On the Cordiality of the t-uniform Homeomorphs - I

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Abstract: Let G be a simple graph with vertex set V and edge set E. A vertex labeling $f : V \longrightarrow \{0,1\}$ induces an edge labeling $\overline{f} : E \longrightarrow \{0,1\}$ defined by $\overline{f}(uv) = | f(u) - f(v) |$. Let $v_f(0), v_f(1)$ denote the number of vertices v with $f(v) = 0$ and $f(v) = 1$ respectively. Let $e_f(0), e_f(1)$ be similarly defined. A graph is said to be **cordial** if there exists a vertex labeling f such that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

A t-uniform homeomorph $P_t(G)$ of G is the graph obtained by replacing all edges of G by vertex disjoint paths of length t. In this paper we investigate the cordiality of $P_t(G)$, when G itself is cordial. We find, wherever possible, a cordial labeling of $P_t(G)$, whose restriction to G is the original cordial labeling of G and prove that for a cordial graph G and a positive integer t, (1) $P_t(G)$ is cordial whenever t is odd, (2) for $t \equiv 2(\bmod 4)$ a cordial labeling g of G can be extended to a cordial labeling f of $P_t(G)$ iff $e_g(0)$ is even, (3) for $t \equiv 0(\bmod 4)$, a cordial labeling g of G can be extended to a cordial labeling f of $P_t(G)$ iff $e_g(1)$ is even.

Introduction

Throughout this paper, all graphs are finite, simple and undirected. A **t-uniform homeomorph** $P_t(G)$ of a graph G is the graph obtained by replacing all edges of G by vertex disjoint paths of length t, i.e. H is obtained from G by introducing $t-1$ new vertices on each edge of G.

Let G be a graph with vertex set V and edge set E. A binary labeling $f : V \longrightarrow \{0,1\}$ induces an edge labeling $\overline{f} : E \longrightarrow \{0,1\}$ defined by $\overline{f}(uv) = | f(u) - f(v) |$. By $v_f(0)$ and $v_f(1)$ we mean the number of vertices with $f(v) = 0$ and $f(v) = 1$ respectively. Similarly by $e_f(0)$ and $e_f(1)$ we mean the number of edges labeled 0 and 1 respectively. A graph G is said to be **cordial**, if there exists a binary vertex labeling f of G such that $| v_f(0) - v_f(1) | \leq 1$ and $| e_(0) - e_f(1) | \leq 1$ and f is called a **cordial labeling** of G. Cahit [1] introduced the concept of cordial graphs as a weaker version of both graceful and harmonious graphs. In this paper we investigate the cordiality of $P_t(G)$, when G itself is cordial. **For simplicity of notation, we use the symbol f itself instead of $\overline{f}$ for the induced edge labeling.**

In what follows, we always take a cordial graph G with a cordial labeling $g : V(G) \longrightarrow \{0,1\}$. Then $| v_g(0) - v_g(1) | \leq 1$ and $| e_g(0) - e_g(1) | \leq 1$. Let $e_g(0) = n$. Then $e_g(1) = n-1, n$ or $n+1$. For an edge uv in G, $\overline{g}(uv) = 1$ iff either $g(u) = 0, g(v) = 1$ or $g(u) = 1, g(v) = 0$. Similarly $\overline{g}(uv) = 0$ iff either $g(u) = 0 = g(v)$ or $g(u) = 1 = g(v)$.

For an edge $e = uv$ of G, by $\gamma(e)$ we mean the path of type $\{u, v_1, v_2, \cdots, v_{t-1}, v\}$ in $P_t(G)$ obtained by introducing the vertices $v_1, \cdots, v_{t-1}$ on the edge uv. We call $\gamma(uv)$ a path of type (I) if $g(u) = g(v) = 0$, of type (II) if $g(u) = g(v) = 1$ and of type (III) if $g(u) = 1, g(v) = 0$.

Theorem 1: If G is cordial, then $P_t(G)$ is cordial for every odd value of t.
Proof: Let $k = t - 1$.

Case (1): $k \equiv 0(\bmod 4)$. Define a binary labeling f of $P_t(G)$ as follows: **(i)** If $g(uv) = 0$, define $f(u) = f(v) = g(u)$ and for vertices on $\gamma(e)$, define $f(v_i) = 1, i \equiv 1, 2(\bmod 4)$, $f(v_i) = 0, i \equiv 0, 3(\bmod 4)$. One can see that on this path having $t = k + 1$ edges, $1 + k/2$ edges have received the label 0 and $k/2$ edges have received the label 1. **(ii)** If $g(u) = 1, g(v) = 0$, then define $f(u) = g(u), f(v) = g(v)$ and $f(v_i) = 0, i \equiv 1, 2(\bmod 4)$, $f(v_i) = 1, i \equiv 0, 3(\bmod 4)$. On this path, $k/2$ edges receive the label 0 and $1 + k/2$ edges receive the label 1.

It is clear that exactly half of the new vertices have received the label 0 and the remaining half have received the label 1. Hence, $| v_f(0) - v_f(1) | = | v_g(0) - v_g(1) | \leq 1$. That f is a cordial labeling of $P_t(G)$ is clear from the following table:

$e_g(1)$	$v_f(0)$	$v_f(1)$	$e_f(0)$	$e_f(1)$
$n-1$	$v_g(0) + \frac{(2n-1)k}{2}$	$v_g(1) + \frac{(2n-1)k}{2}$	$\frac{2kn+2n-k}{2}$	$e_f(0) - 1$
n	$v_g(0) + nk$	$v_g(1) + nk$	$nk + n$	$e_f(0)$
$n+1$	$v_g(0) + \frac{(2n+1)k}{2}$	$v_g(1) + \frac{(2n+1)k}{2}$	$\frac{2nk+n+k}{2}$	$e_f(0) + 1$

Case (2): $k \equiv 2(\bmod 4)$. In this case, for each path γ in $P_t(G)$, we define f as follows: $f(u) = g(u), f(v) = g(v), f(v_i) = 1, i \equiv 1, 2(\bmod 4), i \neq k$ and $f(v_i) = 0, i \equiv 0, 3(\bmod 4), f(v_k) = 0$. This means each path γ has labeling $g(u), 1, 1, 0, 0, \cdots, 1, 1, 0, 0, 1, 0, g(v)$. Whenever $g(uv) = 0$, out of $k + 1$ edges on γ, $k/2$ edges have received the label 0 and $1 + k/2$ edges have received the label 1. If $g(uv) = 1$, out of $k + 1$ edges of γ, $1 + k/2$ edges have received the label 0 and $k/2$ edges have received the label 1. The table below shows that f is a cordial labeling.

$e_g(1)$	$v_f(0)$	$v_f(1)$	$e_f(0)$	$e_f(1)$
$n-1$	$v_g(0)+\frac{(2n-1)k}{2}$	$v_g(1)+\frac{(2n-1)k}{2}$	$\frac{2nk+2n-k}{2}-1$	$e_f(0)+1$
n	$v_g(0)+nk$	$v_g(1)+nk$	$nk+n/2$	$e_f(1)$
$n+1$	$v_g(0)+\frac{(2n+1)k}{2}$	$v_g(1)+\frac{(2n+1)k}{2}$	$\frac{2nk+2n+k}{2}+1$	$e_f(0)-1$

□

When t is even, we cannot easily extend a cordial labeling g of G to a cordial labeling f of $P_t(G)$. The next two theorems give a necessary and sufficient condition for existence of such an extension.

Remark: Let f be labeling of $P_t(G)$ whose restriction to $V(G)$ is the original cordial labeling g of G. This means the intermediate vertices are labeled arbitrarily. If $\gamma(e)$ is a path of type (I), its labeling looks like

$$f(u)=0,0,\cdots,0,1,1,\cdots,1,0,\cdots,0,1,1,\cdots,1,0,\cdots,0=f(v).$$

Each string $1,1,\cdots 1$ gives rise to two edges, one at the beginning and one at the end, with label one. Hence such a $\gamma(e)$ of type (I) gives rise to even number of edges with label one. Similarly, by considering strings of zeros, we can see that each path of type (II), gives rise to even number of edges with label one, where as a path of type (III) gives rise to odd number of edges with label one.

Theorem 2: Let g be a cordial labeling of a graph G and let t be a positive integer, $t \equiv 2(\bmod 4)$. Then there exists a cordial labeling f of $P_t(G)$ whose restriction to G is g iff $e_fg(0)$ is even.

Proof: Let g be a cordial labeling of a graph G, such that $e_g(0)=n$ is even. Let $t=4q+2$ and $k=t-1=4q+1$. We extend the labeling g to a labeling f of $P_t(G)$ as follows: Let uv be an arbitrary edge of G and let $\gamma=\{u,v_1,v_2,\cdots,v_k,v\}$ be the path in $P_t(G)$ given by the edge uv. Define $f(x)=g(x)$ for every $x\in V(G)$. If $g(u)=g(v)=0$, define $f(v_i)=1, i\equiv 1,2(\bmod 4), i\neq k$ and define $f(v_i)=0, i\equiv 0,3(\bmod 4)$. If $g(u)=1$, define $f(v_i)=0, i\equiv 1,2(\bmod 4), i\neq k$ and define $f(v_i)=1, i\equiv 0,3(\bmod 4)$. At this stage one vertex on each path remains to be labeled. We have now n paths of $P_t(G)$ in which $f(v_{k-1})=f(v)$. On $\frac{n}{2}$ of these paths, let

$f(v_k) = f(v)$ and on the remaining $\frac{n}{2}$ paths, let $f(v_k) =| 1 - f(v) |$. So far label distribution has perfectly balanced. Next consider those paths in $P_t(G)$ for which $f(u) = 1, f(v) = 0$. For these paths $f(v_{k-1} = 1$. Define $f(v_k) = 1$ or 0 so as to satisfy the condition $| v_f(0) - v_f(1) |\leq 1$. One can easily see that $e_f(0) =| E(G) | (2t + 1)$
$= e_f(1)$, i.e. f is a cordial labeling of $P_t(G)$ whose restriction to $V(G)$ is g.

Now, let $e_g(0) = n$ be odd. If possible, let f be a cordial labeling of $P_t(G)$ whose restriction to $V(G)$ is the original cordial labeling of G. This means $e_f(0) = e_f(1) = (2q + 1) | E(G) |$. Let $2\delta_1, \cdots, 2\delta_{e_f(0)}$ be the number of edges with label one given by paths of type (I) and (II) and let $2m_1 + 1, \cdots, 2m_{e_f(1)} + 1$ be the number of edges with label one given by paths of type (III). This is justified by the Remark made above.

Case (1): $e_g(1) = n - 1$, i.e. $| E(G) |= 2n - 1$. One finds that $(2n - 1)(2q + 1) = e_f(0) = 2(\delta_1 + \cdots + \delta_n) + 2(m_1 + \cdots, m_{n-1}) + (n - 1)$. This is a contradiction since $(2n - 1(2q + 1)$ is odd and the right hand side is an even number.

Case (2): $e_g(1) = n$, i.e. $| E(G) |= 2n$. One finds that $2n(2q + 1) = e_f(0) == 2(\delta_1 + \cdots + \delta_n) + 2(m_1 + \cdots, m_n) + n$. This is a contradiction since $2n(2q + 1)$ is even and the right hand side is an odd number.

Case (3): $e_g(1) = n + 1$, i.e. $| E(G) |= 2n + 1$. One finds that $(2n + 1(2q + 1) = e_f(0) = 2(\delta_1 + \cdots + \delta_n) + 2(m_1 + \cdots, m_{n+1}) + n + 1$. This is a contradiction since $(2n + 1)(2q + 1)$ is odd and the right hand side is an even number.

Hence we cannot extend g to a cordial labeling f of $P_t(G)$. □

Theorem 3: Let g be a cordial labeling of a graph G and let t be a positive integer,. $t \equiv 4(\bmod 4)$. Then there exists a cordial labeling f of $P_t(G)$ whose restriction to G is g iff $e_g(1)dd$ is even.

Proof: Let g be a cordial labeling of a graph G, such that $e_g(1) = n$ is even. Let $t = 4q + 4$ and $k = t - 1 = 4q + 3$. We extend the labeling g to a labeling f of $P_t(G)$ as follows: Define $f(x) = g(x)$ for every $x \in V(G)$. For an edge $uv \in E(G)$, if $g(u) = g(v) = 0$, for $1 \leq i \leq 4q$, define $f(v_i) = 1, i \equiv 1, 2(\bmod 4)$ and define $f(v_i) = 0, i \equiv 0, 3(\bmod 4)$. If $g(u) = 1$, and $g(v) = 0$ or 1, for $1 \leq i \leq 4q$, define $f(v_i) = 0, i \equiv 1, 2(\bmod 4)$ and define $f(v_i) = 1, i \equiv 0, 3(\bmod 4)$. At this stage three vertices on each of the paths remain to be labeled.

We have now n paths of $P_t(G)$ in which $f(v_{4q}) =. 1, f(v) = 0$. On $\frac{n}{2}$ of these paths, let $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1$ and on the remaining $\frac{n}{2}$ paths, let $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0 = f(v_{4q+3})$. The label distribution is perfectly balanced so far. We complete the remaining labeling as follows:

There are $e_f(0)$ paths $\gamma(e)$ in which either $f(u) = f(v) = 0$ or $f(u) = f(v) = 1$. In both the cases, $f(v_{4q}) = f(u) = f(v)$.

Case (1): $e_g(0) = n$, i.e. $| E(G) |= 2n$. Define $f(v_{4q+1}) = 1 = f(v_{4q+2})$ and $f(v_{4q+3}) = 0$. One can easily see that $v_f(0) = v_g(0)+n(4q+3), v_f(1) = v_g(1)+n(4q+3)$ and $e_f(0) = n(4q+4) = e_f(1)$, i.e. f is a cordial labeling.

Case (2): $e_g(0) = n+1$, i.e. $| E(G) |= 2n+1$. For n paths, with $f(v_{4q}) = f(v)$, define $f(v_{4q+1}) = 1 = f(v_{4q+2}), f(v_{4q+3}) = 0$ and for the remaining path either keep the same labeling or let $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0 = f(v_{4q+3})$; so that the condition $| v_f(0) - v_f(1) |\leq 1$ is satisfied. One can easily see that f is a cordial labeling.

Case (3): $e_g(0) = n-1$, i.e. $| E(G) |= 2n-1$. This means there are $n-1$ paths $\gamma(e)$ of type (I) and (II) together and n paths of type (III). We leave aside one path of type (I) or type (II), if there is no path of type (I) and two paths of type (III). The vertices of all the remaining $n-2$ paths from type (I) and (II) together and $n-2$ from type (III) are labeled as in case (1).

If there is a path of type (I) among the three kept aside, for one of the paths of type (III) define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0 = f(v_{4q+3})$ and for the second path of type (III) define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1$. For the path of type (I) with $f(u) = f(v) = 0$ define $f(v_{4q+1}) = f(v_{4q+2}) = f(v_{4q+3}) = 1$ or $f(v_{4q+1}) = f(v_{4q+2}) = 1, f(v_{4q+3}) = 0$, so as to satisfy the condition $| v_f(0) - v_f(1) |\leq 1$.

If there is no path of type (I) among the three kept aside, for one of the paths of type (III) define $f(v_{4q+1}) = f(v_{4q+2}) = 1, f(v_{4q+3}) = 0$ and for the second path of type (III) define $f(v_{4q+1}) = 0, f(v_{4q+2}) = 1 = f(v_{4q+3})$. For the path of type (II) with $f(u) = f(v) = 1$ define $f(v_{4q+1}) = f(v_{4q+2}) = f(v_{4q+3}) = 1$ or $f(v_{4q+1}) = 1, f(v_{4q+2}) = f(v_{4q+3}) = 0$, so as to satisfy the condition $| v_f(0) - v_f(1) |\leq 1$. It can be easily verified that f is cordial.

Now suppose n is odd. If f is to be cordial we must have $e_f(0) = (2q+2) | E(G) |= e_f(1)$. This is even. By an argument similar to the one in Theorem 2, one can show that any cordial labeling f, of $P_t(G)$ whose restriction to G is the original cordial labeling of g gives an odd number of edges with label zero. Hence there is no such extension of g. □

Remark: Theorem 2 and Theorem 3 do not in anyway prove that $P_t(G)$ is non-cordial in those cases. There may very well exist a cordial labeling of $P_t(G)$ whose restriction to $V(G)$ is not a cordial labeling of G. If G is so richly cordial that it has cordial labeling g, h where $e_g(0)$ is even and $e_h(1)$ is even, then $P_t(G)$ is cordial for every positive integer t.