The $L(2,1)$-labeling problem on ditrees

Gerard J. Chang*
Department of Applied mathematics
National Chiao Tung University
Hsinchu 30050, Taiwan
Email: gjchang@math.nctu.edu.tw

Sheng-Chyang Liaw†
Department of Mathematics
National Central University
Chungli 32054, Taiwan
Email: scliaw@math.ncu.edu.tw

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Abstract

An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d_G(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d_G(x,y)=2$. The $L(2,1)$-labeling problem is to find the smallest number $\lambda(G)$ such that there exists a $L(2,1)$-labeling function with no label greater than $\lambda(G)$. Motivated by the channel assignment problem introduced by Hale, the $L(2,1)$-labeling problem has been extensively studied in the past decade. In this paper, we study this concept for digraphs. In particular, results on ditrees are given.

Keywords. $L(2,1)$-labeling, $L(2,1)$-labeling number, ditree.

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1 Introduction

The $L(2,1)$-labeling problem proposed by Griggs and Roberts [12] is a variation of the frequency assignment problem (or the T-coloring problem) introduced by Hale [8]. Suppose we are given a number of transmitters or stations. The $L(2,1)$-labeling problem is to assign frequencies (nonnegative integers) to the transmitters so that "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies that are at least two frequencies apart.

To formulate the problem in graphs, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent in the graph and "close" if they are of distance two in the graph. More precisely, an $L(2,1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d_G(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d_G(x,y) = 2$. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling such that no label is greater than k. The $L(2,1)$-*labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labeling. The $L(2,1)$-labeling problem has been extensively studied, see the references.

For practical reasons, the transmitters may have the direction constraints. In this case, we consider the $L(2,1)$-labeling on digraphs (directed graphs). In a digraph G, the *distance* $d_G(x,y)$ from vertex x to vertex y is the length of a shortest dipath (directed path) from x to y. We may define $L(2,1)$-labelings, k-$L(2,1)$-labelings and $L(2,1)$-labeling numbers for digraphs in precisely the same way as for graphs. However, to distinguish with the notation for graphs, we use $\lambda^*(G)$ for the $L(2,1)$-labeling number of a *digraph* G. In this paper, we study the $L(2,1)$-labeling numbers of ditrees (directed trees), which are orientations of trees. Recall that an *orientation* of a graph is a digraph obtained from the graph by assigning each edge of the graph an direction. A graph is called the *underline graph* of its orientations.

2 Ditrees

We begin our study by giving the sharp upper bound 4 for the $L(2,1)$-labeling number $\lambda^*(T)$ of a ditree T. Note that this upper bound is quit different from the bounds for a (undirected) tree T: $\Delta(T)+1 \leq \lambda(T) \leq \Delta(T)+2$, where $\Delta(T)$ is the maximum degree of a vertex in T.

Define $N^+(v) = \{u : vu \text{ is an edge}\}$ and $N^-(v) = \{u : uv \text{ is an edge}\}$. If it is necessary to specify G, we use the notion $N_G^+(v)$ for $N^+(v)$ and $N_G^-(v)$ for $N^-(v)$. We call the vertices in $N^+(v)$ the *out-neighbors* of v, in $N^-(v)$ the *in-neighbors* and in $N^+(v) \cup N^-(v)$

the *neighbors*. A *leaf* of a digraph is a vertex v with exactly one neighbor. Note that a ditree of at least two vertices has at lesat two leaves.

Theorem 1 *For any ditree T, we have $\lambda^*(T) \leq 4$. Moreover, $\lambda^*(T) = 4$ if T has a dipath of length 4.*

Proof. To prove $\lambda^*(T) \leq 4$, we shall give a 4-$L(2,1)$-labeling of T. We actually label the vertices of T inductively by $\{0, 2, 4\}$ with the extra condition that if a vertex v is labeled by i then all vertices in $N^+(v)$ are labeled by $(i+2)$ mod 6 and all vertices in $N^-(v)$ by $(i-2)$ mod 6. For the case of $|V(T)| = 1$, we may label the only vertex by 0. For a ditree T of at least two vertices, choose a leaf v of T. Suppose u is the only neighbor of v. By the induction hypothesis, $T - v$ has a 4-$L(2,1)$-labeling f by using only $\{0, 2, 4\}$. Now we can extend the labeling f to $V(T)$ by letting $f(v) := (f(u) + 2)$ mod 6 if $v \in N^+(u)$ and $f(v) := (f(u) - 2)$ mod 6 if $v \in N^-(u)$. It is easy to see that the extended labeling is a 4-$L(2,1)$-labeling of T using $\{0, 2, 4\}$ and satisfying the extra condition.

To prove the second statement, suppose T has a dipath $v_0 \to v_1 \to v_2 \to v_3 \to v_4$, yet there is a 3-$L(2,1)$-labeling. For $1 \leq i \leq 3$, if v_i can be labeled by 1 (respectively, 2) then both v_{i-1} and v_{i+1} should be labeled by 3 (respectively, 0), contradicting to $d_G(v_{i-1}, v_{i+1}) = 2$.

So, v_1, v_2 and v_3 are all labeled by 0 or 3, and so two of them get the same label, a contradiction. This proves $\lambda^*(T) \geq 4$ and so $\lambda^*(T) = 4$.

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Having the upper bound in mind, we now turn to the exact value of $\lambda^*(T)$ according to the length of a longest dipath in T. First, it is trivial that $\lambda^*(T) = 0$ whenever the longest dipath is of length 0, i.e., T has exactly one vertex.

Theorem 2 *For any ditree T whose longest dipath is of length* 1, *we have* $\lambda^*(T) = 2$.

Proof. It is easy to see that $\lambda^*(T) \geq 2$, since T has at least one edge.

Because T has no dipath of length 2, either $N^+(v) = \emptyset$ or $N^-(v) = \emptyset$ for any vertex v in T. We can partition $V(T)$ to be two disjoint vertex sets $A = \{v : N^+(v) = \emptyset\}$ and $B = \{v : N^-(v) = \emptyset\}$. Then labeling all vertices of A by 0 and of B by 2 gives a 2-$L(2,1)$-labeling for T. This completes the proof of the theorem. ∎

Theorem 3 *For any ditree T whose longest dipath is of length* 2, *we have* $\lambda^*(T) = 3$.

Proof. It is easy to check that $\lambda^*(T) \geq 3$, since T has a dipath with length 2. We then only need to prove that any ditree T without dipath of length 3 has a 3-$L(2,1)$-labeling. We in fact prove that for a given $i \in \{0,1,2,3\}$ and a specified vertex v in T with either $N^+(v) = \emptyset$ or $N^-(v) = \emptyset$, there is a 3-$L(2,1)$-labeling with v labeled by i.

The case of $|V(T)| = 1$ is trivial. Without loss of generality, we may assume that the specified vertex v satisfies the condition that $N^-(v) = \emptyset$. If $|N^+(v)| \geq 2$, then split T into two subditrees T_1 and T_2 with $V(T_1) \cap V(T_2) = \{v\}$. By the induction hypothesis, both T_1 and T_2 have a 3-$L(2,1)$-labeling with v labeled by the same value i. Combining them together, we then get a 3-$L(2,1)$-labeling for T with v labeled by i. If $N^+(v) = \{w\}$, then consider $T' = T - v$. For the case of $N^+(w) = \emptyset$, by the induction hypothesis, T' has a 3-$L(2,1)$-labeling with w labeled by $(i+2) \bmod 4$. This labeling can be extended to one for T by labeling v with i. For the case of $N^+(w) \neq \emptyset$, all vertices in $N^+(w)$ have no out-neighbors and all vertices in $N^-(w)$ have no in-neighbors, since T has no dipath of length greater than 2. Delete w from T to get subditrees. Each such subditree T_r has a unique vertex w_r that is a neighbor of w such that $N^+_{T_r}(w_r) = \emptyset$ or $N^-_{T_r}(w_r) = \emptyset$. By the induction hypothesis, each subditree has a 3-$L(2,1)$-labeling with w_r labeled by a specified value. We let w_r be labeled by i if w_r in $N^-(w)$, and by k if w_r in

$N^+(w)$. Let w be labeled by j. Then we can assign (i, j, k) properly by $(0,3,1)$, $(1,3,0)$, $(2,0,3)$ or $(3,0,2)$ to get a 3-$L(2,1)$-labeling of T with v labeled by i. ∎

From the above theorems, we know that $\lambda^*(T) = 3$ or 4 when a longest dipath of T is of length 3. The following two examples show that both are possible. Consider the ditree $T_1 = (V_1, E_1)$ with

$$V_1 = \{v_1, v_2, v_3, v_4\} \text{ and } E_1 = \{v_1v_2, v_2v_3, v_3v_4\};$$

and the ditree $T_2 = (V_2, E_2)$ with

$$V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

and

$$E_2 = \{v_1v_2, v_2v_3, v_3v_4, v_5v_4, v_5v_6, v_6v_7, v_7v_8\}.$$

It is the case that the longest dipaths of both T_1 and T_2 are 3, but $\lambda^*(T_1) = 3$ while $\lambda^*(T_2) = 4$.

We close this paper by raising the question of determining the $L(2,1)$-labeling number of a ditree whose longest dipath is of length 3.

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