma_0 \,|\, V^0. \tag{4.9}$$

From (4.8) and (4.9) it follows that

$$\Gamma_0 \,|\, V^0 = <\rho, \tau^m> \,|\, V^0. \tag{4.10}$$

Now let β be an arbitrary element of Γ_0. By (4.10) there exists an element $\gamma \in <\rho, \tau^m>$ such that $\beta \,|\, V^0 = \gamma \,|\, V^0$. Hence $\gamma^{-1}\beta \,|\, V^0$ is the identity permutation on V^0, i.e., $\gamma^{-1}\beta$ fixes each vertex of V^0. We have shown before that $\{V^0, V^1, \ldots, V^{m-1}\}$ is a complete block system for $\Gamma = \mathrm{Aut}(G)$ and each element of Γ must map a vertical edge to a vertical edge. Further, $v_j^0 v_j^1$ and $v_j^0 v_j^{-1}$ are the only vertical edges incident with v_j^0. Therefore, either

(i) $\gamma^{-1}\beta(v_j^1) = v_j^1$ for each $j \in Z_n$ or

(ii) $\gamma^{-1}\beta(v_j^1) = v_j^{-1}$ for each $j \in Z_n$.

If Case (i) happens, then by reasoning similar to those we have done above it is not difficult to see that $\gamma^{-1}\beta(v_j^i) = v_j^i$ for each $v_j^i \in V(G)$. So $\beta = \gamma \in <\rho, \tau^m>$. If Case (ii) happens, then $\gamma^{-1}\beta(v_j^i) = v_j^{-i}$ for each $v_j^i \in V(G)$, i.e., $\gamma^{-1}\beta = \pi$. So π is also an automorphism of G and $\beta = \gamma\pi \in <\rho, \tau^m, \pi>$.

Assume first that G has $r = 2$. Then π is an automorphism of G. By the assertions proved in the preceding paragraph it is not difficult to see that $\Gamma_0 = <\rho, \tau^m, \pi>$ in this case. Now let δ be an arbitrary element of $\Gamma = \mathrm{Aut}(G)$ and $\delta(V^0) = V^i$. Then $\tau^{-i}\delta(V^0) = V^0$ because $\tau^i(V^0) = V^i$. So $\tau^{-i}\delta \in \Gamma_0 = <\rho, \tau^m, \pi>$. This implies that $\delta \in <\rho, \tau, \pi>$, whence $\Gamma = <\rho, \tau, \pi>$.

Assume next that G has $r > 2$. Then $\pi \notin \Gamma = \mathrm{Aut}(G)$ because we have shown earlier that π is an automorphism of G if and only if G has $r = 2$. Therefore, Case (ii) cannot happen for any $\beta \in \Gamma_0$. It follows that

$\Gamma_0 = <\rho, \tau^m>$. Again let δ be an arbitrary element of Γ and $\delta(V^0) = V^i$. Then $\tau^{-i}\delta(V^0) = V^0$ because $\tau^i(V^0) = V^i$. So $\tau^{-i}\delta \in \Gamma_0 = <\rho, \tau^m>$. This implies that $\delta \in <\rho, \tau>$. Therefore $\Gamma = <\rho, \tau>$.

The proof of Theorem 1 is complete. ■

5. Non-Cayleyness of Graphs in Φ_2

As a corollary of Theorem 1, we now prove the following result.

Theorem 2. *Every graph in the family Φ_2 is a connected non-Cayley tetravalent metacirculant graph.*

Proof. Let $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ be a graph in the family Φ_2. Further, let G have $r = 2$. Then by Theorem 1, $\Gamma = \mathrm{Aut}(G) = <\rho, \tau, \pi>$, where $\rho(v_j^i) = v_{j+1}^i$, $\tau(v_j^i) = v_{\alpha j}^{i+1}$ and $\pi(v_j^i) = v_j^{-i}$. It is easily checked that the following defining relations are satisfied in Γ:

$$\rho^n = \tau^{2m} = \pi^2 = 1\,,\ \tau\rho\tau^{-1} = \rho^\alpha\,,\ \pi\rho = \rho\pi \ \text{ and } \ \pi\tau\pi = \tau^{m-1}\,. \qquad (5.1)$$

Let R be a regular subgroup of Γ. Since $V^i = \{v_j^i : j \in Z_n\}$, $i = 0, 1, \ldots, m-1$, are blocks of Γ, V^i, $i = 0, 1, \ldots, m-1$, are also blocks of R. Since R is regular, for each $j = 0, 1, \ldots, n-1$, there exists exactly one $r_j \in R$ such that $r_j(v_0^0) = v_j^0$. Let $\overline{R} = <r_j : j = 0, 1, \ldots, n-1>$. Then $|\overline{R}| = n$. On the other hand, $\overline{R}$ is contained in the setwise stabilizer Γ_0 of V^0 in Γ. It is not difficult to see by (5.1) that $\Gamma_0 = <\rho, \tau^m, \pi>$ which has order $4n$. Hence $\overline{R}$ is a subgroup of index 4 in Γ_0. There are two cases to consider.

(i) n is odd.

In this case, $<\rho>$ is the unique maximal normal subgoup of odd order n of Γ_0. So $\overline{R} = <\rho>$. Since R is transitive on $V(G)$, there exists an element $\gamma \in R$ mapping V^0 to V^1. By (5.1) it is clear that γ can be represented in one of the following forms:

$$\gamma = \tau\rho^t\,,\ \gamma = \pi\tau^{-1}\rho^t\,,\ \gamma = \tau^{m+1}\rho^t \ \text{ or } \ \gamma = \pi\tau^{m-1}\rho^t. \qquad (5.2)$$

Since $<\rho> = \overline{R} \leq R$, the element $\beta = \gamma\rho^{-t} \in R$. Therefore, $\beta^m \in R$. On the other hand, by (5.2) β has one of the following forms: $\beta = \tau$, $\beta = \pi\tau^{-1}$, $\beta = \tau^{m+1}$ or $\beta = \pi\tau^{m-1}$. Using (5.1) it is not difficult to check that $\beta^m = \tau^m$ in either case. This implies that $\tau^m \in R$ which is impossible because $\tau^m \neq 1$ and it fixes v_0^0. Thus Γ has no regular subgroup if n is odd.

(ii) $n = 2\ell$ with ℓ odd.

Let K be a subgroup of index 4 in Γ_0. Then $|K| = n = 2\ell$. This means that the order of K is not divisible by 4. Since $<\rho>$ is a normal subgroup of Γ_0, we can form the subgroup $K<\rho>$. We have $(\Gamma_0/<\rho>) \geq (K<\rho>)/<\rho> \cong K/(K\cap <\rho>)$ and $|\Gamma_0/<\rho>| = 4$. Since the order of K is not divisible by 4, this implies that $|K/(K\cap<\rho>)| \leq 2$.

Hence, either $|K \cap <\rho>| = 2\ell$ or $|K \cap <\rho>| = \ell$. Therefore, either $K = <\rho>$ or $K \cap <\rho> = <\rho^2>$. It follows that K is one of the following groups:

$$\begin{aligned} K_1 &= <\rho>, \\ K_2 &= \{\rho^i : i \text{ is even}\} \cup \{\tau^m \rho^j : j \text{ is even}\}, \\ K_3 &= \{\rho^i : i \text{ is even}\} \cup \{\tau^m \rho^j : j \text{ is odd}\}, \\ K_4 &= <\pi> \times <\rho^2> \\ &= \{\rho^i : i \text{ is even}\} \cup \{\pi \rho^j : j \text{ is even}\}, \\ K_5 &= <\pi\rho> \\ &= \{\rho^i : i \text{ is even}\} \cup \{\pi \rho^j : j \text{ is odd}\}, \\ K_6 &= \{\rho^i : i \text{ is even}\} \cup \{\pi\tau^m \rho^j : j \text{ is even}\}, \\ K_7 &= \{\rho^i : i \text{ is even}\} \cup \{\pi\tau^m \rho^j : j \text{ is odd}\}. \end{aligned}$$

If $\overline{R} = K_1$, then we can get a contradiction as in (i). It is clear that K_2, K_4 and K_6 are intransitive on V^0, whereas $\overline{R}$ is transitive on it. So $\overline{R} \neq K_2, K_4, K_6$.

Assume now that $\overline{R} = K_3$. Since R is transitive on $V(G)$, there exists an element $\gamma \in R$ mapping V^0 to V^1. By (5.1) it is clear that γ can be represented in one of the forms: $\gamma = \tau\rho^t$, $\gamma = \tau^{m+1}\rho^t$, $\gamma = \pi\tau^{-1}\rho^t$ or $\gamma = \pi\tau^{m-1}\rho^t$. If $\gamma = \tau\rho^t$ with t even, then since ρ^t with t even is in K_3 we have $\gamma\rho^{-t} = \tau \in R$. So $\tau^m \in R$. Since $\tau^m \neq 1$ and τ^m fixes v_0^0, this contradicts the regularity of R. If $\gamma = \tau\rho^t$ with t odd, then since $\tau^m\rho^t$ with t odd is in K_3 we have $\gamma\tau^m\rho^t = (\tau\rho^t)(\tau^m\rho^t) = (\tau\tau^m)(\tau^m\rho^t\tau^m)\rho^t = \tau^{m+1} \in R$. So $(\tau^{m+1})^m = \tau^m \in R$, a contradiction again. Similarly we can get a contradiction if $\gamma = \tau^{m+1}\rho^t$, $\gamma = \pi\tau^{-1}\rho^t$ or $\gamma = \pi\tau^{m-1}\rho^t$. Thus $\overline{R} \neq K_3$. By similar arguments we can show that $\overline{R} \neq K_5$ and K_7. It follows that Γ also has no regular subgroups R if n is even.

Thus the graph G is non-Cayley if $r = 2$. The proof of the fact that a graph G in Φ_2 with $r > 2$ is non-Cayley is similar. The reader is invited to do this in detail. Also, it is trivial that G is a connected tetravalent graph. This completes the proof of Theorem 2. ■

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