Some remarks on the Shannon capacity of odd cycles

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Abstract

We tackle the problem of estimating the Shannon capacity of cycles of odd length. We present some strategies which allow us to find tight bounds on the Shannon capacity of cycles of various odd lengths, and suggest that the difficulty of obtaining a general result may be related to different behaviours of the capacity, depending on the "structure" of the odd integer representing the cycle length. We also describe the outcomes of some experiments, from which we derive the evidence that the Shannon capacity of odd cycles is extremely close to the value of the Lovász theta function.

1 Introduction

The purpose of this note is to study the Shannon capacity of odd cycles, and give some insights into this strikingly difficult problem, which remains open even for the 7-cycle.

The notion of capacity of a graph has been introduced by Shannon in [12], and after that was labeled as *Shannon capacity*. This concept arises in connection with a graph representation of the problem of communicating messages in a zero-error channel. One considers a graph G, whose vertices are letters from a given alphabet, and where adjacency indicates that two letters can be confused. In such a setting, the maximum number of one-letter messages that can be sent without danger of confusion is clearly given by $\alpha(G)$, the independence number of G. If $\alpha(G^k)$ stands for the maximum number of k-letter messages that can be safely communicated, we immediately see that $\alpha(G^k) \geq \alpha(G)^k$, and that equality does not hold in general (see, e.g., [10] for some examples).

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The Shannon capacity of G is the number

$$\Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}\,,$$

which satisfies $\Theta(G) \geq \alpha(G)$, where again equality does not need to occur.

Shannon capacity can be defined in terms of *strong graph products.* We say that two vertices are adjacent if they are either connected by an edge or equal. Then for two graphs G and H, we define their strong product $G \cdot H$ as the graph with vertex set $V(G) \times V(H)$, where (i, j) is adjacent to (i', j') if and only if i is adjacent to i' in G and j is adjacent to j' in H. If G^k denotes the strong product of k copies of G, then $\alpha(G^k)$ is the independence number of G^k.

It was very early recognized that the determination of the Shannon capacity is a very difficult problem, even for small and simple graphs (see [7, 11]). Some advances have been obtained in [4], where a number of nice estimates for the size of the maximum independent set of certain powers of odd cycles have been determined.

In a famous paper of 1979, Lovàsz introduced the "theta function" $\vartheta(G)$, with the explicit goal of estimating $\Theta(G)$ [10].

There are several equivalent definitions of the Lovász theta function [9]. We give here the one that follows from Theorem 5 in [10].

Definition 1 *Let* $\mathbf{A}$ *be the family of matrices* A *such that* $a_{ij} = 0$ *if* i *and* j *are adjacent in* G*, and let* $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ *denote the eigenvalues of* A*. Then*

$$\vartheta(G) = \max_{A\in\mathbf{A}} \left\{1 - \frac{\lambda_1(A)}{\lambda_n(A)}\right\}.$$

Combining the fact that $\Theta(G) \leq \vartheta(G)$ with the easy lower bound $\Theta(C_5) \geq \sqrt{5}$, Lovász has been able to determine exactly the capacity of C_5, the pentagon, which indeed turns out to be $\sqrt{5}$.

The last section of [10] raises a number of interesting questions, e.g., determining the Shannon capacity of odd cycles (which we will denote by C_m, with the subscript indicating the length), and saying whether or not $\vartheta(G) = \Theta(G)$. The latter question was answered in the negative by Haemers, who showed in [8] that the capacity of the complement of the *Schäfli* graph is strictly less than the value of its theta function.

Lovász theta function has the remarkable property of being computable in polynomial time, despite being *sandwiched* between two hard to compute integers, i.e., clique and chromatic number. This property has stimulated a number of studies on using the theta function to analyze the approximability of clique and chromatic number (see, e.g., [6, 13]).

For several families of simple graphs, the value of $\vartheta(G)$ is given by explicit formulas. For instance, in the case of odd cycles of length n we have

$$\vartheta(C_n) = \frac{n\cos(\pi/n)}{1+\cos(\pi/n)}.$$

Lovász observed that the question of the truth of $\vartheta(G) = \Theta(G)$ *pinpoints* the following crucial difficulty: in all cases where the value of $\Theta(G)$ is exactly known, there is some k such that $\alpha(G^k) = \Theta(G)^k$. The key remark is now that, if $\vartheta(G) = \Theta(G)$ for, say, the 7-cycle, then no such k can exist, since no power of $\vartheta(C_7)$ is an integer (as one can easily verify by checking the above expression for $\vartheta(C_m)$).

Shannon capacity and Lovász theta function attracted a lot of interest in the scientific community, because of the applications to communication issues, but also of the connections with some central combinatorial and computational questions in graph theory, like computing the largest clique and finding the chromatic number of a graph. The reader can find in [1, 2, 3, 5] a sample of results and applications of $\vartheta(G)$ and $\Theta(G)$. Despite a lot of work in the field, many basic open questions remain open, notably that of evaluating the Shannon capacity of C_7, and, more in general, of odd cycles.

In this paper, we present some strategies which allow us to improve on the estimates of the Shannon capacity of odd cycles [4], and to point out different behaviours of the capacity, depending on the structure of the odd integer representing the cycle length (see the summary of bounds in Table 1).

Section 2 is centered around a Theorem which provides a tight lower bound on $\alpha(C^{d+1}_{k2^d+1})$, and thus allows us to establish a good estimate of $\Theta(C_{k2^d+1})$.

In Section 3 we make an efficient use of the expansion technique introduced in [4], and derive good estimates on $\alpha(C^d_{n+2})$ starting from $\alpha(C^d_n)$.

In Section 4 we finally present some experimental results which, besides improving some of the bounds obtained theoretically, add further evidence to the fact that the Shannon capacity of odd cycles is in general extremely close, if not equal, to the Lovász theta function.

2 A lower bound on $\alpha(C_n^{d+1})$, for $n = k2^d + 1$

From now on we represent C^d_n as a hypercube H with d dimensions of size n. Each cell of H corresponds to a node of C^d_n. An independent set of C^d_n is given by a subset of the corresponding cells of H, such that no two cells are adjacent. We specify a node of C_n^d, or a cell of H, with a d-tuple of values in the range $0, 1, \ldots, n-1$. Due to the toroidal nature of C_n^d, we will silently assume that the tuple's values are computed modulo n. Note that the cells adjacent to a cell with coordinates $(i_1, i_2, \ldots, i_d)$ are the $3^d - 1$ cells obtained by adding $\{-1, 0, +1\}$ independently to each coordinate.

Example 1 *In [4] it is shown that $\alpha(C_7^5) \geq 7^3 = 343$ and that one such independent set I is given by*

$$I = \{(x_1, x_2, x_3, 2x_1 + 2x_2 + 2x_3, 2x_1 + 4x_2 + 6x_3) : 0 \leq x_1, x_2, x_3 < 7\},$$

where the last two values of the tuples are computed modulo 7.

The exact determination of the size of maximum independent sets in C_n^d seems to be a very hard task, in the general case. When n is of the form $k2^d + 1$, Baumert et al. have proved in [4] that

$$\alpha(C_n^d) = \left(\frac{n}{2}\right)^d \cdot \frac{n-1}{n}.$$

Here we now show that a very good lower bound can be obtained also for the next power of the graph, i.e., C_n^{d+1}. This lower bound will allow us to improve upon several estimates of [4].

Theorem 1 *Let k be an odd positive integer. If $n = k2^d + 1$, then*

$$\alpha(C_n^{d+1}) \geq \left(\frac{n}{2}\right)^{d+1} - \frac{1}{2}\left(\left(\frac{n}{2}\right)^d + n^{d-1}\right).$$

Proof:

We prove the theorem by showing that an independent set for C_n^{d+1} of the required size is given by:

$$\begin{aligned} I = \{(i_1, i_2, \ldots, i_{(d-1)}, i_d, 2t + 2^{d-1}ki_1 + 2^{(d-2)}ki_2 + 2^{(d-3)}ki_3 + \ldots \\ +2ki_{(d-1)} + ki_d - (i_d \mod 2)) | t = 0, \ldots, (k-3)/2 + (i_d \mod 2)\}, \end{aligned} \quad (1)$$

where we have represented C_n^{d+1} as a hypercube with $d+1$ dimensions of size n.

The number of nodes in the above independent set can be easily estimated, observing that from the $d-1$ free indices $(i_1, i_2, \ldots, i_{(d-1)})$ one obtains a contribution to the bound by the factor n^{d-1}, and that such a factor must be multiplied by $\frac{1}{2}(nk-1)$, which can be obtained analyzing the range of variation of the index t, and its combination with the index i_d, which depends on the parity of i_d. Thus the size of the set I turns out to be $\frac{1}{2}n^{d-1}(nk-1)$, which is equal to $\left(\frac{n}{2}\right)^{d+1} - \frac{1}{2}\left(\left(\frac{n}{2}\right)^d + n^{d-1}\right)$.

Let us now consider the following n^d subsets of nodes of C_n^{d+1}:

$$c_{i_1,i_2,\dots,i_d} = \{(i_1, i_2 \dots, i_{d+1}) | i_{d+1} = 0, \dots n-1\}.$$

All the graphs induced by these sets of nodes are isomorphic to C_n.

We now define a new graph G whose nodes all correspond to a set $c_{i_1,i_2,\dots,i_d}$, $i_j = 0, \dots, n-1$, $j = 1, \dots, d$.

In G there is an edge between nodes $(i_1, i_2, \dots, i_d)$ and $(j_1, j_2, \dots, j_d)$, if there exists at least an edge in C_n^{d+1} between the set $c_{i_1,i_2,\dots,i_d}$ and the set $c_{j_1,j_2,\dots,j_d}$. It turns out that G is isomorphic to C_n^d.

Each edge of G belongs to a clique of size 2^d, which is the maximum value of a clique size in G. In particular, a maximum size clique in G can be described as:

$$K_{i_1,i_2,\dots,i_d} = \{(j_1, j_2, \dots, j_d) | j_t \in \{i_t, i_t + 1\}, t = 1, 2, \dots d\}$$

For each of such maximum cliques in G, we consider nodes which are independent in C_n^{d+1}, for each individual maximal clique in G, in a single C_n.

More precisely, we define the following subset I of C_n^d

$$I_{i_1,i_2,\dots,i_d} = \{j_{d+1} | (j_1, j_2, \dots j_{d+1}) \in I \ \text{ and } \ (j_1, \dots, j_d) \in K_{i_1,i_2,\dots,i_d}\}.$$

Let now

$$V_{i_1,i_2,\dots,i_d} = \{(j_1, j_2, \dots, j_{d+1}) | (j_1, \dots, j_d) \in K_{i_1,i_2,\dots,i_d}\}.$$

I is clearly an independent set in C_n^{d+1} if and only if, $\forall i_1, i_2, \dots, i_d$, we have that

$$I_{i_1,i_2,\dots,i_d} \text{ is an independent set, and } |I \cap V_{i_1,i_2,\dots,i_d}| = |I_{i_1,i_2,\dots,i_d}|. \quad (2)$$

We can now estimate the size of the independent set (1). For $i_d \neq n-1$, we have that

$$I_{i_1,i_2,\dots,i_d} = \{x + 2t | x = 2^{(d-1)}ki_1 + 2^{(d-2)}ki_2 \dots + 2ki_{(d-1)} + ki_d - (i_d \mod 2), \quad t = 0, 1, \dots, (n-1)/2)\},$$

while, for $i_d = n-1$, we have

$$I_{i_1,i_2,\dots,i_d} = \{x + s + 2t | x = 2^{(d-1)}ki_1 + 2^{(d-2)}ki_2 \cdots + 2ki_{(d-1)} + ki_d,$$
$$s = \{0, k\}, \quad t = 0, \dots, (k-3)/2\}.$$

Note that all these sets satisfy condition (2).

□

Note that Theorem 1 is a generalization of Theorem 6 in [4], which provides a bound for hypercubes of size 3.

3 Expansion

The *expansion* operation is a technique which allows us to determine an independent set for C^d_{n+2} starting from an independent set for C^d_n, and has been originally introduced by [4]. We build upon the work of [4] to optimize the use of expansion in order to improve the best known bounds on the sizes of some independent sets.

As in the proof of Theorem 1, we represent C^d_n as a hypercube H with d dimensions of size n.

For each subcube c of size 2, uniquely identified by its corner with coordinates $c \equiv (i_1, i_2, \dots, i_d)$ (where each coordinate i_h corresponds to the minimum value for the h-th coordinate of any vertex of the subcube), the outcome of an expansion operation with respect to one coordinate i_k of c is given by the $n+2$ hyperplanes (w.r.t. the k-th dimension) $1, 2, \dots, i_k, i_k + 1, i_k, i_k + 1, \dots, n-1, n$. After repeating this operation w.r.t. each of the d dimensions, we eventually end up with a hypercube of size $n+2$.

It is trivial to verify that the expansion process maps any independent set into a new independent set. Indeed, if we look at the expansion step w.r.t. the i_k-th coordinate, we can see that the cells belonging to all the hyperplanes but i_k and $i_k + 1$ keep the same neighbors, while the two 'extra copies' of the hyperplanes i_k and $i_k + 1$ are adjacent to the two hyperplanes $i_k + 1$ and i_k, respectively, and such proximity does not affect the independent set, since it was also present in the original independent set.

In Figure 1 we have described the expansion process, applied to a cube of dimension 2 (i.e., a square) w.r.t. its subsquare of size two located in the upper-left corner.

Note that the square of dimension $n+2$ can be decomposed into a 2×2 subsquare S equal to the one which had been used for the expansion, an $n \times n$ square equal to the original one, and a copy of the two rows and columns to which S belongs.

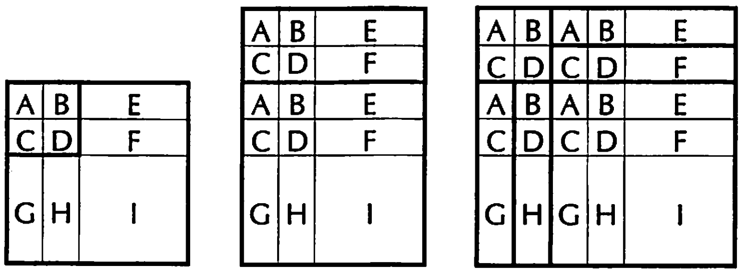

Figure 1: Expansion of a square.

In a similar way, a cube of dimension 3 and size n will be expanded into a cube of size $n+2$ formed by the original cube, the size 2 cube S used for the expansion, and a copy of the 6 planes and the 12 rows and columns to which the 2^3 cubes of S belong.

In general, after expanding a hypercube H corresponding to C_n^d w.r.t. one of its subcubes S of size 2, we will get a hypercube corresponding to C_{n+2}^d which can be decomposed in the union of H, S, and a copy of all the hyperplanes (with dimensions $1, 2, \ldots, d-1$) of H which are incident onto elements belonging to S.

Whenever a given independent set of C_n^d shows a regular structure, we can count the number of its elements belonging to each hyperplane, and thus evaluate the size of the 'expanded' independent set of C_{n+2}^d.

Indeed we have the following Theorems.

Theorem 2 *For each n of the form $k\,2^d+1$, with $k>0$ and $d>1$, we have*

$$\alpha(C_{n+i}^d) \geq k\,\frac{(n+i)^d - i^d}{n} + \left(\frac{i}{2}\right)^d = \left(\frac{n+i}{2}\right)^d \cdot \frac{n-1}{n} + \frac{1}{n} \cdot \left(\frac{i}{2}\right)^d,$$

where i is even.

Proof: An independent set of the required cardinality can be obtained expanding $i/2$ times the maximum independent set of C_n^d, whose cardinality is kn^{d-1}, as we have seen in Section 2.

Such independent set has the following structure: each of its n^{d-1} cycles C_n contains exactly k nodes, and, in general, each hyperplane (of dimension $j>1$, and thus consisting of n^j cells) contains kn^{j-1} elements from the independent set.

The subcube of size 2 located at the intersection of all the hyperplanes that have been 'replicated' is contained in $2^{n-j}\binom{n}{j}$ hyperplanes of dimension j, each contributing to the independent set with $n^j(k/n) = kn^{j-1}$ elements.

W.l.o.g. we may assume that the cube of size 2 contains an element of the independent set. Then the cardinality of the independent set of C^d_{n+2} obtained by means of expansion is

$$1 + kn^{d-1} + \left(\frac{k}{n}\right)\left[2^1\binom{d}{1}n^{d-1} + 2^2\binom{d}{2}n^{d-2} + \cdots + 2^{d-1}\binom{d}{d-1}n\right] =$$

$$1 + kn^d + \frac{k}{n}[(n+2)^d - n^d - 2^d] = \frac{k(n+2)^d + 1}{n}.$$

After executing $i/2$ expansions, with i even, we obtain an independent set for C^d_{n+i} whose cardinality can be evaluated in a similar way. Recalling that $n = k2^d + 1$, we indeed obtain

$$\left(\frac{i}{2}\right)^d + kn^{d-1} + \frac{k}{n}[(n+i)^d - n^d - i^d] = \frac{k(n+i)^d + (i/2)^d}{n}. \quad (3)$$

□

We now apply the expansion process to the independent set of Theorem 1, in order to obtain a lower bound on $\alpha(C^{d+1}_{n+i})$.

Theorem 3 *For each n of the form $k\,2^d + 1$, with $k > 1$ odd and $d > 1$, we have*

$$\alpha(C^{d+1}_{n+i}) \geq \left(\frac{n+i}{2}\right)^{d+1} \cdot \frac{n-1}{n} + \frac{1}{n} \cdot \left[\left(\frac{i}{2}\right)^{d+1} + 2^{d-1}i^d - \frac{(n+i)^d}{2}\right], \quad (4)$$

where i is even.

Proof:

An independent set with the required cardinality can be obtained by expanding $i/2$ times the independent set of C^{d+1}_n, obtained from Theorem 1, and whose cardinality is $n^{d-1}(nk-1)/2$.

The proof is similar to the previous one, but it is slightly more complicated, since the structure of the independent sets of C^{d+1}_n is not perfectly symmetric.

Indeed in this case each C_n does not contain a constant number of elements of the independent set, but rather a number of elements which is either $\lfloor k/2 \rfloor$ or $\lceil k/2 \rceil$. Analogously, pairs of adjacent hyperplanes corresponding to C^2_n can overall contain either $nk-1$ or nk elements. By means of a tedious analysis of the structure of the maximum independent set of C_n^{d+1}, one can eventually derive the following formula, whose shape is similar to that of (3), and which can be simplified in order to get (4).

$$\begin{aligned}\alpha(C_{n+1}^{d+1}) &\geq (i/2)^{d+1} + \frac{nk-1}{2} n^{d-1} + k\frac{(n+i)^{d+1} - n^{d+1} - i^{d+1}}{2n} \\ &- \frac{(n+i)^d - n^d - i^d}{2n}.\end{aligned}$$

□

4 Experimental results

Our investigation towards finding sharp estimates for $\alpha(C_n^d)$ has been also carried out by designing and implementing algorithms for the detection of large independent sets. This experimental activity has made it possible to improve upon many of the known lower bounds on the size of the independent sets.

4.1 Algorithm description

Given the high degree of regularity of C_n^d, our algorithm for searching large independent sets in $G = C_n^d$ does not explicitly store the adjacency matrix of G. In particular, each vertex of G is identified by a d-tuple of numbers between 0 and $n-1$, in such a way that two vertices s and t, with $s \equiv (i_1, \ldots, i_d)$ and $t \equiv (j_1, \ldots, j_d)$, are adjacent if and only if, for each $k = 1, \ldots, d$, either $i_k = j_k$ or $i_k = j_k \pm 1 \pmod n$. Given an independent set IS and a vertex $t \in IS$, we say that 'it is possible to move' t if there exists a vertex t' adjacent to t such that, after substituting t' to t, we still obtain an independent set.

The following is a high level description of our algorithm.

1. Build a 'starting' independent set IS for G. IS can be a previously computed independent set, or it can be generated by randomly adding vertices to an initially empty independent set.

2. For each vertex $t \in IS$, try to 'move' t in a random direction, in order to obtain a new independent set of the same cardinality.

3. With probability P_r, check if there exist vertices in G which can be added to the current independent set.

4. For each $t \in IS$, delete t from IS with probability P_d.

5. Go to Step 2.

After Step 3, we check if the cardinality of the current independent set improves upon the largest known value, and, if this is the case, we store such current independent set. The algorithm terminates after a certain amount of time or an appropriate number of iterations.

The probabilities of adding (P_r) and of deleting (P_d) used in Steps 3 and 4 are not constant. The best experimental results have been obtained by varying them according to the simulated annealing paradigm. In practice, in the initial stages of the algorithm we use larger values for P_d and smaller for P_r, in such a way that, working on a non optimal independent set, Step 2 could more easily move suitable vertices. As time goes by, the value of P_d (P_r) is decreased (increased), in order to give an advantage to a 'stabilization' of the independent set. If, after a given number of iterations, we do not obtain improvements, then we increase again the value of P_d to make it possible for the algorithm to avoid the current local optimum.

The ability of our algorithm to rapidly approximate the maximum independent strongly depends on an expedient imported from tabu search: at Step 2, after having moved a vertex of the independent set from position t to the adjacent position t', we mark vertex t as prohibited. Prohibited vertices cannot be reconsidered for insertion into the independent set for a given number of iterations. In this way, we force the current independent set to 'move' towards sets of vertices not yet considered.

An example of an independent set for C_7^4 obtained by our algorithm is shown in Figure 2.

Our algorithm has been implemented in C, and executed on a Linux workstation based on a 400Mhz Pentium II.

4.2 An upper bound

When dealing with Shannon capacity, it often turns out to be easier to improve lower bounds rather than upper bounds. Our small contribution to the upper bound issue concerns C_{13}^3, for which it was known that $247 \leq \alpha(C_{13}^3) \leq 252$. More precisely, in [4] it was shown that $\alpha(C_{13}^3) \leq 252$. By adopting a search technique – to be described next – we have shown that $\alpha(C_{13}^3) \neq 252$, which implies Theorem 4.

Theorem 4 $\alpha(C_{13}^3) \leq 251$.

Proof:

Any 'candidate' independent set of cardinality 252 can be classified according to the number of vertices belonging to each of the 13 subgraphs C_{13}^2, which, in the geometric representation of the graph, can be viewed as the 'slices' of size 13×13 forming the cube corresponding to C_{13}^3. We denote by $G_0, G_1, \ldots, G_{12}$ these subgraphs, and by g_i, for $i = 0, 1, \ldots, 12$, the number of elements of the independent set belonging to G_i. By assumption,

n	3	4	5	6	d	$\alpha'(C_n^d)^{\frac{1}{d}}$	$\theta(C_n)$
5	10 o	25 o	50 o	125 o	2	2.2361	2.2361
7	33 o	108 e	343	1101 e	4	3.2237	3.3177
9	81 o	324	1458	6561	3	4.3267	4.3601
11	148 o	761 e	3996	21904	3	5.2896	5.3863
13	247	1531 e	9633	61009	3	6.2743	6.4042
15	382 x, e	2770 e	19864 c	145924 c	3	7.2558	7.4171
17	578 o	4913 o	39304	334084	4	8.3721	8.4270
19	807 o	7666 o	68994	651610	4	9.3571	9.4348
21	1092	11441	114660	1201305	4	10.3423	10.4410
23	1437 x, e	16466 x	181126 c	2074716 c	4	11.3278	11.4462
25	1875 o	23125 a	281250	3515625	4	12.3316	12.4505
27	2362 o	31522 x, e	413350	5579044	4	13.3246	13.4542
29	2929	42017 x, e	594587	8579041	4	14.3171	14.4574
31	3580 x, e	54934 x, e	830560	12816400 c	4	15.3095	15.4601
33	4356 o	71874 o	1185921 o	18974736	5	16.3988	16.4626
35	5197 o	90947 o	1591572 o	27056724	5	17.3926	17.4647
37	6142	113586	2101333	37824138	5	18.3865	18.4666
39	7195 x, e	140211 x	2734074 x	51947406 c	5	19.3804	19.4683
41	8405 o	171462 a	3510825 x	70644025	5	20.3743	20.4699
43	9696 o	207514 x, e	4454896 x	94012416	5	21.3682	21.4713
45	11115	249005 x, e	5591997 x	123543225	5	22.3621	22.4726
47	12666 x, e	296439 x, e	6950358 x	160427556 c	5	23.3562	23.4737
49	14406 o	352947 o	8588377 a	207532836	4	24.3740	24.4748
51	16244 o	414196 o	10492965 x	263867536	4	25.3689	25.4758
53	18232	483091	12721391 x	332849699	4	26.3638	26.4767
55	20377 x, e	560244 x	15313309 x	415701048 c	4	27.3586	27.4776
57	22743 o	646551 a	18311493 x	517244049	4	28.3564	28.4783
59	25222 o	742247 x	21761957 x	636149284	4	29.3520	29.4791
61	27877	848193 x	25714075 x	777127129	4	30.3476	30.4798
63	30712 x, e	965097 x	30220701 x	943226944 c	4	31.3432	31.4804

Table 1: For $n = 5, 7, \ldots, 63$, and for $p = 3, 4, 5, 6$ we report the cardinality $\alpha'(C_n^d)$ of the largest known independent set for C_n^d. The alphabetic code which follows the numeric values has the following meaning: 'o' denotes an already known optimal value, while the other codes denote an improvement upon the values obtained in [4]: in particular 'a' indicates that the numeric value has been obtained by applying Theorem 1, 'x' that the numeric value has been improved thanks to the application of the expansion formulas, 'c' that the value has been determined thanks to improvements obtained on smaller powers, and 'e' that the improvement has been obtained experimentally. The column labeled as d indicates, for each n, which is the actual value of d that provides the maximum value for $\alpha'(C_n^d)^{1/d}$, and this value is itself reported in the adjacent column. The last column reports the value of the Lovász theta function of C_n.

n,d	3	4	5	6	Δ
5					
7		8		12	0.0096139
9					
11		21			
13		1			
15	2	66	104	1524	0.0126851
17					
19					
21					
23	2	3	33	378	0.0005160
25		144			0.0005682
27		256			0.0069569
29		405			0.0093310
31	2	599	464	14316	0.0145705
33					
35					
37					
39	2	1	6	114	0.0000085
41		201	29		0.0000337
43		31	92		0.0000883
45		477	232		0.0001856
47	2	694	504	5066	0.0003387
49			28514		
51			37645		
53			48788		
55	2	1	62252	742	0.0000122
57		264	78382		0.0028951
59		366	97562		0.0036190
61		486	120221		0.0043481
63	2	627	146834	122844	0.0050920

Table 2: For $n = 5, 7, \ldots, 63$, and $d = 3, \ldots, 6$, we report the improvements obtained for the cardinality $\alpha'(C_n^d)$ of the largest known independent set for C_n^d. The last column shows the corresponding increment for $\max_{d=3,\ldots,6}\{\alpha'(C_n^d)^{1/d}\}$.

we have $\sum_i g_i = 252$ and $g_i + g_{i+1} \leq 39$ for each i, since $\alpha(C_{13}^2) = 39$. Up to symmetries, there are 22 sequences s_j of values g_i which satisfy these constraints. In the following we list three of them, i.e.,

$$s_1 = (19, 19, 20, 19, 20, 19, 20, 18, 20, 19, 20, 19, 20)\,,$$
$$s_2 = (19, 20, 18, 21, 18, 21, 18, 21, 18, 21, 18, 19, 20)\,,$$
$$s_3 = (18, 18, 21, 18, 21, 18, 21, 18, 21, 18, 21, 18, 21)\,.$$

Starting from an independent set whose slices have cardinality

$$s_j = (g_{j,0}, \ldots, g_{j,12})\,,$$

5			4	2		
					5	
6		6				
				0		
		0			3	1
4	2		3	1		
					4	2

0

		6				0
3	1			0		
					3	1
4	2	4	2			
				5		
		5				6
				6		

1

				4	2	
		4	2			5
	6			5		
						6
	0		0			
					1	
3	1	3	1			

2

	6		6			
					0	
1			0			3
	4			3	1	
2		2				4
	5		5	3		
						5

3

1		1				3
	4		4	2		
		2			5	
6			5			
					6	
0		0				
	3				0	

4

					5	
6		6				
				0		
		0			3	1
	2, 4		3	1		
					4	2
5			4	2		

5

	3	1		0		
						1, 3
	2, 4	.	2, 4			
					5	
6			5			
					6	
0		6				

6

Figure 2: Representation of the independent set of cardinality 108 of C_7^4. Each 7×7 matrix represents one of the 7 subgraphs C_7^3 which form C_7^4. A numeric value k in position i, j in matrix h indicates that vertex (h, i, j, k) belongs to the independent set.

we consider the cardinalities of the 13 independent sets of C_{13}^2 obtained by merging (two by two) the neighboring slices of G. In this was, we obtain 22 sequences

$$t_j = (g_{j,0} + g_{j,1}, g_{j,1} + g_{j,2}, \ldots, g_{j,11} + g_{j,12}, g_{j,12} + g_{j,0}) .$$

Three of them are

$$\begin{aligned} t_1 &= (38, 39, 39, 39, 39, 39, 38, 38, 39, 39, 39, 39, 39)\,, \\ t_2 &= (39, 38, 39, 39, 39, 39, 39, 39, 39, 39, 37, 39, 39)\,, \\ t_3 &= (36, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39)\,. \end{aligned}$$

For each of the 22 sequences t_j, we have generated all the possible sequences $u_1, u_2, \ldots$ of 13 independent sets of C_{13}^2 with cardinality $t_{j,0}, \ldots, t_{j,12}$. For each of these sets we have verified if the intersections between adjacent elements could provide us with an independent set for C_{13}^3 of the appropriate cardinality. Since the distinct independent sets of C_{13}^2 with cardinality 39 and 38 are 52 and 794638, respectively, we have adopted some tricks in order to reduce the number of candidates u_i that must be checked:

1. The independent sets of cardinality 38 are, up to symmetries, 'only' 4702. Therefore one of these can be chosen, w.l.o.g., among 4702 candidates as opposed to 794638.

2. In a sequence $u = (u_0, u_1, \ldots, u_{12})$ derived from t_j, two adjacent elements u_i and u_{i+1} must have an intersection of cardinality $t_{j,i}$.

3. Given three consecutive elements of a sequence $u = (u_0, u_1, \ldots, u_{12})$, the union of the first and the third one must contain the second one.

4. Given a partial sequence $u = (u_0, u_1, \ldots, u_{11})$, there exists at most one way to complete it with an element u_{12}. Indeed, by construction, each element of a alleged independent set must appear an even number of times in the sequence u, whose elements are the union of adjacent slices of C_{13}^3.

The last argument allows us to consider independent sets of cardinality 36 or 37 only at the end of the process, and thus to avoid their explicit generation in advance, as for the cardinalities 38 and 39. This can be done because in each sequence t_j there is at most one 36 or 37.

Building upon the previous arguments, we have carried out the proof of non-existence of an independent set of C_{13}^3 with cardinality 252 by a C program which has run for about 12 hours on a workstation based on a 400 MHz Pentium 2 processor.

The combinatorial explosion of the feasible configurations suggests that this approach cannot be applied to further reduce the upper bound on $\alpha(C_{13}^3)$. □

The bounds obtained by applying our theoretical and experimental results are shown in Tables 1 and 2, where we point out differences and improvements upon previously known results, e.g. those in [4].

5 Conclusions

This paper has provided several improvements in the estimates of the Shannon capacity of odd cycles. We have also obtained some new results on the structure of independent sets in C_m^k, and used them to bound $\Theta(C_m)$.

We believe that our contribution could provide a starting point for new investigations. Both the arguments developed in Section 2 and in Section 3 seem to indicate that the Shannon capacity of odd cycles might depend in a very subtle way on the value of their length, and that different techniques might be needed to handle different lengths.