Domination Graphs of Tournaments with Isolated Vertices

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Abstract. The domination graph dom(D) of a digraph D has the same vertex set as D, and $\{u, v\}$ is an edge if and only if for every w, either (u, w) or (v, w) is an arc of D. In earlier work we have shown that if G is a domination graph of a tournament, then G is either a forest of caterpillars or an odd cycle with additional pendant vertices or isolated vertices. We have also earlier characterized those connected graphs and forests of non-trivial caterpillars that are domination graphs of tournaments. We complete the characterization of domination graphs of tournaments by describing domination graphs with isolated vertices.

Keywords: domination graph, tournament, competition graph

Preliminaries. A tournament is an oriented complete graph. Let $V(D)$ and $A(D)$ denote the vertex and arc sets of a digraph D respectively. An arc from vertex x to y is denoted by (x, y). If D is a digraph and $(x, y) \in A(D)$, we say x beats y. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G respectively. An edge between vertices x and y is denoted by $\{x, y\}$.

Vertices x and y dominate a tournament T if for all vertices $z \neq x, y$ either x beats z or y beats z. We call such vertices x and y a *dominating pair*. The domination graph of a tournament T, denoted dom(T), is the graph on vertices $V(T)$ with $\{x, y\} \in E(\text{dom}(T))$ if and only if x and y dominate T (see Figure 1).

Domination graphs were introduced by Fisher et al. [6] in conjunction with competition graphs. The competition graph of a digraph D is the graph on the same vertices as D with an edge between two vertices if they beat a common vertex in D. The domination graph of a tournament is the complement of the competition graph of its reversal (see [6]). See Lundgren

[11] or Roberts [14] for more about competition graphs and Moon [12] or Reid [13] for more on tournaments.

The domination digraph $\mathcal{D}(T)$ of a tournament T is the digraph with the same vertices as T in which vertex x beats vertex y if x and y dominate T and x beats y in T. Thus, $\mathcal{D}(T)$ is the orientation of dom(T) induced by T (see Figure 1).

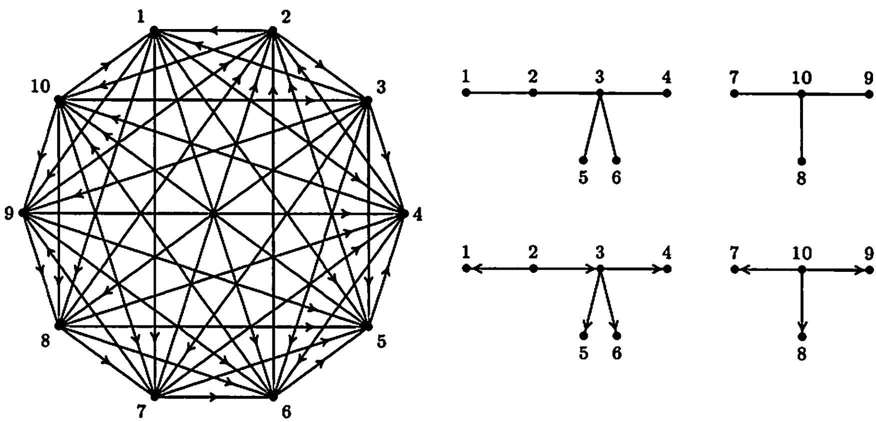


Figure 1 A tournament with its domination graph and domination digraph

In this paper we extend previous work on domination graphs, appearing in [5], [6], [7], and [8]. A vertex of a graph is *pendant* if it has exactly one neighbor. A *caterpillar* is a connected graph whose nonpendant vertices, if any, form a (possibly trivial) path. Any caterpillar may be pictured as a path, called the *spine*, with additional pendant vertices adjacent to the interior vertices of the path. (Many authors define the spine to be the graph that results when all vertices of degree 1 are removed; our definition is more convenient here.) We refer (with some abuse of notation) to the pendant vertices and pendant edges attached to an interior vertex of the spine as *legs*; all legs attached to a single vertex are called a *leg cluster*. A typical caterpillar is shown in Figure 2. We assume that all caterpillars are represented in this way, with the spine drawn horizontally and all legs below it. We may thus refer to vertices as being to the left or right, or up or down, compared to other vertices. In particular, we say that a leg vertex v is to the left of any spine vertex w to the right of the spine vertex to which v is attached, and also v is to the left of any leg vertex adjacent to w; we use "right" similarly. A *star*, denoted $K_{1,n}$, is a caterpillar with exactly one vertex adjacent to the other n vertices. We say that a caterpillar has a *triple end* if at least one vertex adjacent to an endpoint of the spine has degree at least four. Observe that for $n \geq 4$, $K_{1,n}$ has a triple end, but $K_{1,3}$ does not. A *spiked cycle* is a connected graph whose nonpendant vertices form a cycle.

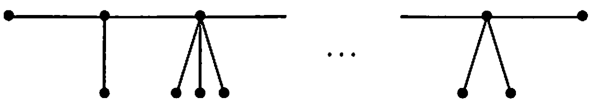


Figure 2 A typical caterpillar

Fisher et al. [6] determined necessary conditions for a graph to be the domination graph of a tournament, and in [8] extended this to a complete characterization of domination graphs without isolated vertices.

THEOREM 1 [8, 6] If G is the domination graph of a tournament, then either G consists of a spiked odd cycle with zero or more isolated vertices, or the components of G are caterpillars. If G is the union of n nontrivial components, then G is the domination graph of a tournament if and only if

$n = 1$ and G is a spiked odd cycle, a star or a caterpillar with a triple end

or

$n \geq 2$, each component is a caterpillar and one of the following holds:

- $n = 2$ and both caterpillars have a triple end or one has a triple end and the other is $K_{1,3}$;
- $n = 3$ and all three caterpillars have a triple end, or one caterpillar is $K_{1,3}$ and the other two have a triple end, or two caterpillars are $K_{1,3}$ and the other one has a triple end;
- $n = 5$ and at least one of the caterpillars has a triple end or is a $K_{1,3}$;
- $n = 4$ or $n \geq 6$.

In this paper we complete the characterization of graphs that are the domination graph of a tournament by extending this result to include trivial components. Cho et al [2, 3] have characterized graphs that are the domination graph of a regular tournament. Jimenez [9] has several results on the domination graphs of near-regular tournaments, but several open problems remain. Two graphs related to competition graphs are niche graphs and competition-resource graphs. For a tournament, the mixed pair graph is the complement of the niche graph, and these graphs have been characterized by Bowser et al. [1]. The domination-compliance graph is the complement of the competition-resource graph, and results on these graphs can be found in Jimenez and Lundgren [10] and in Doherty and Lundgren [4], but open problems remain.

The General Case. We now consider those graphs G that are domination graphs of tournaments and that have one or more isolated vertices. If G is a graph, denote by $i(G)$ the smallest positive integer n such that G plus n isolated vertices is the domination graph of a tournament; if there is no such n, $i(G) = \infty$.

PROPOSITION 2 If G is the domination graph of a tournament in which no vertex has indegree 0, then $i(G) = 1$.

Proof. Let such a tournament be T. Add a new vertex v, and direct all arcs to the new vertex forming T'. Then it is easy to check that $\text{dom}(T') = G \cup \{v\}$. ∎

COROLLARY 3 If G is not connected and G is the domination graph of a tournament, then $i(G) = 1$.

COROLLARY 4 Let $i(G) = n < \infty$. Then for any $m \geq n$, G plus m isolated vertices is the domination graph of a tournament.

This corollary implies that to characterize domination graphs with isolated vertices, it is sufficient to know $i(G)$ for all graphs G.

Now it follows immediately from Theorem 1 that if G consists of any n nontrivial caterpillars, where $n = 4$ or $n \geq 6$, then G together with any number of isolated vertices is the domination graph of a tournament. It is also easy to see that any spiked odd cycle G has $i(G) = 1$. Thus, we need only determine $i(G)$ when G consists of a single caterpillar or consists of 2, 3 or 5 nontrivial caterpillars. (We need not consider forests of caterpillars containing trivial components, except the trivial forest K_1. If G is a forest of caterpillars with k trivial components, and G' is the forest of nontrivial components of G, then $i(G)$ is 1 or $i(G') - k$, whichever is greater, and in neither case do we learn anything new from $i(G)$ about which graphs are domination graphs.)

Following a few more preliminary results, we prove that any forest G of 3 or 5 nontrivial caterpillars has $i(G) = 1$. Most forests G of two nontrivial caterpillars have $i(G) = 1$ as well, but we get this result only by considering three cases. Most single caterpillars G also have $i(G) = 1$, but again three cases are required. Finally, there are nine forests G of one or two caterpillars which have $i(G) > 1$; we treat these individually. We summarize these results in a theorem.

THEOREM 5 Suppose G is a forest of nontrivial caterpillars, or $G = K_1$. Then $i(G) = 1$, except that $i(P_4) = 3$, $i(K_1) = 6$, $i(K_2) = 6$, $i(K_{1,2}) = 4$, $i(K_{1,3}) = 3$, $i(K_{1,2} \cup K_{1,2}) = 2$, $i(K_{1,2} \cup K_2) = 2$, $i(K_2 \cup K_2) = 4$, and $i(K_{1,3} \cup K_2) = 2$.

Suppose that G is the union of nontrivial caterpillars $G_1, \ldots, G_n$. It will be useful to *properly color* such graphs with two colors, say red and blue, where, as usual, a proper coloring is an assignment of one color to each vertex such that if two vertices are adjacent, then they are assigned different colors. Each G_i can be so colored in one way (up to interchanging the two colors). For convenience, we adopt the following conventions: r_i refers to any red vertex in G_i, and similarly b_i denotes a blue vertex in G_i. Thus we may refer to an "(r_1, b_2) pair," meaning any pair of vertices consisting of a red vertex in G_1 and a blue vertex in G_2. We use Lb_i to denote the leftmost blue vertex on the spine of G_i; Rb_i to denote the rightmost blue vertex on the spine of G_i; and similarly for Rr_i.

PROPOSITION 6 [5] In the domination digraph of a tournament, a vertex loses to at most one vertex and beats at most one vertex that beats other vertices.

The next result follows from Proposition 6 and the proof of Theorem 5 in Fisher et al. [7]. If G is the domination graph of T, let T_i denote the subtournament of T induced by $V(G_i)$, and let D_i denote the digraph induced on G_i by T.

PROPOSITION 7 Suppose G, the union of n nontrivial caterpillars $G_1, \ldots, G_n$, is the domination graph of a tournament T and is properly colored with two colors. Then the following conditions hold in T:

1. D_i is as shown in Figure 3 (the leftmost edge on the spine of G_i might be directed either way as indicated by the two arrows).
2. Arcs of T_i are always directed to the right between vertices of different colors and to the left between vertices of the same color, except for arcs between vertices in the same leg cluster and arcs between a pendant vertex v at either end of the spine and the leg cluster (if any) attached to the spine vertex adjacent to v.

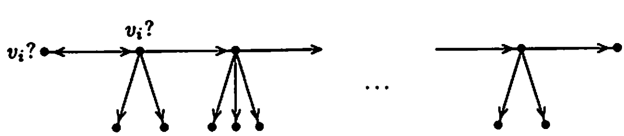


Figure 3 Orientation of a caterpillar

It now follows that the oriented caterpillar D_i has a unique vertex of indegree 0, which we call v_i. As indicated in Figure 3, this vertex will be one of the first two vertices on the spine.

PROPOSITION 8 [8] Suppose G is the union of non-trivial caterpillars $G_1, \ldots, G_n$ and is the domination graph of a tournament T. Suppose that G has been colored so that all vertices v_i have the same color. If an arc is directed from v_i to v_j, and if u and w are vertices of G_i and G_j, respectively, then there is an arc from u to w if and only if u and w are the same color.

Suppose G is the union of non-trivial caterpillars. We define a *standard tournament on* $V(G)$ as follows. In each caterpillar, orient all spine edges to the right and all leg edges down. The vertex v_i is then the leftmost vertex on the spine of the ith caterpillar. Suppose a subtournament on the v_i is specified. Color the vertices of G properly with red and blue, assigning red to all v_i. Within caterpillars, orient all arcs to the right between vertices of different colors and to the left between vertices of the same color. Finally, orient arcs between caterpillars as described in Proposition 8. It is easy to check that if T is a standard tournament on $V(G)$, then G is a subgraph of $\text{dom}(T)$, and if G consists of at least two caterpillars, each caterpillar is an induced subgraph of $\text{dom}(T)$.

The following definitions will prove useful: Two vertices u and v are *dominated* in a digraph if there is a vertex w such that (w, u) and (w, v) are arcs. Two vertices u and v are *paired* in a digraph if there is a vertex w such that (u, w) and (v, w) are arcs, or (w, u) and (w, v) are arcs. Two vertices are *distinguished* if there is a vertex w such that (u, w) and (w, v) are arcs, or (w, u) and (v, w) are arcs. A digraph is *well-covered* if every two distinct vertices u and v are paired and distinguished.

PROPOSITION 9 If G consists of five nontrivial caterpillars then $i(G) = 1$.

Proof. There is a tournament on 5 vertices in which every two vertices are distinguished, and every pair of vertices, except one, is paired; see figure 4, in which v_1 and v_2 are not paired. Let us call such a tournament *almost well-covered.*

Let G be a graph consisting of 5 nontrivial caterpillars G_i, $i = 1, \ldots, 5$, and denote the leftmost spine vertices by v_i. Add arcs among the vertices v_i to form an almost well-covered tournament; without loss of generality, suppose that (v_2, v_1) is an arc and that v_1 and v_2 are not paired. Let T_0 be the standard tournament on $V(G)$.

Add a new vertex v with arcs to v_1 and Rr_2 and arcs from all other vertices, forming tournament $T = T_0 \cup \{v\}$. It is straightforward to check that any pair of vertices from different G_i, except a pair from G_1 and G_2, is

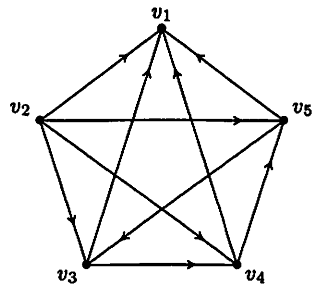


Figure 4 Almost well-covered tournament.

dominated by one of the v_i or its neighbor, Lb_i, on the spine of G_i, because the subtournament on the v_i is almost well-covered. Moreover, since v_1 and v_2 are distinguished by some third v_i, any pair of differently colored vertices from G_1 and G_2 is dominated by v_i or its neighbor, Lb_i, on the spine of G_i. An (r_1, r_2) pair is dominated by Rr_2 or Lb_1 or v; a (b_1, b_2) pair is dominated by v_1. Finally, any pair (v, w) is dominated by v_3, v_4, Lb_3, or Lb_4. Since v does not beat any two vertices that are adjacent in G, the endpoints of each edge of G are a dominating pair. Thus $G \cup \{v\}$ is the domination graph of T. ∎

PROPOSITION 10 If G consists of three nontrivial caterpillars then $i(G) = 1$.

Proof. Let G be a graph consisting of three nontrivial caterpillars G_i, $i = 1, \ldots, 3$, and denote the leftmost spine vertices by v_i. Add arcs (v_3, v_2), (v_2, v_1) and (v_1, v_3), and then let T_0 be the standard tournament on $V(G)$. Add a new vertex v with arcs to all v_i and Rr_i and from all other vertices, forming $T = T_0 \cup \{v\}$.

Now any pair of vertices of different colors on two different caterpillars is dominated by v_i or Lb_i on the third caterpillar. An (r_1, r_2) pair is dominated by Rr_2 or Lb_1 or v, and a (b_1, b_2) pair is dominated by v_1; similarly for pairs (r_2, r_3), (r_3, r_1), (b_2, b_3), and (b_3, b_1). Any pair (v, w) is dominated by Lb_i for some i. Since v does not beat any blue vertices, the endpoints of each edge of G are a dominating pair. Thus $G \cup \{v\}$ is the domination graph of T. ∎

PROPOSITION 11 If G consists of one caterpillar that is not a star and one other nontrivial caterpillar, then $i(G) = 1$.

Proof. Let G_1 be a non-star, and note that this implies that it has at least 4 vertices on its spine. Let (v_2, v_1) be an arc, and then define the standard tournament T_0 on $V(G)$. Add a vertex v with arcs to v_1, v_2, Rb_1, and Rr_2, forming T as before. Now an (r_1, r_2) pair is dominated by Rr_2 or Lb_1 or v; a (b_1, b_2) pair is dominated by v_1; a (b_1, r_2) pair is dominated by Lb_2, Rb_1, or v; and an (r_1, b_2) pair is dominated by v_2. Also, a (v, r_2) pair is dominated by Lb_1; a (v, b_2) pair is dominated by Rr_1; a (v, r_1) pair is dominated by Lb_1 or Rr_1; and a (v, b_1) pair is dominated by any b_2. Since G_1 has at least 4 vertices on its spine, v_1 is not adjacent to Rb_1, so v does not dominate any adjacent vertices in G. Thus, the endpoints of each edge of G are a dominating pair, and so $G \cup \{v\}$ is the domination graph of T. ∎

PROPOSITION 12 If G consists of a star $K_{1,n}$, $n \geq 4$, and one other nontrivial caterpillar then $i(G) = 1$.

Proof. Let G_1 be a nontrivial caterpillar and G_2 the $K_{1,n}$. Let v_1 and v_2 be the leftmost vertices on the spines as usual, and let (v_2, v_1) be an arc. Orient all spine edges to the right and all leg edges down. Color the vertices of G properly with red and blue, assigning red to the v_i. In G_1 add arcs to the right between vertices of different colors and to the left between vertices of the same color. In G_2 add arcs from all red vertices to v_2, and add arcs among the remaining red vertices so that each such red vertex is beat by some other red vertex, as indicated in figure 5. Finally, orient arcs between caterpillars as in Proposition 8. Call the result T_0. As with the standard tournament, G_1 and G_2 are induced subgraphs of dom(T_0). Add a vertex v with arcs to v_2 and Rb_1, forming T.

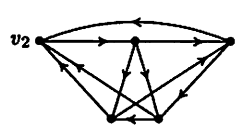


Figure 5 Subtournament on $K_{1,4}$

Now an (r_1, r_2) pair is dominated by some r_2 other than v_2; a (b_1, b_2) pair is dominated by v_1; a (b_1, r_2) pair is dominated by Lb_2, Rb_1, or v; and an (r_1, b_2) pair is dominated by v_2. Also, a (v, r_2) pair is dominated by some r_2 other than v_2; a (v, b_2) pair is dominated by v_1; a (v, r_1) pair is dominated by any r_2 other than v_2; and a (v, b_1) pair is dominated by v_1. Since v does not beat any vertices adjacent in G, the endpoints of each edge of G are a dominating pair. Thus $G \cup \{v\}$ is the domination graph of T. ∎

PROPOSITION 13 If G consists of two stars, a $K_{1,3}$ and a $K_{1,n}$, $n = 2$ or 3, then $i(G) = 1$.

Proof. Let G_1 be a $K_{1,3}$ and G_2 the other star. Let v_1 be the central vertex in G_1 and v_2 be the leftmost vertex on the spine of G_2, and let (v_2, v_1) be an arc. Orient the edges of the caterpillars as shown in figure 6, and color the vertices of G properly with red and blue, assigning red to the v_i. Add arcs to G_1 and G_2 as shown, and then add arcs between the two components as in Proposition 8. Add a vertex v with arcs to v_1 and Rr_2, forming T. It is now easy to verify by inspection that $G \cup \{v\}$ is the domination graph of T. ∎

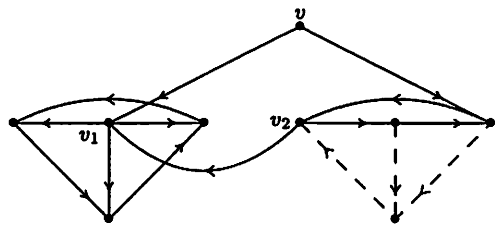


Figure 6 Tournament giving $K_{1,3} \cup K_{1,2}$ or $K_{1,3} \cup K_{1,3}$ (some arcs not shown)

PROPOSITION 14 if G is a star $K_{1,n}$, $n \geq 4$, then $i(G) = 1$.

Proof. By proposition 2, it is sufficient to show that $K_{1,n}$ is the domination graph of a tournament with no vertex of indegree 0. Denote the vertices by $v_0, v_1, \ldots, v_n$, with v_0 the central vertex. Include arcs (v_1, v_0); (v_0, v_i) and (v_i, v_1) for $i > 1$; (v_i, v_{i+1}) for $i \in \{2, \ldots, n-1\}$; and (v_n, v_2). Add other arcs among the vertices $\{v_2, \ldots, v_n\}$ in any way. No vertex has indegree 0 and the domination graph is $K_{1,n}$ as desired. ∎

PROPOSITION 15 If G consists of a single caterpillar with at least 5 vertices on its spine, then $i(G) = 1$.

Proof. Orient the spine arcs to the right, leg arcs down. Let v_1 be the leftmost vertex on the spine; color v_1 red, and complete this to a coloring of $G = G_1$ (we call this single caterpillar G_1 so that we can continue to use our established conventions for referring to vertices). Direct arcs between vertices of the same color to the left, others to the right. Add a vertex v with arcs to v_1 and Rr_1, and arcs from all other vertices, to form the tournament T.

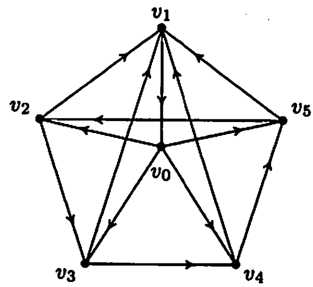


Figure 7 Tournament with domination graph $K_{1,5}$ (some arcs not shown)

Suppose w and x are vertices of different colors and not adjacent in G, with x to the right of w. Then the pair is dominated by the unique vertex y such that (y, x) is an arc in the orientation of G. If w and x are two blue vertices, the pair is dominated by v_1. If w and x are two red vertices, the pair is dominated by Rr_1 or Lb_1 or v. A (v, r_1) pair is dominated by Lb_1 or the red vertex immediately to the right of Lb_1 on the spine. A (v, b_1) pair is dominated by Rb_1 or the red vertex immediately to the right of Lb_1 on the spine. No vertex beats vertices that are adjacent in G, so the endpoints of each edge of G are a dominating pair. Thus $G \cup \{v\}$ is the domination graph of T. ∎

PROPOSITION 16 If G consists of a single caterpillar with 4 vertices on its spine and at least one leg, then $i(G) = 1$.

Proof. Orient the spine arcs to the right, leg arcs down. Let v_1 be the leftmost vertex on the spine; color v_1 red, and complete this to a coloring of $G = G_1$. Direct arcs between vertices of the same color to the left, others to the right. Add a vertex v with arcs to v_1 and Rr_1, and arcs from all other vertices, to form the tournament T. Without loss of generality, assume that Lb_1 has a leg attached, connecting Lb_1 to vertex u.

Suppose w and x are vertices of different colors and not adjacent in G, with x to the right of w. Then the pair is dominated by the unique vertex y such that (y, x) is an arc in the orientation of G. If w and x are two blue vertices, the pair is dominated by v_1. If w and x are two red vertices, the pair is dominated by Rr_1 or Lb_1 or v. A (v, r_1) pair is dominated by Lb_1 or u. A (v, b_1) pair is dominated by Rb_1 or u. No vertex beats vertices that are adjacent in G, so the endpoints of each edge of G are a dominating pair. Thus $G \cup \{v\}$ is the domination graph of T. ∎

Propositions 11 through 13 cover all forests of two nontrivial caterpillars except $K_{1,3} \cup K_2$ and two nontrivial stars on three or fewer vertices, namely

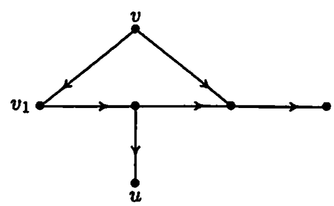
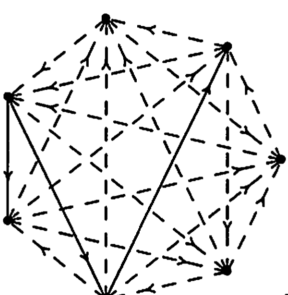


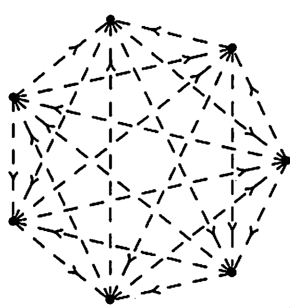
Figure 8 Tournament giving P_4 plus leg(s) (some arcs not shown)

$K_{1,2} \cup K_{1,2}$, $K_{1,2} \cup K_2$, $K_2 \cup K_2$. Propositions 14 through 16 cover all single caterpillars except P_4 and stars on four or fewer vertices, namely K_1, K_2, $K_{1,2}$ and $K_{1,3}$. The values of $i(G)$ for these nine graphs are: $i(P_4) = 3$; $i(K_1) = 6$; $i(K_2) = 6$; $i(K_{1,2}) = 4$; $i(K_{1,3}) = 3$; $i(K_{1,2} \cup K_{1,2}) = 2$; $i(K_{1,2} \cup K_2) = 2$; $i(K_2 \cup K_2) = 4$; $i(K_{1,3} \cup K_2) = 2$.

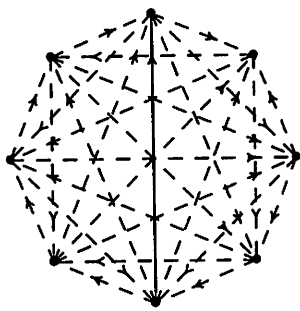
Tournaments verifying that $i(G)$ is no larger than this for each of these nine graphs are shown below. The arcs shown as solid lines are the edges of the domination graph of the tournament. It is a bit tedious, but not difficult, to show that in each case no smaller number of isolated vertices may be added to form a domination graph of a tournament; details are available from David Guichard at guichard@whitman.edu. (For K_1, K_2, and $K_{1,2}$, this follows immediately from theorem 3.1 of [5].) These tournaments originally were found by computer search; we used the excellent "nauty" software by Brendan McKay to generate all tournaments and their domination graphs on 10 or fewer vertices. The tournaments and graphs are available from David Guichard. Professor McKay may be reached at bdm@cs.anu.edu.au.



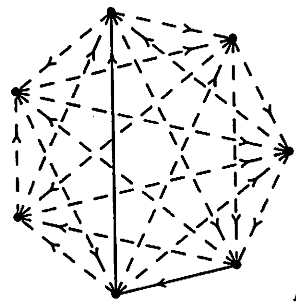
P_4



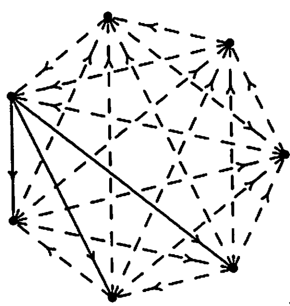
K_1



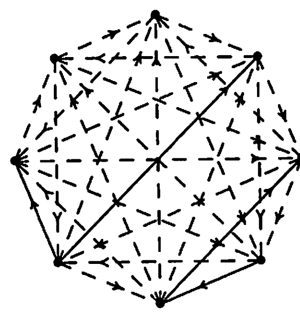
K_2



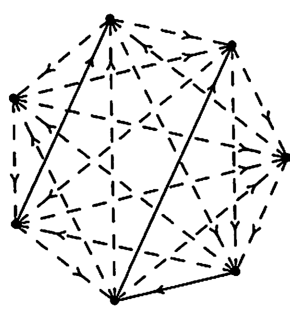
$K_{1,2}$



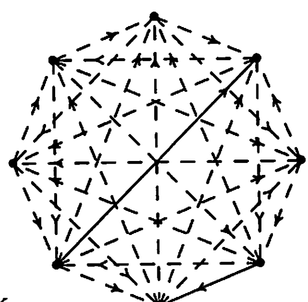
$K_{1,3}$



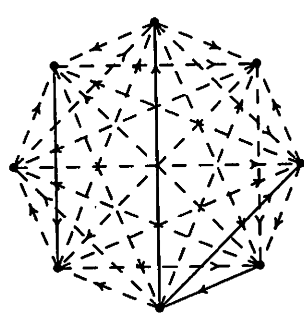
$K_{1,2} \cup K_{1,2}$



$K_{1,2} \cup K_2$



$K_2 \cup K_2$



$K_{1,3} \cup K_2$