Four traps are almost always enough to catch a square-celled animal

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Abstract. In a recent paper [7] Maynard answered a question of Harary and Manvel [3] about the reconstruction of *square-celled animals*. One of his results relied on a general algebraic approach due to Alon, Caro, Krasikov and Roditty [1]. Applying arguments of a more combinatorial nature we improve this result and give an answer to a question raised by him in [7].

Keywords. Reconstruction; Square-celled animal; Deck

1 Introduction

A *square-celled animal* is a finite set of rookwise connected squares which form a simply connected region in the plane. A *sub-animal* arises from a square-celled animal by deleting any one square and two animals will be called *isomorphic* if one can be transformed into the other by translation and/or rotation by a multiple of 90°.

In [4], [6] and [13] the problem of counting square-celled animals was mentioned and studied and in [3] Harary and Manvel considered the problem of reconstructing a square-celled animal from the multiset of all its sub-animals given up to isomorphism. For some examples see the following figure.

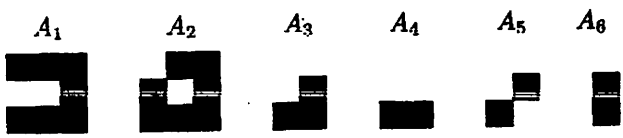

Figure 1.

In Figure 1, A_1, A_3, A_4 and A_6 are square-celled animals, A_2 is not simply connected and A_5 is not rookwise connected. A_4, A_5 and A_6 are the three sub-animals of A_3, A_5 is a sub-animal but not a square-celled animal and A_4 and A_6 are isomorphic.

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The main result of [3] was the following.

Theorem 1 (Harary and Manvel [3]) *Every square-celled animal is uniquely determined (up to isomorphism) by the multiset of all its sub-animals (given up to isomorphism).*

Harary and Manvel [3] asked if Theorem 1 is still true for animals that are not simply connected. Using a general algebraic approach due to Alon, Caro, Krasikov and Roditty [1], Maynard [7] was able to give a nine-lines proof of an affirmative answer to this question.

A set A of m squares of the form $[x, x+1] \times [y, y+1]$ for $x, y \in \mathbf{Z} = \{0, -1, 1, -2, 2, ...\}$ in the plane was called an *m-fig* in [7]. Again, isomorphism is defined with respect to translation and/or rotation by a multiple of 90°.

For some integer k with $1 \leq k < m = |A|$ the *k-deck* of A was defined as the multiset of all subsets of A of cardinality $m-k$ given up to isomorphism. Maynard proved the following theorem which easily implies an affirmative answer to the question in [3].

Theorem 2 (Maynard [7]) *Let m and k be integers. If $m-k \geq 7$, then all m-figs are uniquely determined (up to isomorphism) by their k-decks.*

Maynard remarks that it is not clear whether the bound of 7 in Theorem 2 is best-possible. Using arguments of a more combinatorial nature we will show that it can be improved to 5. In fact, our proof shows that 4 seems to be the right bound.

We will use a different way to formulate the problem. Instead of square-celled animals or m-figs, we will consider finite sets of (distinct) points in the plane $\mathbf{R}^2$. This clearly allows a more general setting for the problem. In order to turn a square-celled animal or an m-fig into a set of points one can e.g. just consider the set of centers of the squares.

Let $A, B \subseteq \mathbf{R}^2$ be two finite sets. We say that A and B are *isomorphic* and write $A \cong B$, if one can be transformed into the other by translation and/or rotation by a multiple of 90°. For some $k \geq 1$ we define the k-deck of A as a function $d_{A,k}$ defined on sets S of k points in $\mathbf{R}^2$ as

$$d_{A,k}(S) = |\{S' \subseteq A \mid S' \cong S\}|.$$

(Note that $m-k$ is replaced by k.) Using this notation, Theorem 2 can be restated as follows.

Theorem 2' (Maynard [7]) *Every finite set $A \subseteq \mathbf{Z}^2$ with $|A| \geq 7$ is uniquely determined (up to isomorphism) by its 7-deck.*

Our main result whose proof will be postponed to the next section is the following.

Theorem 3 *Every finite set $A \subseteq \mathbf{R}^2$ with $|A| \geq 5$ is uniquely determined (up to isomorphism) by its 5-deck.*

Our proof and intuition suggest that the 4-deck should actually be sufficient for reconstruction (only the very last part of the proof uses the 5-deck). It is quite easy to see that the 2-deck can not carry enough information to determine every set uniquely. Hence the best-possible bound for Theorem 3 is at least 3.

The reader who is interested in the graph-theoretic origins of the reconstruction problems is referred to Bondy's excellent survey [2]. It is obvious that similar reconstruction problems can be considered for subsets of $\mathbf{R}^n$ using various notions of isomorphism. Such problems were studied e.g. in [1], [8], [9], [10], [11] and [12].

2 Proof of the Theorem 3

Let $A \subseteq \mathbf{R}^2$ be any finite set. Consider the group G of automorphisms of the plane $\mathbf{R}^2$ that is generated by translations and rotations by multiples of 90°. For some k with $2 \leq k \leq |A|$ let S be a set of representatives of the orbits defined by the action of G on sets $S \subseteq \mathbf{R}^2$ of cardinality k.

For all sets $\tilde{S} \subseteq \mathbf{R}^2$ with $|\tilde{S}| = k - 1$ we have

$$d_{A,k-1}(\tilde{S}) = \frac{1}{|A| - |\tilde{S}|} \sum_{S \in \mathcal{S}, d_{A,k}(S)>0} d_{A,k}(S) \cdot d_{S,k-1}(\tilde{S}),$$

i.e. the k-deck of A uniquely determines the $(k-1)$-deck of A. We note that this observation corresponds to *Kelly's lemma* for finite graphs [5] (see also [2]). During the proof we may therefore use the k-deck of A for all $k < 5$.

Whenever we say that *'A is uniquely determined'* we implicitly mean *'A is uniquely determined up to isomorphism'*. If it is suitable for our exposition, then we will replace A by a set isomorphic to A.

We choose $(\tilde{x}, \tilde{y}) \in \mathbf{R}^2$ such that

(i) $d_{A,2}(\{(0,0),(\tilde{x},\tilde{y})\}) > 0$,

(ii) given condition (i), $\tilde{x}$ is maximum and

(iii) given condition (ii), $\tilde{y}$ is maximum,

i.e. $(\tilde{x}, \tilde{y})$ is the maximal element of $\{(x,y) - (x',y') | (x,y),(x',y') \in A\}$ under lexicographic order.

Considering the smallest *'rectangle'* of the form $[x_1, x_2] \times [y_1, y_2]$ that contains A, it is easy to see that $d_{A,2}(\{(0,0),(\tilde{x},\tilde{y})\}) \leq 2$.

Case 1. $d_{A,2}(\{(0,0),(\tilde{x},\tilde{y})\}) = 1$.

We assume without loss of generality that $(0,0),(\tilde{x},\tilde{y}) \in A$. For all $(x,y) \in \mathbf{R}^2 \setminus \{(0,0),(\tilde{x},\tilde{y})\}$ we have that (see left part of Figure 2)

$$|A \cap \{(x,y),(\tilde{x}-x,\tilde{y}-y)\}| = d_{A,3}(\{(x,y),(0,0),(\tilde{x},\tilde{y})\}).$$

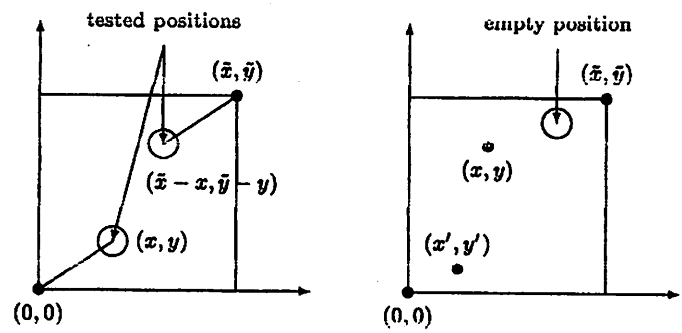

Figure 2.

If

$$|A \cap \{(x,y),(\tilde{x}-x,\tilde{y}-y)\}| \in \{0,2\}$$

for all $(x,y) \in \mathbf{R}^2 \setminus \{(0,0),(\tilde{x},\tilde{y})\}$ with $(x,y) \neq (\tilde{x}-x,\tilde{y}-y)$, then A is uniquely determined. Hence we assume that

$$|A \cap \{(x',y'),(\tilde{x}-x',\tilde{y}-y')\}| = 1$$

for some $(x',y') \in \mathbf{R}^2 \setminus \{(0,0),(\tilde{x},\tilde{y})\}$ with $(x,y) \neq (\tilde{x}-x,\tilde{y}-y)$. We assume without loss of generality that $(x',y') \in A$. Now we have that for all $(x,y) \in \mathbf{R}^2 \setminus \{(0,0),(x',y'),(\tilde{x},\tilde{y})\}$ (see right part of Figure 2),

$$|A \cap \{(x,y)\}| = d_{A,4}(\{(x,y),(0,0),(x',y'),(\tilde{x},\tilde{y})\})$$

and A is uniquely determined.

Case 2. $d_{A,2}(\{(0,0),(\tilde{x},\tilde{y})\}) = 2$.

It is easy to see that the smallest rectangle of the form $[x_1,x_2] \times [y_1,y_2]$ that contains A is a square in this case. We may therefore assume without loss of generality that $A \subseteq [0,\tilde{x}] \times [0,\tilde{x}]$ and that $A \cap (\{0\} \times \mathbf{R}) \neq \emptyset$, $A \cap (\{\tilde{x}\} \times \mathbf{R}) \neq \emptyset$, $A \cap (\mathbf{R} \times \{0\}) \neq \emptyset$ and $A \cap (\mathbf{R} \times \{\tilde{x}\}) \neq \emptyset$. Let

$$y_{\text{left}} = \max\{y \geq 0 \,|\, d_{A,3}(\{(0,y),(x,0),(\tilde{x},\tilde{y}+y)\}) > 0 \text{ for some } x \in \mathbf{R}\}$$

and let

$$x_{\text{bottom}} = \max\{x \geq 0 \,|\, d_{A,3}(\{(0, y_{\text{left}}), (x, 0), (\tilde{x}, \tilde{y} + y_{\text{left}})\}) > 0\}.$$

Let $y_{\text{right}} = \tilde{y} + y_{\text{left}}$ and $x_{\text{top}} = x_{\text{bottom}} - \tilde{y}$. We may assume without loss of generality that (see Figure 3)

$$(0, y_{\text{left}}), (\tilde{x}, y_{\text{right}}), (x_{\text{top}}, \tilde{x}), (x_{\text{bottom}}, 0) \in A$$

and that $(0, y) \notin A$ for all $y < y_{\text{left}}$, $(\tilde{x}, y) \notin A$ for all $y > y_{\text{right}}$, $(x, \tilde{x}) \notin A$ for all $x < x_{\text{top}}$ and $(x, 0) \notin A$ for all $x > x_{\text{bottom}}$.

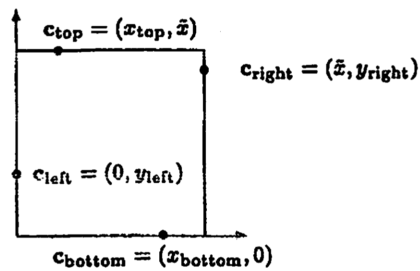

Figure 3.

By the choice of y_{left} and x_{bottom}, we know that

$$y_{\text{left}} \geq \max\{x_{\text{top}}, \tilde{x} - x_{\text{bottom}}, \tilde{x} - y_{\text{right}}\}.$$

Furthermore, if $y_{\text{left}} = \max\{x_{\text{top}}, \tilde{x} - x_{\text{bottom}}\}$ and $y_{\text{left}} > \tilde{x} - y_{\text{right}}$, then $y_{\text{left}} = x_{\text{top}}$ and $\tilde{x} - y_{\text{right}} = \tilde{x} - x_{\text{bottom}}$ and, if $y_{\text{left}} = \tilde{x} - y_{\text{right}}$, then

$$y_{\text{left}} = x_{\text{top}} = \tilde{x} - y_{\text{right}} = \tilde{x} - x_{\text{bottom}}.$$

Let $c_{\text{left}} = (0, y_{\text{left}})$, $c_{\text{right}} = (\tilde{x}, y_{\text{right}})$, $c_{\text{top}} = (x_{\text{top}}, \tilde{x})$ and $c_{\text{bottom}} = (x_{\text{bottom}}, 0)$. Let $C = \{c_{\text{left}}, c_{\text{right}}, c_{\text{top}}, c_{\text{bottom}}\}$.

We will now consider different cases. Note that the definitions of $\tilde{x}$, y_{right}, x_{bottom}, y_{left} and x_{top} just used the 2- or 3-deck of A. We will therefore be able to determine in which case we are.

Case 2.1. $y_{\text{left}} > \max\{x_{\text{top}}, \tilde{x} - x_{\text{bottom}}, \tilde{x} - y_{\text{right}}\}$.

For all $(x, y) \in \mathbf{R}^2 \setminus \{c_{\text{left}}, c_{\text{bottom}}, c_{\text{right}}\}$ we have

$$|A \cap \{(x, y)\}| = d_{A,4}(\{(x, y), c_{\text{left}}, c_{\text{bottom}}, c_{\text{right}}\})$$

and A is uniquely determined.

Case 2.2. $y_{\text{left}} = \max\{x_{\text{top}}, \tilde{x} - x_{\text{bottom}}, \tilde{x} - y_{\text{right}}\}$.

If $y_{\text{left}} \neq \tilde{x} - y_{\text{right}}$, then, as noted above,

$$y_{\text{left}} = x_{\text{top}} \neq \tilde{x} - y_{\text{right}} = \tilde{x} - x_{\text{bottom}}.$$

For all $(x, y) \in \mathbf{R}^2 \setminus \{\mathbf{c}_{\text{left}}, \mathbf{c}_{\text{bottom}}, \mathbf{c}_{\text{right}}\}$ we have

$$|A \cap \{(x, y)\}| = d_{A,4}(\{(x, y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{bottom}}, \mathbf{c}_{\text{right}}\})$$

and A is uniquely determined. Hence we may assume now that $y_{\text{left}} = \tilde{x} - y_{\text{right}}$ which implies that $y_{\text{left}} = x_{\text{top}} = \tilde{x} - y_{\text{right}} = \tilde{x} - x_{\text{bottom}}$.

For $\mathbf{x} = (x, y) \in \mathbf{R}^2$ let $\mathbf{x}_{\text{left}} = (x, y)$, $\mathbf{x}_{\text{right}} = (\tilde{x} - x, y_{\text{right}} - (y - y_{\text{left}}))$, $\mathbf{x}_{\text{top}} = (x_{\text{top}} + (y - y_{\text{left}}), \tilde{x} - x)$, $\mathbf{x}_{\text{bottom}} = (x_{\text{bottom}} - (y - y_{\text{left}}), x)$ (see Figure 4) and

$$X_{\mathbf{x}} = \{\mathbf{x}_{\text{left}}, \mathbf{x}_{\text{right}}, \mathbf{x}_{\text{top}}, \mathbf{x}_{\text{bottom}}\}.$$

It is easy to see that $|X_{\mathbf{x}}| = 1$ for exactly one $\mathbf{x} \in \mathbf{R}^2$ and that $|X_{\mathbf{x}}| = 4$ for all other $\mathbf{x} \in \mathbf{R}^2$.

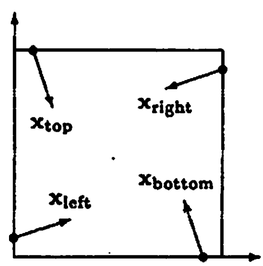

Figure 4.

For all $(x, y) \in \mathbf{R}^2 \setminus C$ we have

$$|A \cap X_{\mathbf{x}}| = d_{A,3}(\{(x, y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}).$$

For the unique $\mathbf{x} = (x, y) \in \mathbf{R}^2 \setminus C$ with $|X_{\mathbf{x}}| = 1$, the intersection $A \cap X_{\mathbf{x}}$ is uniquely determined. Similarly, for all $\mathbf{x} = (x, y) \in \mathbf{R}^2 \setminus C$ with $|X_{\mathbf{x}}| = 4$ and $d_{A,3}(\{(x, y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) \in \{0, 4\}$, the intersection $A \cap X_{\mathbf{x}}$ is uniquely determined. We will therefore not consider these $\mathbf{x} = (x, y) \in \mathbf{R}^2 \setminus C$ again in what follows.

We note the following observation. If $\mathbf{x} = (x, y) \in \mathbf{R}^2 \setminus C$ is such that

$$d_{A,3}(\{(x, y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) = 2,$$

then there are two different possibilities. The set $(A \cap X_{\mathbf{x}}) \cup \{\mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}$ is either isomorphic to

$$\{\mathbf{x}_{\text{left}}, \mathbf{x}_{\text{right}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\} \text{ or to } \{\mathbf{x}_{\text{left}}, \mathbf{x}_{\text{bottom}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}.$$

It is easy to see that we can differentiate these two possibilities by considering

$$d_{A,4}(\{\mathbf{x}_{\text{left}}, \mathbf{x}_{\text{right}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) \text{ and } d_{A,4}(\{\mathbf{x}_{\text{left}}, \mathbf{x}_{\text{bottom}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}).$$

Case 2.2.1. $d_{A,3}(\{(x',y'), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) = 1$ for some $\mathbf{x}' = (x',y') \in \mathbf{R}^2 \setminus C$ with $|X_{\mathbf{x}'}| = 4$.

We assume without loss of generality that $\mathbf{x}' = \mathbf{x}'_{\text{left}} = (x',y') \in A$. Now for all $(x,y) \in \mathbf{R}^2 \setminus (C \cup \{(x',y')\})$ we have that

$$|A \cap \{(x,y)\}| = d_{A,4}(\{(x,y), (x',y'), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}),$$

and A is uniquely determined. Hence we can assume that

$$d_{A,3}(\{(x,y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) \neq 1$$

for all $(x,y) \in \mathbf{R}^2 \setminus C$ with $|X_{\mathbf{x}}| = 4$.

Case 2.2.2. $d_{A,3}(\{(x',y'), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) = 3$ for some $\mathbf{x}' = (x',y') \in \mathbf{R}^2 \setminus C$.

We assume without loss of generality that $\mathbf{x}'_{\text{left}}, \mathbf{x}'_{\text{bottom}}, \mathbf{x}'_{\text{top}} \in A$.

Let $\mathbf{x} = (x,y) \in \mathbf{R}^2 \setminus (C \cup X_{\mathbf{x}'})$. If $d_{A,3}(\{(x,y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) = 3$, then there are the following four possibilities (see Figure 5)

a) $\mathbf{x}_{\text{right}} \notin A$,

b) $\mathbf{x}_{\text{top}} \notin A$,

c) $\mathbf{x}_{\text{left}} \notin A$ or

d) $\mathbf{x}_{\text{bottom}} \notin A$.

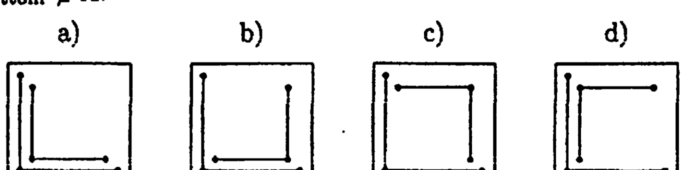

Figure 5.

Let $S_1 = \{\mathbf{x}_{\text{left}}, \mathbf{x}'_{\text{left}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}$, $S_2 = \{\mathbf{x}_{\text{right}}, \mathbf{x}'_{\text{left}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}$ and $S_3 = \{\mathbf{x}_{\text{bottom}}, \mathbf{x}'_{\text{left}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}$.

Given possibility a) we obtain $d_{A,4}(S_1) = 3$. Given possibility b) we obtain $d_{A,4}(S_1) = 2$, $d_{A,4}(S_2) = 2$ and $d_{A,4}(S_3) = 3$. Given possibility c) we obtain $d_{A,4}(S_1) = 2$ and $d_{A,4}(S_2) = 3$. Given possibility d) we obtain $d_{A,4}(S_1) = 2$, $d_{A,4}(S_2) = 2$ and $d_{A,4}(S_3) = 2$. We can therefore differentiate all four possibilities.

If $d_{A,3}(\{(x,y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) = 2$, then a similar case analysis as above again using the *test sets* S_1, S_2 and S_3 leads to the same conclusion. Hence also in this case the set A is uniquely determined and we can assume that

$$d_{A,3}(\{(x,y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) \neq 3$$

for all $(x,y) \in \mathbf{R}^2 \setminus C$.

Case 2.2.3. $d_{A,3}(\{(x,y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) \not\in \{1,3\}$ for all $\mathbf{x} = (x,y) \in \mathbf{R}^2 \setminus C$ with $|X_{\mathbf{x}}| = 4$.

If $d_{A,3}(\{(x,y), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) \neq 2$ for all $(x,y) \in \mathbf{R}^2 \setminus C$, then A is uniquely determined (by the observations at the end of Case 2.2). Hence we assume that

$$d_{A,3}(\{(x',y'), \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}) = 2$$

for some $\mathbf{x}' = (x',y') \in \mathbf{R}^2 \setminus C$. As noted at the end of Case 2.2, there are essentially two different possibilities for the intersection of A with the set $X_{\mathbf{x}'}$. Since both cases can be treated similarly, we may assume without loss of generality, that

$$\mathbf{x}'_{\text{left}}, \mathbf{x}'_{\text{bottom}} \in A.$$

Now for all $(x,y) \in \mathbf{R}^2 \setminus (C \cup \{\mathbf{x}'_{\text{left}}, \mathbf{x}'_{\text{bottom}}\})$ we have that

$$|A \cap \{(x,y)\}| = d_{A,5}(\{(x,y), \mathbf{x}'_{\text{left}}, \mathbf{x}'_{\text{bottom}}, \mathbf{c}_{\text{left}}, \mathbf{c}_{\text{right}}\}),$$

A is uniquely determined and the proof is complete. (Note that this was the only time we actually used the 5-deck.) □

For the analogous question in $\mathbf{R}^1$ algebraic and combinatorial arguments lead to the same result. Both imply that every finite set in $\mathbf{R}^1$ with at least 4 elements is uniquely determined up to translation and reflection, i.e. the mapping $x \mapsto -x$, by its 4-deck. For higher dimensions the combinatorial arguments get more and more involved and difficult whereas the algebraic arguments remain as simple and powerful as was demonstrated in [7].

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