PLANE PARTITION CORES

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1. Introduction

The core C of a plane partition P is an array including all parts of the plane partition [7] which have a part to the right and a part below. For example, the core of the plane partition $\begin{matrix}5431\\21\\1\\1\end{matrix}$ is $\begin{matrix}543\\2\end{matrix}$ and of $\begin{matrix}5431\\2111\\11\end{matrix}$ is $\begin{matrix}543\\21\end{matrix}$.

The number of plane partitions of the positive integer m having core C is denoted N(m,C). Some values of N(m,C) can be found in Table I in Section 2; Section 2 also includes applications of triangular cores to enumeration of certain types of linear partitions and certain restricted walks. Enumeration of linear partitions and "random" walks have long been of interest [1], [8]. Lemmas used in evaluating N(m,C) are stated and proved in Section 3, along with notational remarks and examples. And the Theorem (the main result in this paper), useful in evaluation of N(m,C) for triangular cores C (which are related to linear partition and walk applications), is stated and proved in Section 4; notes and examples involving triangular cores and Fibonacci numbers [5] are included in Section 4.

2. Applications

A plane partition has no core if and only if all of its parts lie in one row or all of its parts lie in one column. The conjugate of a plane partition P is denoted P' and is a plane partition whose rows are the columns of P, respectively. Table I includes values of N(m,C) if C=C' and N(m,C)+N(m,C')=2N(m,C) if C≠C', as well as values of p(m), the number of linear partitions of M [3], for m=1 to m=20.

<u>Example T.</u> If a core C_1 is represented by $\begin{matrix} c & c \\ c & \\ c & \end{matrix}$ (where the c's are positive integers which are not necessarily equal), and is identified by $\begin{matrix} 3 \\ 1 \end{matrix}$ (using row lengths), then C_1' is $\begin{matrix} c & c \\ c & \\ c & \end{matrix}$, $C_1 \neq C_1'$, and the Table I value corresponding, to m=15 is 18; therefore, $N(15,C_1)=18/2=9$. If a core C_2 is $\begin{matrix} c & c & c & c \\ c & c & c & \\ c & c & & \\ c & & & \end{matrix}$, identified by $\begin{matrix} 4 \\ 3 \\ 2 \\ 1 \end{matrix}$, then $C_2=C_2'$ and $N(19,C_2)=17$.

TABLE I

CORE m		1	2	2 1	3	3 1	2 2	4	3 2	4 1	3 2 1	5	4 2	3 3	5 1	4 3	4 2 1	3 3 1	6	5 2	4 3 1	3 3 1	p(m)
1	1																						1
2	2																						2
3	2	1																					3
4	2	3																					5
5	2	3	2																				7
6	2	4	4	1																			11
7	2	5	2	4	2																		15
8	2	6	2	5	4	2	1																22
9	2	7	2	6	2	6	1	2	2														30
10	2	8	2	7	2	8	0	4	6	2	1												42
11	2	9	2	8	2	10	0	2	4	6	5	2	2	2									56
12	2	10	2	9	2	12	0	2	4	8	8	4	4	2	2	2	2	2					77
13	2	11	2	10	2	14	0	2	4	10	9	2	2	0	6	6	8	4	2	2	2	1	101
14	2	12	2	11	2	16	0	2	4	12	11	2	2	0	8	4	12	4	4	4	8	2	
15	2	13	2	12	2	18	0	2	4	14	13	2	2	0	10	4	12	4	2	2	12	0	
16	2	14	2	13	2	20	0	2	4	16	15	2	2	0	12	4	14	4	2	2	16	0	
17	2	15	2	14	2	22	0	2	4	18	17	2	2	0	14	4	16	4	2	2	20	0	
18	2	16	2	15	2	24	0	2	4	20	19	2	2	0	16	4	18	4	2	2	24	0	
19	2	17	2	16	2	26	0	2	4	22	21	2	2	0	18	4	20	4	2	2	28	0	
20	2	18	2	17	2	28	0	2	4	24	23	2	2	0	20	4	22	4	2	2	32	0	

TABLE I (cont.)

C	6	5	5	4	4	4	7	6	5	5	4	4	3	7	6	6	5	5	5	4	4	p(m)
O	1	3	2	4	3	2		2	4	3	4	3	3	1	3	2	4	3	2	4	3	
R E			1		2	1				1	1	2	3			1	1	2	1	2	3	
						1						1							1			
m																						
14	2	2	2	2	2	1																135
15	6	4	8	2	8	4	2	2	2	2	2	1	1									176
16	8	2	12	0	8	6	4	4	6	6	4	6	1	2	2	2	2	2	2	2	2	231
17	10	2	12	0	6	6	2	2	4	8	4	12	0	6	4	8	8	6	8	2	4	
18	12	2	14	0	6	7	2	2	4	10	4	14	0	8	2	12	12	6	12	0	4	
19	14	2	16	0	6	8	2	2	4	12	4	17	0	10	2	12	16	4	12	0	4	
20	16	2	18	0	6	9	2	2	4	14	4	20	0	12	2	14	20	4	14	0	4	

C	8	7	6	6	5	5	5	4	4	4	8	7	7	6	6	6	6	5	5	5	5	4	p(m)
O		2	4	3	5	4	3	4	4	3	1	3	2	5	4	3	2	5	4	4	2	4	
R E				1		2	2	3	2	3			1		1	2	1	1	3	2	1	3	
							1		1	1							1			1	1	1	
m																					1		
17	2	2	2	2	2	2	2	2	2	1													297
18	4	4	4	6	2	6	10	4	6	3	2	2	2	2	2	2	2	2	2	2	1	2	385
19	2	2	2	8	0	4	18	0	6	5	6	4	8	6	6	6	8	4	8	10	4	6	
20	2	2	2	10	0	4	20	0	4	7	8	2	12	4	8	6	12	4	8	18	6	6	

C	9	8	7	7	6	6	6	5	5	5	5	4	4	9	8	8	7	7	7	7	6	6	6	6	6	5	5	5	5	5	4	p(m)
O		2	4	3	5	4	3	5	4	3	3	4	4	1	3	2	5	4	3	2	6	5	4	4	2	5	5	4	4	4	4	
R E				1	1	2	2	2	3	2	2	4	3			1		1	2	1		2	3	2	1	3	2	4	3	2	4	
							1		1	2	1		2							1				1	1		1		2	1	1	
											1														1					1		
m																																
19	2	2	2	2	2	2	2	2	2	2	1	2	1																			490
20	4	4	4	6	8	4	10	2	10	6	4	2	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	627

A triangular core C is one with t rows of length t, t-1, ..., 1, respectively. The minimal number of parts in a plane partition with triangular core C (containing t rows) is denoted M(C) ; and M(C)= (t+1)(t+2)/2 . Core C_2 in Example T is triangular, t = 4 , and $M(C_2)$= 5(6)/2=15 ; the plane partition

cccc1
ccc1
cc1
c1
1

has the smallest number of parts, 15, of any plane partition having core C_2 .

Let all parts in a plane partition having m parts and triangular core C (with t rows) be 1; row sums of this plane partition are parts in a linear partition of m ; and N(m,C) is the number of linear partitions of m of the following type:

a) the linear partition has at least t+1 parts if part t+1 is 1 , with all parts 1 thereafter, and has exactly t+1 parts if part t+1 is 2;

(1) b) part 1 is at least t+1 , and part 2 is t if part 1 is greater than t+1 and is t or t+1 if part 1 is t+1;

c) for t = 1, ..., t-1, if part i equals part i+1 then part i+2 is 2 less than part i+1 , and if part i is greater than part i+1 then part i+2 is equal to part i+1 or is one less than part i+1 .

<u>Example P.</u> The number of linear partitions of m=12 satisfying (1) with t=3 is 8. The 8 possibilities are shown (all with core

ccc
cc
c

; c=1 in all instances).

ccc111 ,	ccc11 ,	ccc11 ,	ccc11 ,	ccc1 ,	ccc1 ,	ccc1 ,	ccc1 .
cc1	cc1	cc1	cc1	cc11	cc11	cc1	cc1
c1	c11	c1	c1	c1	c1	c11	c1
1	1	11	1	11	1	1	1
		1		1		1	1
							1

They correspond to the linear partitions 6+3+2+1, 5+3+3+1, 5+3+2+2, 5+3+2+1+1, 4+4+2+2, 4+4+2+1+1, 4+3+3+1+1, 4+3+2+1+1+1 of 12, respectively; all satisfy (1). Using the Theorem (stated in Section 4), the value of N(12,C), t=3, is A+B+E+D , where $A=A_2=C(3,2)+F_2=4$ since $2\leq[4/2]$, B=0 since $2\leq[5/2]$, E=t+1=4, D=0 (since $t\ngeq 4$); and N(12, C)=4+0+0+4=8 .

Consider a walk (on a plane) from R' to R to S to S' (with R to the right of Q , S below Q , $\overline{QR}=\overline{QS}=t+1$, QR horizontal and perpendicular to QS at Q , R=R' or R' to the right of R , S=S' or S' below S) of the following type:

a) if R'≠R , start at R', walk left to R and down 1, left 1, and if R'=R , walk from R down 1 then either left 0 or left 1;

(2) b) from row 2 to row 3, ..., row t to row t+1: if on line RS, walk down 1, then either left 0 or left 1, and if not on line RS (i.e. if 1 unit to the right of RS), walk down 1 and left 2;

c) from row t to row t+1: if on line RS, walk down 1, then either left 0 or left 1 (if left 0, the walk actually terminates one unit to the right of S, and if left 1, continue down from S to S'), and if not on line RS, walk down 1 and left 2 to S (and the walk terminates at S).

Note that the walker must be on line RS at least every other row level from R to S. For example, the diagram

```
Q11R111R'
111
11
S
1
S'
```

corresponds to t=3, $\overline{RR'}=4$, $\overline{SS'}=2$.

An m-walk (given t and the corresponding triangular core C) is one satisfying (2) and such that $m-M(C)=\overline{RR'}+\overline{SS'}+j$, where j is the number of left 0's taken from R to S (or from R to one unit to the right of S). Then, for given t, the number of m-walks is N(m,C).

Example W. The number of 12-walks, t=3, is 8 . The 8 possibilities are shown:

$j=0,\ 12-10=\overline{RR'}+\overline{SS'}+0,\ \overline{RR'}+\overline{SS'}=2,$

```
ccc111 , ccc11 , cccl ,
ccl      ccl     ccl
cl       cl      cl
1        1       1
         1       1
                 1
```

$j=1,\ 12-10=\overline{RR'}+\overline{SS'}+1,\ \overline{RR'}+\overline{SS'}=1,$

```
cccl ,  ccc11 ,  cccl ,  ccc11 ,
cc11    ccl      ccl     ccl
cl      cl1      cl1     cl
1       1        1       11
1                1
```

$j=2,\ 12-10=\overline{RR'}+\overline{SS'}+2,\ \overline{RR'}+\overline{SS'}=0,$

```
cccl .
cc11
cl
11
```

N(12,C)=3+4+1=8. The 8 possible plane partitions here correspond to those in Example P (but are given in a different order); and N(12,C) can be computed using Table I (as was done in Example P).

3. Evaluation of N(m,C).

A linear partition of a positive integer n, as defined by Andrews [1], is a finite nonincreasing sequence of positive integers $n_1, n_2, \ldots, n_r$ such that $\sum_{i=1}^{r} n_i=n$; the n_i are the parts of the partition and p(n) is the number of linear partitions of n. A plane partition of the positive integer n is a representation of n in the form $n = \sum_{i,j=1}^{\infty} n_{i,j}$, where the $n_{i,j}$ are nonnegative integers such that $n_{i,j} \geq n_{i,j+1}$ and $n_{i,j} \geq n_{i+1,j}$, as in works by MacMahon [4], Gordon and Houton [2], and Stanley [7]; the nonzero $n_{i,j}$ are the parts of the plane partition.

The core of a plane partition is defined to be an array including all parts of the plane partition which have a part to the right and a part below; if the given plane partition has no such parts, we have an "empty" array and say the plane partition has no core. Each plane partition either has a unique core or has no core. But not every plane partition array can serve as a core of a plane partition.

Example 1. The plane partitions (1) $\begin{array}{l} 53321 \\ 4221 \\ 111 \end{array}$ and (2) $\begin{array}{l} 53321111 \\ 4221 \\ 111 \end{array}$

both have the same core, $\begin{array}{l} 5332 \\ 422 \end{array}$. The plane partitions (3) 6211

and (4) $\begin{array}{l} 1 \\ 1 \\ 1 \end{array}$ have no core. The plane partition array (5) $\begin{array}{l} 5432 \\ 32 \\ 22 \end{array}$ is not a core

of any plane partition since if a minimum number of parts are appropriately adjoined to the right and below, a plane partition $\begin{array}{l} 54321 \\ 3211 \\ 221 \\ 11 \end{array}$ with core $\begin{array}{l} 5432 \\ 32 \\ 22 \end{array}$

(different from (5)) is obtained.

A plane partition has no core if and only if all of its parts lie in one row or all of its parts lie in one column. Conditions under which a plane partition array is not a core of any plane partition are given in Lemma 2 below.

In this paper we will include only the case in which all parts of a plane partition not in the core of the plane partition are 1's; this array of 1's will be called the fence of the plane partition. A plane partition having no core must have all parts 1, and is its own fence. For example, the plane partition 1111 has no core and the fence of this plane partition is 1111.

The lengths of the rows in a core (or in any plane partition array) having t rows will be denoted by $r_1, r_2, \ldots, r_t$, respectively (with $t, r_1, r_2, \ldots, r_t$ positive integers and $r_1 \geq r_2 \geq \ldots \geq r_t$). The number of rows in a given core (or in any given plane partition array) of length r_1 will be denoted v_1, of length r_{v_1+1} will be denoted v_2, and so forth, with the number of rows of length $r_{v_1+v_2+\cdots+v_{s-1}+1}=r_t$ denoted by v_s (where $s, v_1, \ldots, v_s$ are positive integers); and $v_1+v_2+\ldots+v_s=t$. For a plane partition having m parts (m a positive integer), core C and fence F, denote by N(C) the number of parts in the core and by N(F) the number of parts (1's) in the fence; then N(C)+ N(F)=m. If C is a core, denote by M(C) the minimal number of parts in any plane partition having core C. M(C) is unique, and for any plane partition having core C and m parts, $M(C) \leq m$. Define F(C)=M(C)-N(C); F(C) is the minimal number of 1's in a fence for a given core C. For any positive integer m and core C define N(m,C) to be the number of plane partitions having m parts and core C. If m<M(C) then N(m,C)=0. N(M(C),C)=1.

For convenience, each part of a core of a plane partition will be labeled c, where different c's in a core can assume different (positive integer) values. In Example 1, (1) can be represented as

```
cccc1 .
ccc1
111
```

<u>Example 2</u>. Given the core C:

```
ccccc
cccc
cccc
cc
```

, we have $t=4$, $r_1=5$, $r_2=r_3=4$, $r_4=2$, $v_1=1$, $v_2=2$, $v_3=1$, with $v_1+v_2+v_3=t=4$. The plane partition having core C and the smallest possible number of parts is

(1)
```
ccccc1
cccc1
cccc1
cc11
11
```
; N(C)+N(F)=15+7=22; M(C)= 22. The only plane partition having C as a core and M(C) parts is (1), and N(M(C),C)=1.

Lemma 1. If C is a core, then $M(C)=2r_1+r_2+\dots r_t+1+t-s$.

Proof. F(C) is equal to the number of columns in C plus the number of rows to which a 1 must be adjoined; a 1 must be adjoined to the first row and to any row of the same length as the row above it. $F(C)=r_1+1+(v_1-1+v_2-1+\dots+v_s-1)=r_1+1+t-s$. And $M(C)=N(C)+F(C)=r_1+r_2+\dots+r_t+r_1+1+t-s=2r_1+r_2+\dots+r_t+1+t-s$. ||

In Example 2, $M(C)=2(5)+4+4+2+1+4-3=22$.

The evaluation of N(m,C) is simplified by consideration of special cases.

Lemma 2. If A is a plane partition array with $r_i \geq r_{i+1}+2=r_{i+2}\,2\geq 3$ for $i+1>v_1$ and $t\geq i+2$, then no plane partition having A as a core exists.

Proof. Assume a plane partition having such an array as core does exist. Then a portion of rows i, i+1, i+2 can be represented as

```
...cccc... .
...cc11
...cc1
```

The first 1 in row i+1 has a 1 to the right and below, so must be in the core of the plane partition. Therefore, A cannot serve as a core of any plane partition. ||

The conjugate of a plane partition P is denoted P' and is a plane partition whose rows are the columns of P, respectively. The next lemma follows directly.

Lemma 3. There is a one-to-one correspondence between the set of plane partitions having a given core C and the set of plane partitions having the core C'. If no plane partition having a given plane partition array A as core exists, then no plane partition having A' as core exists.

Example 3. The plane partitions represented by (1) $\begin{array}{l} ccc11 \\ cc1 \\ 11 \end{array}$ and (1') $\begin{array}{l} cc1 \\ cc1 \\ c1 \\ 1 \\ 1 \end{array}$ correspond under the correspondence noted in Lemma 3; the cores, $\begin{array}{l} ccc \\ cc \end{array}$ and $\begin{array}{l} cc \\ cc \\ c \end{array}$ are conjugates.

Since the array A: $\begin{array}{l} cccc \\ cc \\ cc \end{array}$ cannot serve as a core for any plane partition (by Lemma 2), then there exists no plane partition having A' : $\begin{array}{l} ccc \\ ccc \\ c \\ c \end{array}$ as core.

Lemma 4. Let C be a core with $t \geq 2$.

1. If $r_1 = r_2$ or if $r_t \geq 2$ then there exists a positive integer k such that N(m,C) attains a constant value for all $m \geq M(C)+k$. If $r_1 = r_2$ and $r_t \geq 2$ then this constant value is 0.

2. If $r_1 > r_2$ and $r_t = 1$ then there exist positive integers k and q such that if $m \geq M(C)+k$ then $N(m+1,C) - N(m,C) = q$.

Proof. To simplify consideration of cases, a - will be used to indicate a position in which the adjoining of a 1 to the plane partition having the given core and M(C) parts does not change the core, and a * will be used to indicate a position in which the adjoining of a 1 does change the core.

1. Suppose that $r_1=r_2$ and $r_t\geq 2$. A portion of the plane partition with core C and M(C) parts can be represented as

```
ccc...c1*
ccc...c1*
   ...
cc
11
**
```

with at least one - present. N(M(C),C)=1 . (There is only one plane partition with core C if M(C)=m since no 1's are adjoined.) N(M(C)+1,C) is the number of -'s . (One 1 is adjoined, and this 1 can be adjoined in a position indicated by a - .) If -'s are in adjacent rows, they must be in adjacent columns, and adjoining 1's in both indicated positions will result in altering the core; therefore, 1's cannot be adjoined in both indicated positions if the core is to be left unchanged. There exist at most a finite number of -'s , say w . If k>w then N(M(C)+k,C)=0 . (More than w 1's cannot be placed in w positions. If some -'s are in adjacent rows, k might be $\leq$w .)

If $r_t\geq 2$ and $r_1>r_2$, we can represent a portion of the plane partition having the given core and M(C) parts as

```
cc...cc...c1- .
cc...c1...1-
   ...
cc
**
```

Again, N(M(C),C)=1 and N(M(C)+1,C) is the number of -'s . Any number of 1's can be adjoined to the first row so that N(m,C)>0 for all m$\geq$M(C) . If k exceeds the number of -'s , say w , then N(M(C)+k,C) =N(M(C)+h,C) for h$\geq$k (where h is an integer, since at most w-1 1's can be placed in positions indicated by -'s below the first row).

Similarly, the result holds if $r_t=1$ and $r_1=r_2$.

2. Suppose that $r_1>r_2$ and $r_t=1$. A portion of the plane partition having the given core and M(C) parts can be represented as

```
cc...cc...c1- .
cc...c1...1-
     ...
c1
1
-
```

N(M(C),C)=1 and N(M(C)+1,C) is the number of -'s. Any number of 1's can be adjoined to the first row and to the first column, and N(m,C)>0 for all m ≥ M(C). If w is the number of -'s, N(M(C)+k+1,C)-N(M(C)+k,C)= N(M(C)+k+2,C)-N(M(C)+k+1,C)=N(M(C)+h+1,C)-N(M(C)+h,C) for k>w and for all integers h≥k+1 . ||

Example 4.

1. Consider the plane partition with core C and M(C) parts represented by

```
cccccc1*
cccccc1*
cc1111-
c1*
1-
-
```

. N(M(C),C)=N(23,C)=1 .

N(M(C)+1,C)=3. (There are 3 -'s). N(M(C)+2,C)=3. (There are 2 cases with one 1 adjoined to row 3 and 1 case with both 1's adjoined to column 1.) N(M(C)+3,C)=2 . (There is 1 case with one 1 adjoined to row 3 and 1 case with all three 1's adjoined to column 1.) And N(M(C)+k,C)=2 for k≥3 .

2. Consider the plane partition with core C and M(C) parts represented by

```
ccccc1-
cccc1-
cc11-
c1*
1-
-
```

. N(M(C),C)=N(18,C)=1 . N(M(C)+1,C)=5.

N(M(C)+2,C)=1+3+5=9. (There is 1 case with two 1's adjoined to row 1, 3 cases with one 1 adjoined to row 1, and 5 cases with no 1's adjoined to row 1.) Similarly, N(M(C)+3,C)=1+3+3+3=10, N(M(C)+4,C)= 1+3+3+2+3=12 , N(M(C)+5,C)=1+3+3+2+2+3=14; and N(M(C)+k+!,C)-N(M(C)+k,C)=2=q if k≥3.

Lemma 5. If m is an integer, m≥3 , then $p(m)=2+\sum_{C} N(m,C)$ where the summation is taken over all cores C having all parts 1.

Proof. There is a one-to-one correspondence between the set of plane partitions having all parts 1 and the set of linear partitions. A plane partition with b columns and c_1 parts in column 1, c_2 parts in column 2, ..., c_b parts in column b (where b, $c_1, c_2, \ldots, c_b$ are positive integers, $c_1 \geq c_2 \geq \ldots \geq c_b$) corresponds to the linear partition with parts $c_1, c_2, \ldots, c_b$. The linear partitions of m are those corresponding to the two plane partitions with all parts 1 and m parts and no core (111 ... 1 and $\begin{matrix}1\\1\\\vdots\\1\end{matrix}$) and those corresponding to plane partitions with all parts 1 and m parts which have a core. The number of linear partitions of m is $2+\sum_{C} N(m,C)$ where C is a core with all parts 1 (since $N(m,C)=0$ if $m<M(C)$). ||

Table I includes values of N(m,C) or 2N(m,C) for $1 \leq m \leq 20$ and cores C with all parts 1 and $M(C) \leq m$. The cores are specified by $\begin{matrix}r_1\\r_2\\\vdots\\r_t\end{matrix}$.

If C and C' are the same, the entry is N(m,C), and if C is not the same as C', the entry is N(m,C)+N(m,C')=2N(m,C) which is the number of plane partitions having m parts and core C or C'. Table entries were computed by using the results and techniques discussed above. Values of p(m) [3] are also given.

4. Triangular Cores.

A triangular core C is specified by
$$\begin{matrix} t \\ t-1 \\ \vdots \\ 1 \end{matrix}$$
(with t a positive integer and $r_1=t$, $r_2=t-1$, ..., $r_t=1$) and has $t(t+1)/2$ parts, and a representation of the plane partition having this core and $M(C)=(t+1)(t+2)/2$ parts is

```
cc...cc1- .
cc...c1-
  ...
c1-
1-
-
```

The number of plane partitions with $m=M(C)+v$ parts and triangular core (having $t(t+1)/2$ parts) and first row of length $t+1+s$ will be denoted by $n(t,v,s)$. Note that $n(t,0,0)=N(M(C),C)=1$ and $n(t,1,1)+n(t,1,0)=N(M(C)+1,C)=1+(t+1)=t+2$. The Fibonacci numbers, F_1, F_2, ..., the greatest integer function [ ], and $C(n,r)$ (with $C(n,r)=n(n-1)\ldots(n-r+1)/r!$ for positive integers n and r, $C(n,0)=C(0,0)=1$, $C(n,r)=0$ for $n<r$) appear in the statement of the following lemma and the main Theorem [5], [6].

<u>Lemma 6.</u> B1. $F_w+F_{w-1}+\ldots+F_2+F_1+1=F_{w+2}$.

B2. If $v\geq[(t+4)/2]$, $t\geq2$, then $n(t,v,1)=F_t$.

B3. If $v>[(t+1)/2]$ then $n(t,v,0)=F_{t+1}$.

Proof. Mathematical induction is used. B1: $F_1+1=2=F_3$; assume that $F_{w-1}+\ldots+F_2+F_1+1=F_{w+1}$; then $F_w+F_{w-1}+\ldots+F_2+F_1+1=F_w+F_{w+1}=F_{w+2}$.
B2: $n(2,3,1)=1=F_2=n(2,4,1)=\ldots=n(2,v,1)$ if $v\geq[(2+4)/2]$ and $n(3,3,1)=2=F_3=n(3,4,1)=\ldots=n(3,v,1)$ if $v\geq[(3+4)/2]$; assume that $n(T,v,1)=F_T$ if $v\geq[(T+4)/2]$, $2\leq T\leq t-1$; then $n(t,v,1)=n(t-2,v-1,1)+n(t-3,v-1,1)+\ldots+n(2,v-1,1)+n(1,v-1,1)+1=F_{t-2}+F_{t-3}+\ldots+F_2+F_1+1=F_t$ (by B1) if $v-1\geq[(t-2+4)/2]$ iff $v\geq[(t-2+4+2)/2]=[(t+4)/2]$.
B3: $n(2,2,0)=2=F_3=n(2,3,0)=\ldots=n(2,v,0)$ if $v>[(2+1)/2]$ and $n(3,2,0)=3=F_4=n(3,3,0)=\ldots=n(3,v,0)$ if $v>[(3+1)/2]$;

assume that $n(T,v,0)=F_{T+1}$ if $v>[(T+1)/2], 2\leq T\leq t-1$; then $n(t,v,0)=n(t-1,v,1)+n(t-1,v,0)=F_{t-1}+F_t=F_{t+1}$ (by B2) if $v\geq[(t-1+4)/2]$ and $v>[(t-1+1)/2]$, thus if $v>[(t+1)/2]$. ||

<u>Theorem.</u> Let C be a triangular core having $t(t+1)/2$ parts. Then $N(M(C)+0,C)=1$, $N(M(C)+1,C)=t+2$, and,
for $v\geq 2$, $N(M(C)+v,C)= A + B + E + D$, where

$A = A_1 = F_{t+1}$ if $v > [(t+1)/2]$ and

$A = A_2 = C(t-(v-2),v)+F_2 \cdot C(t-v,v-2)+F_4 \cdot C(t-(v+1), v-3) +$
$F_6 \cdot C(t-(v+2),v-4)+\ldots +F_{2(v-3)} \cdot C(t-(2v-4),2) +$
$F_{2(v-2)} \cdot C(t-(2v-3),1)+F_{2(v-1)}$ if $v\leq[(t+1)/2]$,

$B = F_t \cdot (v-[(t+2)/2])$ if $v>[(t+2)/2]$ and

$B = 0$ if $v\leq[(t+2)/2]$,

$E = t+1$,

$D = D_1 + D_2 + D_3 + D_4 + \ldots + D_{[t/2]-1}$ with

$D_1= F_3+t(t-3)/2$ if $t\geq 4$, $v>2$ and

$D_1 = 0$ otherwise,

$D_2 = F_5+(F_4+1)(t-5)/1!+(t-6)(t+2)/3!$ if $t\geq 6$, $v>3$ and

$D_2 = 0$ otherwise,

$D_3 = F_7+(F_6+1)(t-7)/1!+(F_4+1)(t-8)(t-7)/2! +$
$(t-8)(t-7)(t-6)(t+3)/4!$ if $t\geq 8$, $v>4$ and

$D_3 = 0$ otherwise,

$D_4 = F_9+(F_8+1)(t-9)/1!+(F_6+1)(t-10)(t-9)/2! +$
$(F_4+1)(t-10)(t-9)(t-8)/3! +$
$(t-10)(t-9)(t-8)(t-7)(t+4) /5!$ if $t\geq 10$, $v>5$ and

$D_4 = 0$ otherwise,

$D_{[t/2]-1} = F_{2[t/2]-1}+(F_{2[t/2]-2}+1)(t-(2[t/2]-1))/1! +$
$(F_{2[t/2]-4}+1)(t-(2[t/2]))(t-(2[t/2]-1))/2! +$
$(F_{2[t/2]-6}+1)(t-(2[t/2]))(t-(2[t/2]-1))(t-(2[t/2]-2))/3! + \ldots +$
$(F_6+1)(t-(2[t/2]))(t-(2[t/2]-1)) \ldots (t-([t/2]+5))(t-([t/2]+4))/([t/2]-3)! +$
$(F_4+1)(t-(2[t/2]))(t-(2[t/2]-1)) \ldots (t-([t/2]+4))(t-([t/2]+3))/([t/2]-2)! +$

$(t-(2[t/2]))(t-(2[t/2]-1))\ldots(t-([t/2]+3))(t-([t/2]+2)) \cdot$
$(t+([t/2]-1))/[t/2]!$ if $v>[t/2]$ and
$D_{[t/2]-1} = 0$ otherwise.

Proof. As noted above, $N(M(C)+0,C)=1$, $N(M(C)+1,C)=t+2$. Assume that $v\geq 2$. $N(M(C)+v,C)=n(t,v,v)+n(t,v,v-1)+n(t,v,v-2)+\ldots+n(t,v,2)+n(t,v,1)+n(t,v,0)$. It will be proved that (1) $E=n(t,v,v)+n(t,v,v-1)$, (2) $B=n(t,v,1)+n(t,v,2)+\ldots+n(t,v,v-[(t+2)/2])$, (3) $A=n(t,v,0)$ and (4) $D=n(t,v,v-2)+n(t,v,v-3)+\ldots+n(t,v,v-[t/2])$.

(1). $n(t,v,v)=1$, $n(t,v,v-1)=t$ (since $v-1$ 1's are adjoined to cc ... c1 in row 1 and the other 1 can be adjoined to row 3 or 4 or ... or t+2) so that $n(t,v,v)+n(t,v,v-1) = E = t+1$.
(2). $n(t,v,1)+n(t,v,2)+n(t,v,3)+\ldots+n(t,v,v-[(t+2)/2])=n(t,v,1)+n(t-2,v-2,1)+n(t-2,v-2,0)+n(t-2,v-3,1)+n(t-2,v-3,0)+\ldots+n(t-2,[(t+2)/2],1)+n(t-2,[(t+2)/2],0)$
$=F_t+F_{t-2}+F_{t-1}+F_{t-2}+F_{t-1}+\ldots+F_{t-2}+F_{t-1}$ (by B2 and B3) which equals $(v-[(t+2)/2]) \cdot F_t=B$ if $v>[(t+2)/2]$ (where $v-[(t+2)/2]$ is the number of $F_t=F_{t-2}+F_{t-1}$ terms). If $v<[(t+2)/2]$ there are no terms and the value is $0=B$ and if $v=[(t+2)/2]$ then $(v-[(t+2)/2]) \cdot F_t=0=B$.
(3). If $v>[(t+1)/2]$ then $n(t,v,0)=F_{t+1}=A_1$ (by B3).
Suppose that $v\leq[(t+1)/2]$.
$n(t,2,0)=n(t-1,2,1)+n(t-2,2,1)+\ldots+n(2,2,1)+n(1,2,1)+1=t-1+t-2+\ldots+2+1+1$
$=C(t,2)+1$ (by B2).
$n(t,3,0)=n(t-1,3,1)+n(t-2,3,1)+\ldots+n(5,3,1)+n(4,3,1)+n(3,3,1)+n(2,3,1)+n(1,3,1)+1=n(t-3,2,1)+n(t-3,2,0)+n(t-4,2,1)+n(t-4,2,0)+\ldots+n(3,2,1)+n(3,2,0)+n(2,2,1)+n(2,2,0)+n(1,2,1)+n(1,2,0)+n(2,3,1)+n(1,3,1)+1=$
$n(t-3,2,1)+n(t-4,2,1)+\ldots+n(3,2,1)+n(2,2,1)+n(1,2,1)+1-1+n(t-3,2,0)$
$+n(t-4,2,0)+\ldots+n(3,2,0)+n(2,2,0)+n(1,2,0)+F_4$ where $n(2,3,1)+n(1,3,1)+1$
$=1+1+1=F_4$.
Since $n(t-3,2,1)+n(t-4,2,1)+\ldots+n(3,2,1)+n(2,2,1)+n(1,2,1)+1-1=$

$n(t-2,2,0)-1$, $v=2$, $n(2,2,0)=F_3$ (because $2>[3/2]$) and $F_3=C(2,2)+1$, and $n(1,2,0)=1$, then $n(t,3,0)=C(t-2,2)+1+C(t-3,2)+1+...+C(3,2)+1+C(2,2)+1+F_4+1-1=C(t-1,3)+1(t-3)+F_4=C(t-1,3)+F_2\cdot C(t-3,1)+F_4$.

Assume that $n(t,v-1,0)=C(t-(v-3),v-1)+F_2\cdot C(t-(v-1),v-3)+F_4\cdot C(t-v,v-4)+F_6\cdot C(t-(v+1),v-5)+\ldots+F_{2(v-5)}\cdot C(t-(2v-7),3)+F_{2(v-4)}\cdot C(t-(2v-6),2)+F_{2(v-3)}\cdot C(t-(2v-5),1)+F_{2(v-2)}\cdot n(t,v,0)=n(t-1,v,1)+n(t-2,v,1)+...+n(2v-1,v,1)+n(2v-2,v,1)+n(2v-3,v,1)+...+n(3,v,1)+n(2,v,1)+n(1,v,1)+1=n(t-3,v-1,1)+n(t-3,v-1,0)+n(t-4,v-1,1)+n(t-4,v-1,0)+...+n(2v-3,v-1,1)+n(2v-3,v-1,0)+n(2v-4,v-1,1)+n(2v-4,v-1,0)+n(2v-5,v-1,1)+n(2v-5,v-1,0)+...+n(1,v-1,1)+1-1+n(1,v-1,0)+n(2,v,1)+n(1,v,1)+1=n(t-3,v-1,1)+n(t-4,v-1,1)+\ldots+n(2v-3,v-1,1)+n(2v-4,v-1,1)+n(2v-5,v-1,1)+\ldots+n(1,v-1,1)+1-1+n(t-3,v-1,0)+n(t-4,v-1,0)+...+n(2v-3,v-1,0)+n(2v-4,v-1,0)+n(2v-5,v-1,0)+...+n(1,v-1,0)+F_4$ where $n(2,v,1)+n(1,v,1)+1=F_4$. Note that $n(t-3,v-1,1)+n(t-4,v-1,1)+...+n(2v-3,v-1,1)+n(2v-4,v-1,1)+n(2v-5,v-1,1)+\ldots+n(1,v-1,1)+1=n(t-2,v-1,0)$, that $n(2v-4,v-1,0)=F_{2v-3}$ (since $v-1>[(2v-3)/2]$), that $F_{2v-3}=C(v-1,v-1)+F_2\cdot C(v-3,v-3)+F_4\cdot C(v-4,v-4)+...+F_{2v-8}\cdot C(2,2)+F_{2v-6}\cdot C(1,1)+F_{2v-4}$ (because $1+F_2+F_4+...+F_{2v-8}+F_{2v-6}+F_{2v-4}=F_{2v-3}$), and that $n(2v-5,v-1,0)+...+n(1,v-1,0)+F_4-1=F_{2v-4}+\ldots+F_2+F_4-1=F_{2v-4}+\ldots+F_4+F_3+F_4+1-1=F_{2(v-1)}$.

Then
$$
\begin{aligned}
n(t,v,0)=\;&C(t-2-(v-3),v-1)+F_2\cdot C(t-2-(v-1),v-3)+F_4\cdot C(t-2-v,v-4)\\
&+F_6\cdot C(t-2-(v+1),v-5)+...+F_{2(v-5)}\cdot C(t-2-(2v-7),3)\\
&+F_{2(v-4)}\cdot C(t-2-(2v-6),2)+F_{2(v-3)}\cdot C(t-2-(2v-5),1)+F_{2(v-2)}\\
+\;&C(t-3-(v-3),v-1)+F_2\cdot C(t-3-(v-1),v-4)+F_4\cdot C(t-3-v,v-4)\\
&+F_6C(t-3-(v+1),v-5)+...+F_{2(v-5)}\cdot C(t-3-(2v-7),3)\\
&+F_{2(v-4)}\cdot C(t-3-(2v-6),2)+F_{2(v-3)}\cdot C(t-3-(2v-7),1)+F_{2(v-2)}\\
+\;&\ldots\\
+\;&C(v,v-1)+F_2\cdot C(v-2,v-3)+F_4\cdot C(v-3,v-4)+F_6\cdot C(v-4,v-5)+...\\
&+F_{2(v-4)}\cdot C(3,2)+F_{2(v-3)}\cdot C(2,1)+F_{2(v-2)}\\
+\;&C(v-1,v-1)+F_2\cdot C(v-3,v-3)+F_4\cdot C(v-4,v-4)+F_6\cdot C(v-5,v-5)+...\\
&+F_{2(v-4)}\cdot C(2,2)+F_{2(v-3)}\cdot C(1,1)+F_{2(v-2)}\\
+\;&F_{2(v-1)}=
\end{aligned}
$$

$C(t-(v-2),v)+F_2 \cdot C(t-v,v-2)+F_4 \cdot C(t-(v+1),v-3)+F_6 \cdot C(t-(v+2),v-4)$
$+\ldots+F_{2(v-5)} \cdot C(t-(2v-6),4)+F_{2(v-4)} \cdot C(t-(2v-5),3)$
$+F_{2(v-3)} \cdot C(t-(2v-4),2)+F_{2(v-2)} \cdot C(t-(2v-2),1)+F_{2(v-1)}$
$= A_2$.

(4). $n(t,v,v-2) = n(t-1,2,0) = n(t-2,2,1)+n(t-2,2,0)$.
If $t=4$, $v=2$, then $n(t-2,2,0)=n(2,2,0)=F_3$ (by A_1) and
$n(4,v,v-2)=t-2+2=4=F_3+4(4-3)/2=D_1$.
If $t>4$, $n(t-2,2,0)=n(t-2,2)+F_2$ (by A_2) and $n(t,v,v-2)=t-2+C(t-2,2)+F_2$
$=F_3+(t-3)(1+(t-2)/2)=F_3+t(t-3)/2=D_1$.

Mathematical induction will be used to prove the formulas for
$D_2, D_3, \ldots, D_{[t/2]-1}$. Let $D_r=D_r(t)$, $r=1, \ldots, [t/2]-1$.
$D_1 = D_1(t) = n(t,v,v-2)$ (above).
To show that $D_2 = D_2(t) = n(t,v,v-3)$, with $D_2(t) = n(t-1,3,0) =$
$n(t-2,3,1)+n(t-2,3,0) = n(t-3,2,0)+n(t-2,3,0)$, it will suffice to show that
$D_2(t) - D_1(t-2) = n(t-2,3,0)$ (since $D_1(t) = n(t,v,v-2) = n(t-1,2,0)$ implies that
$n(t-3,2,0) = D_1(t-2)$).
$D_2(t) - D_1(t-2) = F_5 - F_3 +(t-5)(F_4+1-(t-2)/2)+(t-6)(t-5)(t+2)/3!$
$= F_4+C(t-5,1) \cdot F_2 +(t-5)(F_3 +1-(t-2)/2)+(t-5)(t-4)(t+2)/3! -2(t-5)(t+2)/3!$
$= F_4+C(t-5,1) \cdot F_2 +C(t-3,3)+(t-5)(F_3 +1-(t-2)/2+5(t-4)/3! - 2(t+2)/3!)$
$= F_4 +F_2 \cdot C(t-5,1)+C(t-3,3)+0$
$= n(t-2,3,0)$ by A_2. Note that $t-6$ was replaced by $t-4-2$ and that $t+2$ was replaced by $t-3+5$ in the next step.
And $D_2(t) = n(t,v,v-3)$.
Similarly, to show that $D_3 = D_3(t) = n(t,v,v-4)$, it suffices to show that
$D_3(t) = D_2(t-2) = n(t-2,4,0)$.
$D_3(t) - D_2(t-2) = F_7 - F_5+(t-7)F_5+(t-8)(t-7)((F_4+1)/2! -t/3!)$
$+(t-8)(t-7)(t-6)(t+3)/4!$
$= F_6 + F_4 \cdot C(t-7,1)+(t-7)F_3 +(t-6)(t-7)(F_2/2!+(F_3+1)/2!-t/3!)$
$-2(t-7)((F_3+F_2+1)/2!-t/3!)+(t-5)(t-7)(t-6)(t+3)/4!$
$-3(t-7)(t-6)(t+3)/4!$

$= F_6 + F_4 \cdot t(t\text{-}7,1)+C(t\text{-}6,2) \cdot F_2+C(t\text{-}4,4)+(t\text{-}6)(t\text{-}7)(0)$
$= n(t\text{-}2,4,0)$ (by A_2) since $3/2!\text{-}t/3!+2/3!+(7(t\text{-}5)\text{-}3(t+3))/4!$
$= \text{-}4(t\text{-}11)/4!+(4t\text{-}44)/4! = 0$. Note that t-8 was replaced by t-6-2 and by t-5-3 , respectively, and that t+3 was replaced by t-4+7 in the next step.

And $D_3(t) = n(t,v,v\text{-}4)$.

Assume that $D_{m-1} = D_{m-2}(t\text{-}2)+n(t\text{-}2,m,0)$.

Show that $D_m(t)\text{-}D_{m-1}(t\text{-}2) = n(t\text{-}2,m+1,0)$.

$D_m\text{-}D_{m-1}(t\text{-}2)=F_{2m+1}\text{-}F_{2m-1}+(t\text{-}(2m+1))(F_{2m}+1\text{-}F_{2m-2}\text{-}1)$
$+(t\text{-}(2m+2))(t\text{-}(2m+1))(F_{2m-2}+1+F_{2m-4}\text{-}1)/2!$
$+(t\text{-}(2m+2))(t\text{-}(2m+1))(t\text{-}(2m))(F_{2m-4}+1\text{-}F_{2m-6}\text{-}1)/3!$
$+\ldots$
$+(t\text{-}(2m+2))(t\text{-}(2m+1)) \ldots (t\text{-}(m+5))(F_6+1\text{-}F_4\text{-}1)$
$+(t\text{-}(2m+2))(t\text{-}(2m+1)) \ldots (t\text{-}(m+4))((F_4+1)/(m\text{-}1)!\text{-}(t+m\text{-}3)/m!)$
$+(t\text{-}(2m+2))(t\text{-}(2m+1)) \ldots (t\text{-}(m+4))(t\text{-}(m+3))(t+m)/(m+2)!$
$= F_{2m}+(t\text{-}(2m+1))(F_{2m-1})+(t\text{-}(2m+2))(t\text{-}(2m+1))(F_{2m-3})/2!$
$+(t\text{-}(2m+2))(t\text{-}(2m+1))(t\text{-}(2m))(F_{2m-5})/3!$
$+\ldots$
$+(t\text{-}(m+4)\text{-}(m\text{-}2))(t\text{-}(2m+1)) \ldots (t\text{-}(m+5))F_5/(m\text{-}2)!$
$+(t\text{-}(2m+2))(t\text{-}(2m+1)) \ldots (t\text{-}(m+4))((F_4+1)/(m\text{-}1)!\text{-}(t\text{-}(t+m\text{-}3)/m!)$
$+(t\text{-}(m+2)\text{-}m)(t\text{-}(2m+1)) \ldots (t\text{-}(m+4))(t\text{-}(m+3))(t\text{-}(m+1)+2m+1)/(m+1)!$
$= F_{2m}+F_{2m-2} \cdot C(t\text{-}(2m+1),1)+F_{2m-3}(t\text{-}2m)(t\text{-}(2m+1))/2!$
$\text{-}F_{2m-3} \cdot 2(t\text{-}(2m+1))/2!+F_{2m-3}(t\text{-}(2m+1))$
$+F_{2m-5} \cdot (t\text{-}(2m\text{-}1)\text{-}3)(t\text{-}(2m+1))(t\text{-}2m)/3!$
$+\ldots$
$+F_5 \cdot (t\text{-}(m+4))(t\text{-}(2m+1)) \ldots (t\text{-}(m+5))/(m\text{-}2)!$
$\text{-}F_5 \cdot (m\text{-}2)(t\text{-}(2m+1)\ldots(t\text{-}(m+5))/(m\text{-}2)!$
$+(t\text{-}(m+3)\text{-}(m\text{-}1))(t\text{-}(2m+1))\ldots(t\text{-}(m\text{-}4))((F_4+1)/(m\text{-}1)!\text{-}(t+m\text{-}1)/m!)$
$+(t\text{-}(2m+2))(t\text{-}(2m+1))\ldots(t\text{-}(m+4))(t\text{-}(m+3))(t+m)/(m+1)!$
$= F_{2m}+F_{2m-2} \cdot C(t\text{-}(2m+1),1)+F_{2m-4} \cdot C(t\text{-}2m,2)+\ldots+F_4 \cdot C(t\text{-}(m+4),m\text{-}2)$
$+F_{2m-5}(t\text{-}2m)(t\text{-}(2m+1))/2! +F_{2m-5}(t+(2m\text{-}1))(t\text{-}(2m+1))(t\text{-}2m)/3!$

$$-F_{2m-5} \cdot 3(t-(2m+1))(t-2m)/3!$$

$$+ \ldots$$

$$+F_5(t-(2m+1))\ldots(t-(m+5))/(m-3)!$$

$$+F_3(t-(m+4))(t-(2m+1)) \ldots (t-(m+5))/(m-2)!$$

$$-F_5(m-2)(t-(2m+1)) \ldots (t-(m+5))/(m-2)!$$

$$+(t-(m+3))(t-(2m+1)) \ldots (t-(m+4))((F_4+1)/(m-1)! -(t+m-1)/m!)$$

$$+(t-(m+2)-m)(t-(2m+1))\ldots(t-(m+4))(t-(m+3))(t+m)/(m+1)!$$

$$= F_{2m}+F_{2m-2} \cdot C(t-(2m+1),1)+F_{2m-4} \cdot C(t-2m,2)+\ldots+F_4 \cdot C(t-(m+4),m-2)$$

$$+F_2 \cdot C(t-(m+3),m-1)+C(t-(m+1),m+1)$$

$$+(t-(2m+1)) \ldots (t-(m+4))(t-(m+3))((F_3+1)/(m-1)!-(t+m-3)/m!)$$

$$-(m-1)(t-(2m+1))\ldots(t-(m+4))((F_2+1)/(m-1)!-(t+m-3)/m!)$$

$$+(t-(2m+1)) \ldots (t-(m+3))(t-(m+2))(2m+1)/(m+1)!$$

$$-m(t-(2m+1))\ldots(t-(m+3))(t+m)/(m+1)!$$

$$=F_{2m}+F_{2m-2} \cdot C(t-(2m+1),1)+F_{2m-4} \cdot C(t-2m,2)+\ldots+F_4 \cdot C(t-(m+4),m-2)$$

$$+F_2 \cdot C(t-(m+3),m-1)+C(t-(m+1),m+1)+(t-(2m+1)) \ldots(t-(m+3))(0)$$

$= n(t-2,m+1,0)$ (by A_2) since $(m-1)/m!+(F_3+1)/(m-1)! - (t+m-3)/m!$
$+(2m+1)(t-(m+2))/(m+1)!-m(t+m)(m+1)!$
$= -(m+1)(t-3m-2)/(m+1)!+(m+1)(t-3m-2)/(m+1)! = 0$.
And $D_m(t) = n(t,v,v-(m+1))$ for $m=1, 2, \ldots , [t/2]-1$. ||

Reference to diagrams lead to relations among n(t,v,s) expressions as well as results given in the theorem.

<u>Example 5.</u> For a triangular core C, with t=4, v=5,

cccc111111	cccc11111	cccc1111	cccc111	cccc11	cccc1
ccc1	ccc1	ccc1	ccc1	ccc1	ccc1
cc1	cc1-	cc1-	cc1-	cc1-	cc1-
c1	c1-	c1-	c1-	c1-	c1-
1	1-	1-	1	1	1
	-	-	-	-	-
E=n(4,5,5)+n(4,5,4)		D=D_1	B=n(4,5,2)+		A=n(4,5,0)
=n(3,0,0)+n(3,1,0)			n(4,5,1)		=n(3,5,1)
		=n(3,2,0)			+n(3,5,0)
= 1 + 4		=4	=3+3		=n(2,4,0)
					+n(3,5,0)
					=2+3

$N(5 \cdot 6/2+5,C) = n(4,5,5)+n(4,5,4)+n(4,5,3)+n(4,5,2)+n(4,5,1)+n(4,5,0)$
$= A + B + E + D = 5+2(3)+(1+4)+4=20$, which equals
$F_5 + F_4 (5-3)+(4+1)+(F_3 +2)$ (as given by the theorem).

Example 6. Let C be a triangular core. Applications of the theorem for t=7 and t=12 are given.

1. t = 7.

$N(8 \cdot 9/2+0, C) = 1.$

$N(36+1,C)=7+2=9.$

$N(36+2,C)=A+B+E+D=(C(7,2)+F_2)+0+(7+1)+0=22+0+8+0=30.$

$N(36+3,C)=A+B+E+D=(C(6,3)+F_2 \cdot C(4,1)+F_4)+0+(7+1)+(F_3 +7 \cdot 4/2)$
$=27+0+8+16 =51.$

$N(36+4,C) =A+B+E+(D_1+D_2) =(C(5,4)+F_2 \cdot C(3,2)+F_4 \cdot C(2,1)+F_6)+0$
$+(7+1)+(16+F_5+(F_4 +1)2/1!+1 \cdot 2 \cdot 9/3!) =22+0+8+(16+16) =62 .$

$N(36+5,C) =F_8+F_7 \cdot (5-4)+8+(32) =21+13+8+32 =74 .$

For $v >[(t+2)/2] =4$,

$N(36+v,C) =F_8+F_7 (v-4)+8+32 =21+13(v-4)+8+32 =3v+9.$

2. t=12 .

$N(13 \cdot 14/2+0,C) = 1.$

$N(91+1,C) =12+2 = 14 .$

$N(91+2,C) =A+B+E+D =(C(12,2)+F_2) +0+(12+1)+0 =67+0+13+0 =80.$

$N(91+3,C) =A+B+E+D_1 =(C(11,3)+F_2 \cdot C(9,1)+F_4) +0+13+(F_3+12 \cdot 9/2)$
$=177+0+13+56 =246.$

$N(91+4,C) =A+B+E+(D_1+D_2) =(C(10,4)+F_2 \cdot C(8,2)+F_4 \cdot C(7,1)+F_6)$
$+0+13+(56+F_5+(F_4+!) \cdot 7/1!+6 \cdot 7 \cdot 14/3!)$
$=267+0+13+(56+131) =467.$

$N(91+5,C) =A+B+E+(D_1+D_2+D_3) =(C(9,5)+F_2 \cdot C(7,3)+F_4 \cdot C(6,2)$
$+F_6 \cdot C(5,1)+F_8)+0+13+(56+131+F_7+(F_6+1) \cdot 5/1!+(F_4+1)4 \cdot 5/2!$
$+4 \cdot 5 \cdot 6 \cdot 15/4!) =267+0+13+(56+131+173) =640 .$

$N(91+6,C) =A+B+E+(D_1+D_2+D_3+D_4) =(C(8,6)+F_2 \cdot C(6,4)+F_4 \cdot C(5,3)$
$+F_6 \cdot C(4,2)+F_8 \cdot C(3,1)+F_{10})+0+13+(56+131+173+F_9+(F_8+1)3/1!$

$+(F_6+1)2\cdot 3\cdot 4/3!+2\cdot 3\cdot 4\cdot 5\cdot 16/5!)$

$=239+0+13+(56+131+173+159)=771$.

$N(91+7,C)=A+B+E+(D_1+D_2+D_3+D_4+D_5)=F_{13}+0+13$

$+(56+131+173+159+F_{11}+(F_{10}+1)(1)/1!+0)=233+0+13$

$+(56+131+173+159+145)=233+0+13+664=910$.

$N(91+8,C)=F_{13}+F_{12}(8-7)+13+664=1054$.

For $v>[(t+2)/2]=7$,

$N(91+v,C)=F_{13}+F_{12}(v-7)+13+664=233+144(v-7)+13+664$

$=144v-98$.

Some values of $N((t+1)(t+2)/2+v,C)$, where C is a triangular core and $v>[(t+2)/2]$, are shown.

t	1	2	3	4	5	6	7
N(M(C)+v,C)	v+1	v+3	2v+3	3v+5	5v+6	8v+8	13v+9

t	8	9	10	11	12
N(M(C)+v,C)	21v+9	34v+5	55v-7	89v-36	144v-98 .