Gracefulness of n - cone $C_m \vee K_n^c$

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Abstract: It is proved that the n-cone $C_m \vee K_n^c$ is graceful for any $n \geq 1$ and $m \equiv 0$ or $3(mod\ 12)$. The gracefulness of the following n-cones is also established : $C_4 \vee K_n^c, C_5 \vee K_2^c, C_7 \vee K_n^c, C_9 \vee K_2^c, C_{11} \vee K_n^c, C_{19} \vee K_n^c$. This partially answers the question of gracefulness of n-cones which is listed as an open problem in the survey article by J.A. Gallian.

1. INTRODUCTION

Our notation and terminalogy are as in [1]. If a, b are two integers, $a \leq b$, then $[a, b]$ will denote the set of all integers from a to b . A *vertex labeling* of a (p, q) graph $G = G(V, E)$ is an injective function f from V to a set of integers and the *induced edge labeling* f^* is the function from E to positive integers defined by $f^*(xy) = \mid f(x) - f(y) \mid$ for any edge $xy \in E$. A vertex labeling $f : V \longrightarrow [0, q]$ of G is said to be *graceful* if $f^*(E) = [1, q]$. A graph is said to be *graceful* if it has a graceful labeling. The graph $C_m \vee K_n^c$ is known as a *n-cone*. This graph is obtained by considering the cycle C_m, n new independent vertices and joining each one of them to every vertex of C_m. For $n = 1$, the graph is known as a *wheel* and it is known to be graceful [3]. In his survey, Gallian[2] mentions that the case $n = 2$, the graph in this case being called *double cone*, is still unsettled for gracefulness. Our main result is that the n-cones $C_m \vee K_n^c$ is graceful for $n \geq 1$ and $m \equiv 0$ or 3 (mod 12). We also establish the gracefulness of the n-cones $C_4 \vee K_n^c, C_5 \vee K_2^c, C_7 \vee K_n^c, C_9 \vee K_2^c, C_{11} \vee K_n^c$, and $C_{19} \vee K_n^c$.
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2. SPECIAL LABELINGS OF C_m

A vertex labeling f of a (p,q) graph $G = (V,E)$ is said to be a *special labeling* if it satisfies the following conditions.

Condition 1 For any $i, 1 \leq i \leq p$, there exists a vertex u_i of G such that $f(u_i)$ is either $2i-1$ or $2i$.

Condition 2 $f^*(E) = [1,2p] \backslash f(V)$.

Condition 3 Both $f(x)$ and $f^*(xy)$ are odd implies that $f(x) < f(y)$.

We note that Conditions 1 and 2 imply that $p = q$.

The following theorem connects existence of a special labeling of a graph with $p = q$ to the existence of a graceful labeling of the graph $G \vee K_n^c$. All throughout this paper, u_i, $1 \leq i \leq p$, will be a vertex of G and v_j, $1 \leq j \leq n$, will be a vertex of K_n^c.

Theorem M If a graph G with $p = q$ has a special labeling, then the graph $G \vee K_n^c$ is graceful, $p \geq 3, n \geq 1$.

Proof Suppose f is a special labeling of G. Define a vertex labeling g on $G \vee K_n^c$ as follows:

$$g(v_j) = j - 1, \quad 1 \leq j \leq n$$

and for $1 \leq i \leq p$,

$$\begin{aligned} g(u_i) &= i(n+1) \quad \text{if} \quad f(u_i) = 2i \\ &= i(n+1) - 1 \quad \text{if} \quad f(u_i) = 2i-1. \end{aligned}$$

We claim that g is a graceful labeling of $G \vee K_n^c$. Clearly,

$$g : V(G \vee K_n^c) \longrightarrow [0, p(n+1)].$$

It is obvious that g is injective. We prove $g^*(E(G \vee K_n^c)) = [1, p(n+1)]$ by showing that for each $i, 1 \leq i \leq p$ and for each k, $1 \leq k \leq n+1$, there is an edge e such that $g^*(e) = (i-1)(n+1) + k$.

Consider such a pair (i,k). Since f is a special labeling of G, by Condition 1, there is a vertex u_i on G such that $f(u_i)$ is either $2i-1$ or $2i$.

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Case 1 $f(u_i) = 2i - 1$ and $1 \le k \le n$.

Here $g(u_i) = i(n+1) - 1$. We have $g(v_{n-k+1}) = n - k$. Clearly $n - k < i(n+1) - 1$ and hence the weignt of edge $u_i v_{n-k+1}$ is $i(n+1) - 1 - n + k = (i-1)(n+1) + k$.

Case 2 $f(u_i) = 2i - 1$ and $k = n + 1$.

By Condition 2, there is an edge $e_i = xy$ on G such that $f^*(xy) = 2i$. But then both $f(x), f(y)$ are odd or both of them are even. Suppose $f(x) = 2a, f(y) = 2b$. Then $2i = f^*(xy) = | 2a - 2b |$ giving $i = | a - b |$. But then $g^*(e_i) = g^*(xy) =$
$| b(n+1) - a(n+1) | = (n+1) | a - b | = i(n+1) = (i-1)(n+1) + (n+1)$. As $k = n + 1, e_i$ is a required edge. Similarly, when both $f(x)$ and $f(y)$ are odd, we get e_i to be a required edge.

Case 3 $f(u_i) = 2i$ and $2 \le k \le n + 1$.

Here $g(u_i) = i(n+1)$ and $g^*(u_i v_{n-k+2}) = (i-1)(n+1) + k$ as $g(v_{n-k+2}) = n - k + 1 < i(n+1) = g(u_i)$.

Case 4 $f(u_i) = 2i$ and $k = 1$.

Let $e_i = xy$ be the edge on G such that $f^*(e_i) = 2i - 1$. Now one of $f(x), f(y)$ is odd, say $f(x)$ is odd. By Condition 3, $f(x) < f(y)$. Let $f(x) = 2a - 1, f(y) = 2b$. Then $f^*(e_i) = f(y) - f(x)$ giving $2i - 1 = 2b - (2a - 1)$. So $i - 1 = b - a$. Also, as $f(x) < f(y), g(x) < g(y)$ and hence

$$\begin{aligned} g^*(e_i) &= g(y) - g(x) \\ &= b(n+1) - a(n+1) + 1 \\ &= (b-a)(n+1) + 1 \\ &= (i-1)(n+1) + 1. \end{aligned}$$

As $k = 1$, e_i is a required edge.

Thus we have proved that $g^*\,(E(G \vee K_n^c)) = [1, p(n+1)]$ and therefore g is a graceful labeling of $G \vee K_n^c$. ■

3. GRACEFULNESS OF $C_{24m} \vee K_n^c$

By Theorem M, in order to establish that $C_{24m} \vee .K_n^c$ is graceful, $m \ge 1, n \ge 1$, it is enough to show that C_{24m} has a special labeling. We obtain a special labeling of C_{24m} by constructing suitable labeling of paths of certain lengths having certain labels to their end-vertices and then joining these labeled paths to form a specially labeled C_{24m}.

Let P_m denote a path of order m.

LEMMA 1.1 For $m \geq 2$, P_{4m-3} has a vertex labeling f such that $f(V) = [m+2, 2m] \cup [2m+3, 3m+1] \cup [5m+1, 7m-1]$ and $f^*(E) = [2m+1, 6m-4]$ where the end vertices of P_{4m-3} receive the labels $5m+1$ and $7m-1$.

PROOF Let $P_{4m-3} = w_1 w_2 \cdots w_{4m-3}$. Define a vertex labeling f as follows:

$$\begin{aligned} f(w_{2i-1}) &= 5m+i \quad \text{for} \quad 1 \leq i \leq 2m-1; \\ \text{and} \qquad f(w_{2i}) &= m+2 \quad \text{for} \quad i=1 \\ &= 3m+3-i \quad \textit{for} \quad 2 \leq i \leq m \\ &= 3m+1-i \quad \text{for} \quad m+1 \leq i \leq 2m-2. \end{aligned}$$

It can be directly verified that $f^*(E) = [2m+1, 6m-4]$. Thus f is a required vertex labeling. ■

REMARK Lemma 1.1 is used later on and whenever we have $m = 1$ we give label 6 to the unique vertex of P_1.

LEMMA 1.2 For any $m \geq 1, P_{8m-1}$ has a vertex labeling f such that $f(V) = [1, m] \cup [m+2, 8m]$ and $f^*(E) = [1, 8m-2]$ where the end labels are $2m+1$ and $8m$.

PROOF Let $P_{8m-1} = w_1 w_2 \cdots w_{8m-1}$. Define a vertex labeling f as follows:

$$\begin{aligned} f(w_1) &= 2m+1; \\ f(w_{2i+1}) &= 4m+1+i \quad \text{for} \quad 1 \leq i \leq m; \\ f(w_{2i}) &= 4m+2-i \quad \text{for} \quad 1 \leq i \leq m; \\ f(w_{8m+1-2i}) &= 8m+1-i \quad \text{for} \quad 1 \leq i \leq m+2; \\ f(w_{8m-2}) &= 2m+2; \\ \text{and} \qquad f(w_{8m-2-2i}) &= i \quad \text{for} \quad 1 \leq i \leq m. \end{aligned}$$

Thus we have used the labels $[1, m] \cup [2m+1, 2m+2] \cup [3m+2, 5m+1] \cup [7m-1, 8m]$ and obtained the edge labels $[1, 2m] \cup [6m-3, 8m-2]$. Therefore, it is enough to label the remaining sub path P_{4m-3} using the labels $[m+2, 2m] \cup [2m+3, 3m+1] \cup [5m+1, 7m-1]$ to obtain the edge labels $[2m+1, 6m-4]$. Lemma 1.1 and the subsequent Remark says that this can be done. ■

LEMMA 1.3 For any $m \geq 1, P_{8m-1}$ has a vertex labeling g· such that $g(V) =$
$\{16m+2, 16m+4, \cdots, 18m\} \cup \{18m+4, 18m+6, \cdots, 32m\}$ and $g^*(E) =$
$\{2, 4, 6, \cdots,$
$16m-4\}$ where the end labels are $20m+2$ and $32m$..

PROOF Let f be a vertex labeling defined on P_{8m-1} as in Lemma 1.2. Then a required vertex labeling g is defined by $g(w) = 2f(w) + 16m$ for all $w \in P_{8m-1}$. ■

LEMMA 1.4 For any $m \geq 1, P_{16m+3}$ has a vertex labeling f such that $f(V) = \{1, 3, 5, \cdots, 16m-1; 18m+2; 20m+2; 32m, 32m+2, \cdots, 48m\}$ and $f^*(E) =$
$\{16m-2, 16m; 16m+1, 16m+3, \cdots, 48m-1\}$ where the end labels are $20m+2$ and $32m$.

PROOF Let $P_{16m+3} = w_1 w_2 \cdots w_{16m+3}$. Define f as follows:

$$\begin{aligned} f(w_{2i-1}) &= 20m+2 \quad \text{for} \quad i = 1 \\ &= 48m+4-2i \quad \text{for} \quad 2 \leq i \leq 8m+2 \\ \text{and} \qquad f(w_{2i}) &= 2i-1, \quad \text{for} \quad 1 \leq i \leq 7m \\ &= 18m+2 \quad \text{for} \quad i = 7m+1 \\ &= 2i-3, \quad \text{for} \quad 7m+2 \leq i \leq 8m+1. \end{aligned}$$

It can be directly verified that f is a required vertex labeling. ■

PROPOSITION 1 C_{24m} has a special labeling for any $m \geq 1$.

PROOF Consider the paths P_{8m-1} and P_{16m+3} with the corresponding labels defined in Lemmas 1.3 and 1.4 respectively. Concatenating the end vertices having the same labels, we have a vertex labeling f of C_{24m} such that $f(V) = \{1, 3, 5, \cdots,$
$16m-1; 16m+2, 16m+4, \cdots, 48m\}$ and $f^*(E) = \{2, 4, 6, \cdots, 16m; 16m+$
$1,$
$16m+3, \cdots, 48m-1\}$. Therefore, $f^*(E) = [1, 48m] \backslash f(V)$ and for each $i, 1 \leq i \leq 24m, f(V)$ contains either $2i-1$ or $2i$. So, Conditions 1 and 2 of special labeling are clearly satisfied. Also, in $f(V)$ the largest odd label is

less than the smallest even label and hence Condition 3 of special labeling is satisfied. Thus, f is a special labeling for C_{24m}. ■

THEOREM 1 $C_{24m} \vee K_n^c$ is graceful for any $m \geq 1$ and $n \geq 1$.

PROOF This follows from Theorem M and Proposition 1. ■

4. GRACEFULNESS OF $C_{24m+3} \vee K_n^c$

As in the case of C_{24m}, we give a special labeling of C_{24m+3} by constructing suitable labelings of paths.

LEMMA 2.1 For any $m \geq 2, P_{4m-3}$ has a vertex labeling g such that $g(V) = [m+2, 2m] \cup [2m+3, 3m+1] \cup [5m+2, 7m]$ and $g^*(V) = [2m+2, 6m-3]$ where the end labels are $5m+2$ and $7m$.

PROOF Consider the vertex labeling f of P_{4m-3} defined in Lemma 1.1. Define g as follows:

$$g(w_{2i-1}) = f(w_{2i-1}) + 1 \quad \text{for} \quad 1 \leq i \leq 2m-1$$

and

$$g(w_{2i}) = f(w_{2i}) \quad \text{for} \quad 1 \leq i \leq 2m-2.$$

Then g is a required vertex labeling of P_{4m-3}. ■

REMARK For $m = 1$, give label 7 to the unique vertex of P_1 when the above lemma is used subsequently.

LEMMA 2.2 For any $m \geq 1, P_{8m}$ has a vertex labeling g such that $g(V) = [1, m] \cup [m+2, 8m+1]$ and $g^*(E) = [1, 8m-1]$ where the end labels are $2m+1$ and $8m+1$.

PROOF Let $P_{8m} = w_1 w_2 \cdots w_{8m}$. Define a vertex labeling g on P_{8m} as follows:

$$\begin{aligned} g(w_1) &= 2m+1; \\ g(w_{2i}) &= 4m+1+i \quad \text{for} \quad 1 \le i \le m+1; \\ g(w_{2i+1}) &= 4m+2-i \quad \text{for} \quad 1 \le i \le m; \\ g(w_{6m-4+2i}) &= 7m-1+i \quad \text{for} \quad 1 \le i \le m+2; \\ g(w_{6m-3+2i}) &= m+1-i \quad \text{for} \quad 1 \le i \le m; \\ \text{and} \qquad g(w_{8m-1}) &= 2m+2. \end{aligned}$$

Thus we have used the vertex labels $[1, m] \cup \{2m+1, 2m+2\} \cup [3m+2, 5m+2] \cup [7m, 8m+1]$ and obtained the edge labels $[1, 2m+1] \cup [6m-2, 8m-1]$. Therefore, it is enough to label the remaining sub path P_{4m-3} (with end labels $5m+2$ and $7m$) using the labels $[m+2, 2m] \cup [2m+3, 3m+1] \cup [5m+2, 7m]$ to obtain the edge labels $[2m+2, 6m-3]$. This follows from Lemma 2.1 and the subsequent Remark. ∎

LEMMA 2.3 For any $m \ge 1, P_{8m}$ has a vertex labeling f such that $f(V) = \{16m+4, 16m+6, \cdots, 18m+2; 18m+6, 18m+8, \cdots, 32m+4\}$ and $f^*(E) = \{2, 4, 6, \cdots, 16m-2\}$ where the end labels are $20m+4$ and $32m+4$.

PROOF Consider the vertex labeling g of P_{8m} defined in Lemma 2.2. Define a new vertex labeling f as $f(w) = 2g(w)+16m+2$ for any vertex w of P_{8m}. It is easy to check that f is a required vertex labeling of P_{8m}. ∎

LEMMA 2.4 For any $m \ge 1, P_{16m+5}$ has a vertex labeling f such that $f(V) = \{1, 3, 5, \cdots, 16m+1; 18m+4; 20m+4; 32m+4, 32m+6, \cdots, 48m+6\}$ and $f^*(E) = \{16m, 16m+2; 16m+3, 16m+5, \cdots, 48m+5\}$ where the end labels are $20m+4$ and $32m+4$.

PROOF Let $P_{16m+5} = w_1 w_2 \cdots w_{16m+5}$. Define f as follows:

$$\begin{aligned} f(w_1) &= 20m+4; \\ f(w_{2i}) &= 2i-1 \quad \text{for} \quad 1 \le i \le 7m+1; \\ f(w_{14m+4}) &= 18m+4; \\ f(w_{14m+4+2i}) &= 14m+1+2i \quad \text{for} \quad 1 \le i \le m; \\ f(w_{2i+1}) &= 48m+8-2i \quad \text{for} \quad 1 \le i \le 8m+2. \end{aligned}$$

Clearly, f is a required vertex labeling. ∎

PROPOSITION 2 C_{24m+3} has a special labeling for any $m \ge 0$.

PROOF When $m = 0$, give labels $1, 4, 6$ to the vertices of C_3.

When $m \geq 1$, consider the paths P_{8m} and P_{16m+5} with the corresponding labels defined in Lemmas 2.3 and 2.4 respectively. Obtain C_{24m+3} by concatenating their end vertices having the same labels. This gives a special labeling of C_{24m+3}. ■

THEOREM 2 $C_{24m+3} \vee K_n^c$ is graceful for any $m \geq 0, n \geq 1$. ■

5. GRACEFULNESS OF $C_{24m-9} \vee K_n^c$

As in the earlier cases, here also we prove the gracefulness of $C_{24m-9} \vee K_n^c$ by showing the existence of a special labeling for C_{24m-9}.

LEMMA 3.1 For any $m \geq 3, P_{4m-5}$ has a vertex labeling f such that $f(V) = [m+2, 2m] \cup [2m+3, 3m-1] \cup \{3m+1\} \cup \{5m-1, 7m-4\}$ and $f^*(E) = [2m, 6m-7]$ where the end labels are $5m-1$ and $7m-4$.

PROOF Let $P_{4m-5} = w_1 w_2 \cdots w_{4m-5}$. Define a vertex labeling f on P_{4m-5} by

$$\begin{aligned} f(w_{2i-1}) &= 7m-3-i \quad \text{for} \quad 1 \leq i \leq 2m-2 \\ \text{and} \quad f(w_{2i}) &= 3m+1 \quad \text{for} \quad i = 1 \\ &= m+i \quad \text{for} \quad 2 \leq i \leq m \\ &= m+2+i \quad \text{for} \quad m+1 \leq i \leq 2m-3. \end{aligned}$$

Then it is easy to check that f is a required vertex labeling. ■

LEMMA 3.2 For any $m \geq 3, P_{2m-1}$ has a vertex labeling f such that $f(V) = \{3m\} \cup [3m+2, 5m-1]$ and $f^*(E) = [1, 2m-2]$ where the end labels are $4m$ and $5m-1$.

PROOF We prove the result by induction on m. When $m = 3$, label the vertices of P_5 with the labels 14, 11, 9, 13, 12 in order and when $m = 4$, label the vertices of P_7 with the labels 19, 14, 18, 12, 15, 17, 16 in order.

Assume that the result is true for $m-2, m \geq 5$. That is, the vertices of P_{2m-5} can be labeled with the labels $\{3m-6\} \cup [3m-4, 5m-11]$ such that the edge induced labels are $[1, 2m-6]$ and the end vertex labels are $4m-8$ and $5m-11$. This implies that by adding 8 to each label, the vertices of P_{2m-5} get labeled with the labels $\{3m+2\} \cup [3m+4, 5m-3]$ such that the edge induced labels are $[1, 2m-6]$ and the end vertex labels are $4m$ and $5m-3$. With such a labeled path P_{2m-5} concatenate a path P_5 in which the labels $5m-1, 3m+3, 5m-2, 3m, 5m-3$ are in order. This results a vertex labeling f of P_{2m-1} such that $f(V) = \{3m\} \cup [3m+2, 5m-1]$ and $f^*(E) = [1, 2m-2]$ where the end labels are $4m$ and $5m-1$ and hence the result is true for any $m \geq 3$. ■

If we add a new vertex with the label $2m+1$ and make it adjacent to the vertex of P_{2m-1} that has received the label $4m$, we get

LEMMA 3.3 For any $m \geq 3, P_{2m}$ has a vertex labeling f such that $f(V) = \{2m+1\} \cup \{3m\} \cup [3m+2, 5m-1]$ and $f^*(E) = [1, 2m-1]$ where the end labels are $2m+1$ and $5m-1$. ■

LEMMA 3.4 For $m \geq 2$, P_{6m-6} has a vertex labeling f such that $f(V) = [m+2, 2m+1] \cup [2m+3, 7m-4]$ and $f^*(E) = [1, 6m-7]$ where the end labels are $2m+1$ and $7m-4$.

PROOF For $m = 2$, label the vertices of P_6 with the labels 10, 8, 7, 4, 9, 5 in order. For any $m \geq 3$, concatenating the paths P_{4m-5} and P_{2m} together with the labels obtained in Lemmas 3.1 and 3.3 at the vertex having the same label, we get a required vertex labeling of P_{6m-6}. ■

LEMMA 3.5 For any $m \geq 2, P_{2m+3}$ has a vertex labeling f such that $f(V) = [1, m] \cup \{2m+2\} \cup [7m-4, 8m-3]$ and $f^*(E) = [6m-6, 8m-5]$ where the end labels are $8m-3$ and $7m-4$.

PROOF Let $P_{2m+3} = w_1 w_2 \cdots w_{2m+3}$. Define a vertex labeling f on P_{2m+3} by

$$\begin{aligned} f(W_{2i-1}) &= 8m-2-i \quad \text{for} \quad 1 \leq i \leq m+2 \\ \text{and} \qquad f(w_{2i}) &= 2m+2 \quad \text{for} \quad i = 1 \\ &= i-1 \quad \text{for} \quad 2 \leq i \leq m+1. \end{aligned}$$

One can easily check that f is a required vertex labeling. ■

LEMMA 3.6 For any $m \geq 1, P_{8m-4}$ has a vertex labeling f such that $f(V) = [1, m] \cup [m+2, 8m-3]$ and $f^*(E) = [1, 8m-5]$ where the end labels are $2m+1$ and $8m-3$.

PROOF When $m = 1$, label the vertices of P_4 with the labels 1, 4, 3, 5 in order.

When $m \geq 2$, the result follows from Lemmas 3.4 and 3.5 by concatenating the end vertices having the same label. ■

By replacing each label $f(w)$ with $2f(w) + 16m - 6$, the above lemma implies the following.

LEMMA 3.7 For any $m \geq 1, P_{8m-4}$ has a vertex labeling f such that $f(V) = \{16m-4, 16m-2, \cdots, 18m-6; 18m-2, 18m, \cdots, 32m-12\}$ and $f^*(E) = \{2, 4, 6, \cdots, 16m-10\}$ where the end labels are $20m-4$ and $32m-12$. ■

LEMMA 3.8 For any $m \geq 1, P_{16m-3}$ has a vertex labeling f such that $f(V) = \{1, 3, 5, \cdots, 16m-7; 18m-4; 20m-4; 32m-12, 32m-10, \cdots, 48m-18\}$ and $f^*(E) = \{16m-8, 16m-6; 16m-5, 16m-3, \cdots, 48m-19\}$ where the end labels are $20m-4$ and $32m-12$.

PROOF Let $P_{16m-3} = w_1 w_2 \cdots w_{16m-3}$. Define a vertex labeling f on P_{16m-3} by

$$\begin{aligned} f(w_1) &= 20m-4; \\ f(w_{2i}) &= 2i-1 \quad \text{for} \quad 1 \leq i \leq 7m-3 \\ &= 18m-4 \quad \text{for} \quad i = 7m-2 \\ &= 2i-3 \quad \text{for} \quad 7m-1 \leq i \leq 8m-2 \\ \text{and } f(w_{2i+1}) &= 48m-16-2i \quad \text{for} \quad 1 \leq i \leq 8m-2. \end{aligned}$$

It is easy to check that f is a required vertex lebeling of P_{16m-3}. ■

PROPOSITION 3 C_{24m-9} has a special labeling for any $m \geq 1$.

PROOF Follows from Lemmas 3.7 and 3.8. ■

THEOREM 3 $C_{24m-9} \vee K_n^c$ is graceful for any $m, n \geq 1$. ■

6. GRACEFULNESS OF $C_{24m-12} \vee K_n^c$

Our proof technique is similar to that used in the earlier sections.

LEMMA 4.1 For any $m \geq 4, P_{2m-4}$ has a graceful labeling f such that $f(V) = [0, 2m-5]$ and $f^*(E) = [1, 2m-5]$ where $2m-6$ and $m-4$ are the end labels.

PROOF Our proof is by induction on m. When $m = 4$, label the vertices of P_4 with the labels 2, 1, 3, 0 in order. When $m = 5$, label the vertices of P_6 with labels 4, 0, 5, 2, 3, 1 in order.

Assume that the result is true for $m-2, m \geq 6$. Then P_{2m-8} can be labeled with the labels $[0, 2m-9]$ and the edge induced labels are $[1, 2m-9]$ where the end labels are $2m-10$ and $m-6$. Adding 2 to each vertex label, we get a vertex labeling of P_{2m-8} in which the vertex labels are from the set $[2, 2m-7]$ and the edge induced labels are from $[1, 2m-9]$ where the end labels are $2m-8$ and $m-4$.

Concatenating a path P_5 with the labels $2m-6, 1, 2m-7, 0, 2m-8$ in order, with the above labeled path P_{2m-8} at the vertex which has received the label $2m-8$, we get a vertex labeling f of P_{2m-4} such that $f(V) = [0, 2m-5]$ and $f^*(E) = [1, 2m-5]$ where $2m-6$ and $m-4$ are the end labels. Thus f is a required vertex labeling. ■

The above lemma can be restated as follows.

LEMMA 4.2 For any $m \geq 4, P_{2m-4}$ has a vertex labeling f such that $f(V) = [3m+2, 5m-3]$ and $f^*(E) = [1, 2m-5]$ where $5m-4$ and $4m-2$ are the end labels.

■

LEMMA 4.3 For any $m \geq 3, P_{4m-7}$ has a vertex labeling f such that $f(V) = \{2m+1\} \cup [2m+3, 3m] \cup [3m+2, 6m-5]$ and $f^*(E) = [1, 4m-8]$ where $2m+1$ and $6m-5$ are the end labels.

PROOF When $m = 3$, label the vertices of P_5 with the labels 13, 12, 9, 11, 7 in order.

When $m \geq 4$, label the first two vertices of P_{4m-7} with $2m+1$ and $4m-2$ in order and the last $2m-3$ vertices of P_{4m-7} with $5m-4, 3m, 5m-2, 3m-1, 5m-1, 3m-2, 5m, 3m-3, \cdots, 6m-6, 2m+3, 6m-5$ in order.

Now it is enough to label the remaining path P_{2m-4} with the labels $[3m+2, 5m-3]$ to obtain the edge labels $[1, 2m-5]$ where $5m-4$ and

$4m-2$ are the end labels of P_{2m-4}. This can be done as stated in Lemma 4.2.

Hence the proof. ■

LEMMA 4.4 For any $m \geq 3, P_{4m+3}$ has a vertex labeling f such that $f(V) = [1,m] \cup [m+2,2m] \cup \{2m+2, 3m+1\} \cup [6m-5, 8m-4]$ and $f^*(E) = [4m-7, 8m-6]$ where the end labels are $8m-4$ and $6m-5$.

PROOF Let $P_{4m+3} = w_1 w_2 \cdots w_{4m+3}$. Define a vertex labeling f on P_{4m+3} by

$$\begin{aligned} f(w_{2i-1}) &= 8m-3-i \quad \text{for} \quad 1 \leq i \leq 2m+2; \\ \text{and} \qquad f(w_{2i}) &= 2m+2 \quad \text{for} \quad i=1 \\ &= i-1 \quad \text{for} \quad 2 \leq i \leq m+1 \\ &= 3m+1 \quad \text{for} \quad i=m+2 \\ &= i-1 \quad \text{for} \quad m+3 \leq i \leq 2m+1. \end{aligned}$$

This f is a required vertex labeling. ■

LEMMA 4.5 For any $m \geq 2, P_{8m-5}$ has a vertex labeling f such that $f(V) = [1,m] \cup [m+2, 8m-4]$ and $f^*(E) = [1, 8m-6]$ where the end labels are $2m+1$ and $8m-4$.

PROOF When $m=2$, label the vertices of P_{11} by the labels 12, 6, 11, 1, 10, 2, 9, 8, 4, 7, 5 in order. When $m \geq 3$, the result follows from Lemmas 4.3 and 4.4 by concatenating the vertices that have received the label $6m-5$.

■

The above lemma can be rewritten as

LEMMA 4.6 For any $m \geq 2, P_{8m-5}$ has a vertex labeling f such that $f(V) = \{16m-6, 16m-4, \cdots, 18m-8; 18m-4, 18m-2, \cdots, 32m-16\}$ and $f^*(E) = \{2, 4, 6, \cdots, 16m-12\}$ where $20m-6$ and $32m-16$ are the end labels. ■

LEMMA 4.7 For any $m \geq 2, P_{16m-5}$ has a vertex labeling f such that $f(V) = \{1, 3, 5, \cdots, 16m-9;\ 18m-6;\ 20m-6;\ 32m-16, 32m-14, \cdots, 48m-24\}$ and $f^*(E) = \{16m-10, 16m-8; 16m-7, 16m-5, \cdots, 48m-25\}$ where the end labels are $20m-6$ and $32m-16$.

PROOF Let $P_{16m-5} = w_1 w_2 \cdots w_{16m-5}$. Define a vertex labeling f on

P_{16m-5} by

$$f(w_{2i-1}) = 20m-6 \quad \text{for} \quad i=1$$
$$= 48m-20-2i \quad \text{for} \quad 2 \le i \le 8m-2$$

and

$$f(w_{2i}) = 2i-1 \quad \text{for} \quad 1 \le i \le 7m-4$$
$$= 18m-6 \quad \text{for} \quad i=7m-3$$
$$= 2i-3 \quad \text{for} \quad 7m-2 \le i \le 8m-3.$$

This f is a required vertex labeling. ■

PROPOSITION 4 C_{24m-12} has a special labeling for any $m \geq 1$.

PROOF When $m = 1$, label the vertices of C_{12} with the labels

$$1, 24, 3, 22, 5, 20, 7, 16, 10, 18, 14, 12$$

in order which gives a special labeling of C_{12}.

When $m \geq 2$, the result follows from Lemmas 4.6 and 4.7 by concatenating the vertices with the same labels. ■

THEOREM 4 $C_{24m-12} \vee K_n^c$ is graceful for any $m, n \geq 1$. ■

7. CONCLUSIONS

In Theorems 1 to 4, we have proved that $C_{24m} \vee K_n^c$, $C_{24m+3} \vee K_n^c$, $C_{24m-9} \vee K_n^c$ and $C_{24m-12} \vee K_n^c$ are graceful. Consequently we have proved the following.

THEOREM 5 $C_m \vee K_n^c$ is graceful for $m \equiv 0, 3 (mod\ 12), m \geq 3$ and $n \geq 1$. ■

We remark that for $n = 1$, our graceful labelings (for wheels) are different from those given by Hoede and Kuiper[3]. Regarding the nongracefulness of the n-cone, for even values of n and $m \equiv 2(mod\ \ 4)$, the n-cones $C_m \vee K_n^c$ cannot be graceful by parity condition of Rosa[4] which states that any graph with all vertices of even degree cannot be graceful when its size $q \equiv 1 or 2\ (mod\ 4)$. Thus we have the following result.

THEOREM 6 $C_m \vee K_n^c$ is not graceful for n even and $m \equiv 2, 6, 10 (mod\ 12)$. ■

We list below some further results on n-cones with short proofs.

PROPOSITION 5 The n-cones $C_7 \vee K_n^c, C_{11} \vee K_n^c$ and $C_{19} \vee K_n^c$ are graceful, $n \geq 1$.

PROOF We give a special labeling for C_7 and C_{11} and four special labelings for C_{19}.

C_7 : 1 14 5 7 10 4 12

C_{11} : 1 22 5 18 7 15 9 12 14 4 20

C_{19} :

(a) 1 36 3 34 5 32 7 30 12 26 16 22 20 24 13 28 9 17 38

(b) 1 36 3 34 5 32 7 30 12 26 16 22 20 28 9 24 13 17 38

(c) 1 36 3 34 5 32 7 30 12 28 9 22 14 24 20 26 15 17 38

(d) 1 36 4 31 8 28 13 24 17 19 22 16 26 12 30 9 34 5 38

■

PROPOSITION 6 The n-cone $C_4 \vee K_n^c$ is graceful, $n \geq 1$.

PROOF Here we give directly a graceful labeling of $C_4 \vee K_n^c, n \geq 1$. Give labels $1, 2, \cdots, n$ to the n vertices of K_n^c; give labels $0, 3(n+1)$, $2(n+1), 4(n+1)$ cyclically to the four vertices of C_4. ■

PROPOSITION 7 The doubel cones $C_5 \vee K_2^c$ and $C_9 \vee K_2^c$ are graceful.

PROOF In both the cases we directly give graceful labelings.

For $C_5 \vee K_2^c$, label the vertices of K_2^c with 1 and 3 and label the vertices of C_5 with $0, 15, 5, 14, 8$ in order.

In $C_9 \vee K_2^c$, allot the labels 1 and 3 for the vertices of K_2^c and give labels $0, 27, 17, 26, 21, 9, 16, 5, 22$ in cyclic order to the nine vertices of C_9. ■