Some Decomposition Invariants of Crowns

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ABSTRACT. Suppose G is a graph. The minimum number of paths (trees, forests, linear forests, star forests, complete bipartite graphs, respectively) needed to decompose the edges of G is called the path number (tree number, arboricity, linear arboricity, star arboricity and biclique number, respectively) of G. These numbers are denoted by $p(G)$, $t(G)$, $a(G)$, $la(G)$, $sa(G)$, $\tau(G)$, respectively. For integers $1 \leq k \leq n$, let $C_{n,k}$ be the graph with vertex set $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ and edge set $\{a_i b_j : i = 1, 2, \cdots, n,\ j \equiv i+1, i+2, \cdots, i+k \pmod n\}$. We call $C_{n,k}$ a crown. In this paper, we prove the following results.

(1) $$p(C_{n,k}) = \begin{cases} n & \text{if } k \text{ is odd,} \\ (k/2)+1 & \text{if } k \text{ is even.} \end{cases}$$

(2) $a(C_{n,k}) = t(C_{n,k}) = la(C_{n,k}) = \lceil (k+1)/2 \rceil$ if $k \geq 2$.

(3) For $k \geq 3$ and $k \in \{3, 5\} \cup \{n-3, n-2, n-1\}$,

$$sa(C_{n,k}) = \begin{cases} \lceil k/2 \rceil + 1 & \text{if } k \text{ is odd,} \\ \lceil k/2 \rceil + 2 & \text{if } k \text{ is even.} \end{cases}$$

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(4) $\tau(C_{n,k}) = n \quad$ if $k \leq (n+1)/2$ or $\gcd(k,n) = 1$.

Due to (3), (4), we propose the following conjectures.

Conjecture A. For $3 \leq k \leq n-1$,

$$sa(C_{n,k}) = \begin{cases} \lceil k/2 \rceil + 1 & \text{if } k \text{ is odd}, \\ \lceil k/2 \rceil + 2 & \text{if } k \text{ is even}. \end{cases}$$

Conjecture B. For $1 \leq k \leq n-1$, $\tau(C_{n,k}) = n$.

1 Introduction

Suppose G is a graph. The *path number* $p(G)$ of G is the minimum number of paths needed to decompose the edges of G. The *tree number* $t(G)$ of G is the minimum number of trees needed to decompose the edges of G. The *arboricity* $a(G)$ of G is the minimum number of forests needed to decompose the edges of G. A *linear forest* is a forest each component of which is a path. The *linear arboricity* $la(G)$ of G is the minimum number of linear forests needed to decompose the edges of G. A *star forest* is a forest each component of which is a star. The *star arboricity* $sa(G)$ of G is the minimum number of star forests needed to decompose the edges of G. The *biclique number* $\tau(G)$ of G is the the minimum number of complete bipartite graphs needed to decompose the edges of G. These decomposition invariants of general graphs satisfy the following properties.

Lemma 1.1 *For any nontrivial graph G,*

(1) $\lceil |E(G)|/(|V(G)|-1) \rceil \leq a(G) \leq t(G) \leq p(G)$,

(2) $max\{a(G), \lceil \Delta(G)/2 \rceil\} \leq la(G) \leq min\{p(G), \lceil 3\lceil \Delta(G)/2 \rceil /2 \rceil\}$,

(3) $u/2 \leq p(G)$ *where u is the number of vertices with odd degree in G,*

(4) $a(G) \leq sa(G) \leq min\{\Delta(G), 2a(G)\}$,

(5) $a(G) = max \lceil |E(H)|/(|V(H)|-1) \rceil$ *where the maximum is taken over all the nontrivial induced subgraphs H of G.*

Proof. (1) The first inequality follows from the fact that any forest in G has at most $|V(G)| - 1$ edges. The other inequalities follow directly from the definitions.

(2) The inequalities $a(G) \leq la(G) \leq p(G)$ follow directly from the definitions. The inequality $\lceil \Delta(G)/2 \rceil \leq la(G)$ follows from the fact every linear forest has maximum degree no more than two. The proof of the inequality $la(G) \leq \lceil 3 \lceil \Delta(G)/2 \rceil /2 \rceil$ can be found in [2].

(3) Let $\mathscr{D}$ be a path decomposition of G. Suppose x is a vertex with odd degree in G. Then there exists at least a path in $\mathscr{D}$ with x as one of its ends. Since a nontrivial path has two ends, $2|\mathscr{D}| \geq u$, which implies $|\mathscr{D}| \geq u/2$. Thus $p(G) \geq u/2$.

(4) The first inequality follows from the definitions. The inequality $sa(G) \leq 2a(G)$ [12] follows from the fact that every tree can be decomposed into no more than two star forests. The inequality $sa(G) \leq \Delta(G)$ [20] can be proved by induction on $\Delta(G)$ using the facts that every connected graph contains a spanning tree, and that every nontrivial tree contains a spanning star forest without isolated vertices.

(5) The formula is a well known result of Nash-Williams [17]. □

The arboricity, linear arboricity and star arboricity of regular graphs have been investigated by many researchers. Using Lemma 1.1(5), J. Akiyama and T. Hamada determined the arboricity of regular graphs in the following.

Theorem 1.2 [3] *Suppose G is a d-regular graph. Then $a(G) = \lceil (d+1)/2 \rceil$.* □

The linear arboricity of d-regular graphs was conjectured in [1] to be $\lceil (d+1)/2 \rceil$, the same value as the arboricity. Star arboricities of d-regular graphs have lower bounds as follows.

Theorem 1.3 [20] *Suppose G is a d-regular graph where $d \geq 2$. Then*

$$sa(G) \geq \lceil d/2 \rceil + 1.$$

Moreover, if $d = 2p$ for some integer $p \geq 2$, and $sa(G) = p+1$, then there is a partition of $V(G)$ into $p+1$ equal size independent sets $A_1, A_2, \cdots, A_{p+1}$ such that for every i, j, $1 \leq i < j \leq p+1$, there are exactly $2c$ edges between A_i and A_j, where c is the common cardinality of all sets A_i. □

An immediate consequence follows from Theorem 1.3.

Corollary 1.4 [20] *Suppose G is a d-regular graph where $d = 2p$ for some integer $p \geq 2$. If $p+1$ does not divide $|V(G)|$, then $sa(G) \geq p+2$.* □

For a survey we also list the results about the above mentioned decomposition invariants for complete graphs and complete bipartite graphs.

Theorem 1.5

(1)[3, 10] $a(K_n) = la(K_n) = t(K_n) = p(K_n) = \lceil n/2 \rceil$.

(2)[4] $sa(K_n) = \lceil n/2 \rceil + 1$ *if* $n \geq 4$.

(3)[21] $\tau(K_n) = n - 1$.

(4)[10, 19]
$$p(K_{m,n}) = \begin{cases} \dfrac{m+n}{2} & \text{if } mn \text{ is odd,} \\ \left\lceil \dfrac{mn}{2n - \delta(m,n)} \right\rceil & \text{if } mn \text{ is even and } m \geq n, \end{cases}$$
$$\text{where } \delta(m,n) = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

(5)[6, 11] $a(K_{m,n}) = t(K_{m,n}) = \lceil mn/(m+n-1) \rceil$.

(6)[1] $la(K_{m,n}) = \lceil (m + \delta(m,n))/2 \rceil$ *if* $m \geq n$.

(7)[8, 20]
$$sa(K_{n,n}) = \begin{cases} \lceil n/2 \rceil + 2 & \text{if } n = 4 \text{ or } n \geq 6, \\ 4 & \text{if } n = 5. \end{cases}$$

(8)[9] $sa(K_{n,kn}) \leq min\{n, \lceil (kn+2)/(k+1) \rceil + 1\}$

if $(k,n) \neq (1,5)$.

Proof. Here we only prove (1) and (5).

(1) By Lemma 1.1(1), we have $\lceil n/2 \rceil \leq a(K_n) \leq t(K_n) \leq p(K_n)$. By Lemma 1.1(2), we have $a(K_n) \leq la(K_n) \leq p(K_n)$. And it was proved in [10] that $p(K_n) = \lceil n/2 \rceil$. Thus $a(K_n) = la(K_n) = t(K_n) = p(K_n) = \lceil n/2 \rceil$.

(5) By Lemma 1.1(1), $\lceil mn/(m+n-1) \rceil \leq a(K_{m,n}) \leq t(K_{m,n})$. It was proved in [11] that $t(K_{m,n}) = \lceil mn/(m+n-1) \rceil$. Thus $a(K_{m,n}) = t(K_{m,n}) = \lceil mn/(m+n-1) \rceil$. □

The results about trees are as follows.

Theorem 1.6 *Suppose T is a tree. Then*

(1)[10] $p(T) = u/2$ *where u is the number of vertices of T with odd degree.*

(2)[1] $la(T) = \lceil \Delta(T)/2 \rceil$.

(3) $sa(T) = 2$ *if T is not a star.*

Proof. (3) Since $a(T) = 1$, the result follows immediately from Lemma 1.1 (4). □

A well-known conjecture about path numbers was posed by Gallai. He conjectured that for any connected graph G on n vertices, $p(G) \leq \lceil n/2 \rceil$. In [7], Donald proved the following result using a construction of Lovász.

Theorem 1.7 [7] *Let G be a graph with u vertices of odd degree and g vertices of even degree. Then $p(G) \leq u/2 + \lfloor 3g/4 \rfloor \leq \lfloor 3n/4 \rfloor$ where $n = u + g$.*

□

The following result follows immediately from Theorem 1.7 and Lemma 1.1(3).

Theorem 1.8 [16] *Let G be a graph with vertices of odd degree only. Then $p(G) = u/2$ where u is the number of vertices of G.* □

For integers $1 \leq k \leq n$, let $C_{n,k}$ denote the graph with vertex set $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ and edge set $\{a_i b_j : i = 1, 2, \cdots, n, j \equiv i + 1, i + 2, \cdots, i + k \pmod{n}\}$. We call $C_{n,k}$ a *crown*. Decompositions of edges of the crown into isomorphic cycles, isomorphic stars and isomorphic complete bipartite graphs have been investigated in [13], [15] and [14], respectively. In this paper, we will study path numbers, tree numbers, arboricities, linear arboricities, star arboricities and biclique numbers of the crown.

2 The path number, tree number, arboricity and linear arboricity of $C_{n,k}$

In this section, we determine the path number, tree number, arboricity, and linear arboricity of the crown.

For our discussions we use the following terminology and notation. The *label* of the edge $a_i b_j$ $(1 \leq i, j \leq n)$ in $C_{n,k}$ is $j - i$ if $i < j$ and

$j+n-i$ if $i \geq j$. For integers p, q, s, t with $s \leq t$, we use the notation $(a_{p+j}b_{q+j})_{s \leq j \leq t}$ to denote the walk $a_{p+s}b_{q+s}a_{p+s+1}b_{q+s+1} \cdots a_{p+t}b_{q+t}$ in $C_{n,k}$; here and in the sequel the subscripts of a, b are taken modulo n. Similarly, the notation $(a_{p-j}b_{q+j})_{s \leq j \leq t}$ is used to denote the walk $a_{p-s}b_{q+s}a_{p-s-1}b_{q+s+1} \cdots a_{p-t}b_{q+t}$ in $C_{n,k}$.

Since $C_{n,k}$ is k-regular, we have, by Theorem 1.2, $a(C_{n,k}) = \lceil (k+1)/2 \rceil$. As to the other invariants, let us begin with the following lemma.

Lemma 2.1 *If k is even, then*

$$a(C_{n,k}) = t(C_{n,k}) = la(C_{n,k}) = p(C_{n,k}) = (k/2) + 1.$$

Proof. Suppose k is even. By Lemma 1.1(1) and (2), we have

$$\begin{aligned} a(C_{n,k}) &\leq t(C_{n,k}) \leq p(C_{n,k}) \text{ and} \\ a(C_{n,k}) &\leq la(C_{n,k}) \leq p(C_{n,k}). \end{aligned}$$

Since $a(C_{n,k}) = \lceil (k+1)/2 \rceil = (k/2) + 1$, it suffices to show that $p(C_{n,k}) \leq (k/2) + 1$.

For $1 \leq \mu \leq k/2$, let $F^{(\mu)}$ be the cycle $(a_j b_{2\mu+j})_{1 \leq j \leq n} a_1$.

Note that each $F^{(\mu)}$ consists of those edges with labels $2\mu - 1$ and 2μ. We see that $C_{n,k}$ is decomposed into cycles $F^{(1)}, F^{(2)}, \cdots, F^{(\frac{k}{2})}$.

For $1 \leq \mu \leq k/2$, let $Q^{(\mu)}$ be the path obtained from $F^{(\mu)}$ by removing the path $a_{\frac{k}{2}-\mu+1} b_{\frac{k}{2}+\mu} a_{\frac{k}{2}-\mu}$.

Let $Q^{(0)}$ be the path $(a_{\frac{k}{2}-j+1} b_{\frac{k}{2}+j})_{1 \leq j \leq \frac{k}{2}} a_n$.

We see that $Q^{(0)}$ consists of those paths removed from $F^{(1)}, F^{(2)}, \cdots, F^{(\frac{k}{2})}$. Thus $C_{n,k}$ is decomposed into paths $Q^{(0)}, Q^{(1)}, \cdots, Q^{(\frac{k}{2})}$. Hence $p(C_{n,k}) \leq (k/2) + 1$. □

Theorem 2.2 $p(C_{n,k}) = \begin{cases} n & \text{if } k \text{ is odd,} \\ (k/2)+1 & \text{if } k \text{ is even.} \end{cases}$

Proof. By Theorem 1.8, the case for odd k holds. By Lemma 2.1, the case for even k holds. □

Theorem 2.3 $t(C_{n,k}) = \lceil (k+1)/2 \rceil$ *if* $k \geq 2$.

Proof. By Lemma 2.1, the result holds for even k. Suppose k is odd. Since $\lceil (k+1)/2 \rceil = a(C_{n,k}) \leq t(C_{n,k})$, it suffices to show that $t(C_{n,k}) \leq \lceil (k+1)/2 \rceil$.

Case 1. n is odd.

Let $\mathscr{T}^{(0)}$ be the subgraph of $C_{n,k}$ induced by the edge set

$$\{a_n b_2, a_1 b_2, a_1 b_3, a_2 b_3\} \cup \{a_i b_j : j = 5, 7, 9, \cdots, n; i = j-3, j-2, j-1\}.$$

Let $\mathscr{T}^{(1)}$ be the subgraph of $C_{n,k}$ induced by the edge set

$$\{a_i b_1 : i = n-2, n-1, n\} \cup \{a_{n-1} b_2\} \cup \{a_n b_j : j = 3, 5, 7, \cdots, k\} \cup$$
$$\{a_i b_j : j = 4, 6, 8, \cdots, n-1; i = j-3, j-2, j-1\}.$$

Note that $\mathscr{T}^{(0)}$ and $\mathscr{T}^{(1)}$ are trees. For $i = 2, 3, 4, 5, \cdots, (k-1)/2$, let $\mathscr{T}^{(i)}$ be the path $(b_{2i+j} a_j)_{1 \le j \le n}$. Then $C_{n,k}$ is decomposed into trees $\mathscr{T}^{(0)}$, $\mathscr{T}^{(1)}$, $\mathscr{T}^{(2)}, \cdots, \mathscr{T}^{(\frac{k-1}{2})}$. (An example of this decomposition for $C_{11,9}$ is given in Table 1.) Thus $t(C_{n,k}) \le (k+1)/2 = \lceil (k+1)/2 \rceil$.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	×	0	0	1	2	2	3	3	4	4	×
a_2	×	×	0	1	0	2	2	3	3	4	4
a_3	4	×	×	1	0	1	2	2	3	3	4
a_4	4	4	×	×	0	1	0	2	2	3	3
a_5	3	4	4	×	×	1	0	1	2	2	3
a_6	3	3	4	4	×	×	0	1	0	2	2
a_7	2	3	3	4	4	×	×	1	0	1	2
a_8	2	2	3	3	4	4	×	×	0	1	0
a_9	1	2	2	3	3	4	4	×	×	1	0
a_{10}	1	1	2	2	3	3	4	4	×	×	0
a_{11}	1	0	1	2	1	3	1	4	1	×	×

(The mark × in the row a_i, column b_j indicates that $a_i b_j$ is not an edge. The number s in the row a_i, column b_j indicates that the edge $a_i b_j$ belongs to the tree $\mathscr{T}^{(s)}$.)

Table 1: The tree decomposition of $C_{11,9}$

Case 2. n is even.

Let $\mathcal{T}^{(0)}$ be the subgraph of $C_{n,k}$ induced by the edge set

$\{a_ib_1 : i = n-2, n-1, n\} \cup \{a_1b_2, a_1b_3, a_2b_3\} \cup$
$\{a_ib_j : j = 5, 7, 9, \cdots, n-1; i = j-3, j-2, j-1\}.$

Let $\mathcal{T}^{(1)}$ be the subgraph of $C_{n,k}$ induced by the edge set

$\{a_{n-1}b_2, a_nb_2\} \cup \{a_nb_j : j = 3, 5, 7, \cdots, k\} \cup$
$\{a_ib_j : j = 4, 6, 8, \cdots, n; i = j-3, j-2, j-1\}.$

Note that $\mathcal{T}^{(0)}$ and $\mathcal{T}^{(1)}$ are trees. For $i = 2, 3, 4, 5, \cdots, (k-1)/2$, let $\mathcal{T}^{(i)}$ be the path $(b_{2i+j}a_j)_{1 \le j \le n}$. Then $C_{n,k}$ is decomposed into trees $\mathcal{T}^{(0)}$, $\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}, \cdots, \mathcal{T}^{(\frac{k-1}{2})}$. (An example of this decomposition for $C_{12,9}$ is given in Table 2.) Thus $t(C_{n,k}) \le (k+1)/2$. □

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
a_1	×	0	0	1	2	2	3	3	4	4	×	×
a_2	×	×	0	1	0	2	2	3	3	4	4	×
a_3	×	×	×	1	0	1	2	2	3	3	4	4
a_4	4	×	×	×	0	1	0	2	2	3	3	4
a_5	4	4	×	×	×	1	0	1	2	2	3	3
a_6	3	4	4	×	×	×	0	1	0	2	2	3
a_7	3	3	4	4	×	×	×	1	0	1	2	2
a_8	2	3	3	4	4	×	×	×	0	1	0	2
a_9	2	2	3	3	4	4	×	×	×	1	0	1
a_{10}	0	2	2	3	3	4	4	×	×	×	0	1
a_{11}	0	1	2	2	3	3	4	4	×	×	×	1
a_{12}	0	1	1	2	1	3	1	4	1	×	×	×

Table 2: The tree decomposition of $C_{12,9}$

As mentioned in Section 1, it was conjectured [1] that the linear arboricity of d-regular graphs is $\lceil (d+1)/2 \rceil$. Now we show that this is true for crowns.

Theorem 2.4 $la(C_{n,k}) = \lceil (k+1)/2 \rceil$.

Proof. By Lemma 2.1, the result holds for even k. Suppose k is odd. Since $\lceil (k+1)/2 \rceil = a(C_{n,k}) \le la(C_{n,k})$, it suffices to show that $la(C_{n,k}) \le \lceil (k+1)/2 \rceil$.

If $k = 1$, it is trivial that $la(C_{n,k}) = 1$. Assume that $k \ge 3$.

For $1 \le \mu \le (k-1)/2$, let $C^{(\mu)}$ be the cycle $(a_j b_{2\mu+j+1})_{1 \le j \le n} a_1$. Note that each $C^{(\mu)}$ consists of those edges with labels 2μ and $2\mu + 1$. We see that $C_{n,k}$ is decomposed into cycles $C^{(1)}, C^{(2)}, \cdots, C^{(\frac{k-1}{2})}$ and edges $a_1 b_2, a_2 b_3, \cdots, a_{n-1} b_n, a_n b_1$. Now we reconstruct this decomposition to obtain a linear forest decomposition.

For $1 \le \mu \le (k-1)/2$, let $P^{(\mu)}$ be the path obtained from $C^{(\mu)}$ by removing the edge $a_{\frac{k-1}{2}-\mu+1} b_{\frac{k-1}{2}+\mu+1}$.

Let $F^{(0)}$ be the subgraph of $C_{n,k}$ induced by the edge set

$$\{a_i b_{i+1} : 1 \le i \le n\} \cup \{a_{\frac{k-1}{2}-\mu+1} b_{\frac{k-1}{2}+\mu+1} : 1 \le \mu \le (k-1)/2\}.$$

Note that $F^{(0)}$ is a linear forest consisting of paths of length 3 and paths of length 1. We see that $C_{n,k}$ is decomposed into linear forests $F^{(0)}, P^{(1)}, P^{(2)}, \cdots, P^{(\frac{k-1}{2})}$. Thus $la(C_{n,k}) \le (k+1)/2 = \lceil (k+1)/2 \rceil$. □

3 The star arboricity of $C_{n,k}$

Recall that a star forest is a forest each component of which is a star, and the star arboricity $sa(G)$ of a graph G is the minimum number of star forests needed to decompose the edges of G. In this section, we investigate the star arboricity of the crown.

Theorem 3.1 $sa(C_{n,k}) = \lceil k/2 \rceil + 1$ *if* $k \ge 3$ *is odd and* $n = a\lceil k/2 \rceil + b$ *where* $a \ge b$ *are nonnegative integers.*

Proof. Since $C_{n,k}$ is a regular graph with degree k, we have, by Theorem 1.3, $sa(C_{n,k}) \ge \lceil k/2 \rceil + 1$. To prove the reverse inequality, let $k = 2t - 1$ for some integer $t \ge 1$. For convenience, we consider $C_{n,k}$ as the graph with vertex set $\{c_1, c_2, \cdots, c_n\} \cup \{d_1, d_2, \cdots, d_n\}$ and edge set $\{c_i d_j : i = 1, 2, \cdots, n; j \equiv i-t+1, i-t+2, \cdots, i+t-2, i+t-1 \ (mod\ n)\}$. In the sequel of the proof, the subscripts of c and d are taken modulo n. Suppose S is a subgraph of $C_{n,k}$ and l is a positive integer. We use $S + l$ to denote the subgraph of $C_{n,k}$ induced by the edge set $\{c_{i+l} d_{j+l} : c_i d_j \in E(S)\}$. In the following, note that $n = b(t+1) + (a-b)t$.

Let G be the star forest of $C_{n,k}$ induced by the edge set

$$
\begin{aligned}
E(G) = \quad & \{c_i d_i \ : i = t+1, 2(t+1), \cdots, b(t+1)\} \\
& \cup \{c_v d_j : v = 1, (t+1)+1, 2(t+1)+1, \cdots, (b-1)(t+1)+1; \\
& \qquad j = v+1, v+2, \cdots, v+t-1\} \\
& \cup \{c_j d_v : v = 1, (t+1)+1, 2(t+1)+1, \cdots, (b-1)(t+1)+1; \\
& \qquad j = v+1, v+2, \cdots, v+t-1\} \\
& \cup \{c_v d_j : v = b(t+1)+1, b(t+1)+t+1, b(t+1)+2t+1, \\
& \qquad \cdots, b(t+1)+(a-b-1)t+1; \\
& \qquad j = v+1, v+2, \cdots, v+t-1\} \\
& \cup \{c_j d_v : v = b(t+1)+1, b(t+1)+t+1, b(t+1)+2t+1, \\
& \qquad \cdots, b(t+1)+(a-b-1)t+1; \\
& \qquad j = v+1, v+2, \cdots, v+t-1\}.
\end{aligned}
$$

Then $G, G+1, G+2, \cdots, G+(t-1)$ are edge disjoint star forests in $C_{n,k}$. (As an example, the star forests $G, G+1, G+2$ in $C_{14,5}$ are given in Fig. 1.)

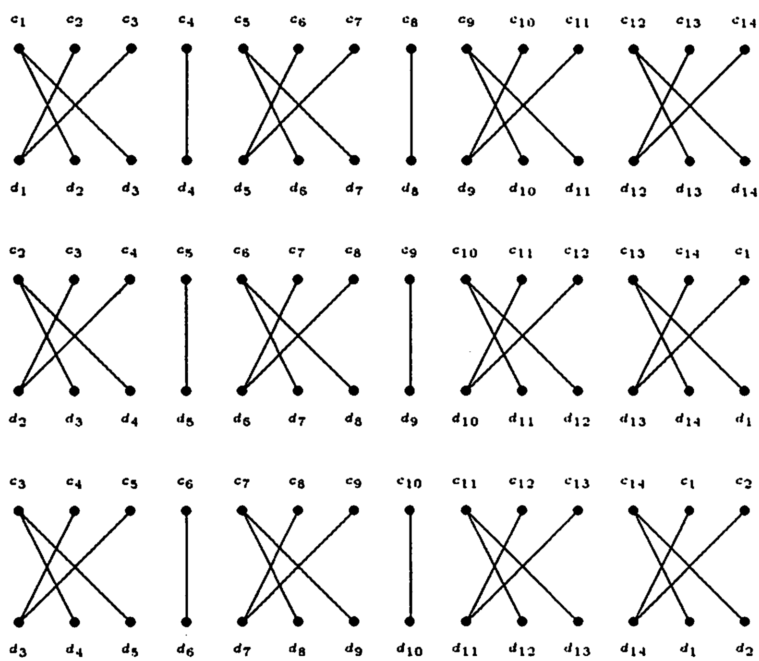

Figure 1: The star forests $G, G+1, G+2$ in $C_{14,5}$

Let H be the star forest induced by the edge set

$$
\begin{aligned}
E(H) = \quad & \{c_v d_j : v = t+1, 2(t+1), \cdots, b(t+1); \\
& \qquad j = v+1, v+2, \cdots, v+t-1\} \\
& \cup \{c_j d_v : v = t+1, 2(t+1), \cdots, b(t+1); \\
& \qquad j = v+1, v+2, \cdots, v+t-1\} \\
& \cup \{c_i d_i : i = 1, 2, 3, \cdots, t-1, t \text{ or } i = t+(t+1), \\
& \qquad t+2(t+1), t+3(t+1), \cdots, t+b(t+1) \\
& \qquad \text{or } i = t+b(t+1)+1, t+b(t+1)+2, \\
& \qquad t+b(t+1)+3, \cdots, n\}.
\end{aligned}
$$

(As an example, the star forest H in $C_{14,5}$ is given in Fig. 2.) We can see that $C_{n,k}$ can be decomposed into star forests $G, G+1, G+2, \cdots, G+(t-1)$ and H. Thus $sa(C_{n,k}) \leq t+1 = \lceil k/2 \rceil + 1$. This completes the proof. □

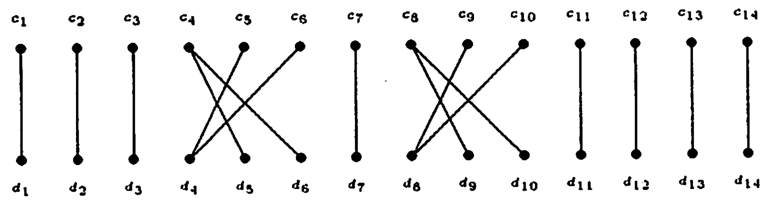

Figure 2: The star forest H in $C_{14,5}$

Theorem 3.2 *For $k = 3, 5$,*

$$sa(C_{n,k}) = \lceil k/2 \rceil + 1.$$

Proof. It is easy to see the sufficient condition of Theorem 3.1 is satisfied for the case $n \geq k = 3$ and the case $n \geq 6, k = 5$. Hence, by Theorem 3.1, $sa(C_{n,k}) = \lceil k/2 \rceil + 1$ holds for these cases. The only case left is $n = k = 5$. But this is also true since $sa(C_{5,5}) = sa(K_{5,5}) = 4$ by Theorem 1.5(7). □

Theorem 3.3 *For $k = n-1, n-2, n-3 \geq 3$,*

$$sa(C_{n,k}) = \begin{cases} \lceil k/2 \rceil + 1 & \text{if } k \text{ is odd.} \\ \lceil k/2 \rceil + 2 & \text{if } k \text{ is even.} \end{cases}$$

Proof. For the proof we reformulate the theorem as follows.

$$(1)\quad sa(C_{n,n-1}) = \lceil n/2 \rceil + 1 \quad \text{if } n \geq 4,$$

$$(2)\quad sa(C_{n,n-2}) = \begin{cases} \lceil n/2 \rceil & \text{if } n \geq 5 \text{ is odd} \\ (n/2) + 1 & \text{if } n \geq 6 \text{ is even.} \end{cases}$$

$$(3)\quad sa(C_{n,n-3}) = \lceil n/2 \rceil \quad \text{if } n \geq 6.$$

Firstly we prove (1).

Since $K_{n,n}$ can be decomposed into the crown $C_{n,n-1}$ and a matching of size n, and obviously a matching is a star forest, we have $sa(K_{n,n}) \leq sa(C_{n,n-1}) + 1$. Thus $sa(C_{n,n-1}) \geq sa(K_{n,n}) - 1$. Using Theorem 1.5(7), we then have $sa(C_{n,n-1}) \geq \lceil n/2 \rceil + 1$ for $n = 4$ or $n \geq 6$. As to the case $n = 5$, it is easy to see $sa(C_{5,4}) \geq 4$ by applying Corollary 1.4. Thus $sa(C_{n,n-1}) \geq \lceil n/2 \rceil + 1$ for $n = 5$, and hence for $n \geq 4$. Now prove the reverse inequality. Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. For a star G in K_n with center v_i and terminal vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_m}$, we use $G^{(1)}$ to denote the star in $C_{n,n-1}$ with center a_i and terminal vertices $b_{i_1}, b_{i_2}, \cdots, b_{i_m}$ and $G^{(2)}$ the star with center b_i and terminal vertices $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$. For a star forest $\mathscr{F}$ in K_n with components $G_1, G_2, \cdots, G_k$, we use $\mathscr{F}'$ to denote the star forest in $C_{n,n-1}$ with components $G_1^{(1)}, G_1^{(2)}, G_2^{(1)}, G_2^{(2)}, \cdots, G_k^{(1)}, G_k^{(2)}$. Let $l = \lceil n/2 \rceil + 1$. By Theorem 1.5(2), K_n can be decomposed into l star forests, say $\mathscr{F}_1, \mathscr{F}_2, \cdots, \mathscr{F}_l$. It is easy to see that $C_{n,n-1}$ can be decomposed into star forests $\mathscr{F}'_1, \mathscr{F}'_2, \cdots, \mathscr{F}'_l$. Thus $sa(C_{n,n-1}) \leq l = \lceil n/2 \rceil + 1$.

Secondly we show that $sa(C_{n,n-2}) = (n/2) + 1$ if $n \geq 6$ is even.

Since $sa(C_{n,n-2}) \leq sa(C_{n,n-1}) \leq (n/2) + 1$, we need to show

$sa(C_{n,n-2}) \geq (n/2) + 1$. By the first part of Theorem 1.3,

$sa(C_{n,n-2}) \geq \lceil (n-2)/2 \rceil + 1 = n/2$. Now apply the "moreover" part of Theorem 1.3. Suppose $sa(C_{n,n-2}) = n/2$. Then there is a partition of $V(C_{n,n-2})$ into independent sets $A_1, A_2, \cdots, A_{\frac{n}{2}}$ such that $|A_i| = 2n/(n/2) = 4$ for $1 \leq i \leq n/2$, and there are exactly 8 edges between A_i and A_j for $1 \leq i < j \leq n/2$. Since each A_i is an independent set in $C_{n,n-2}$ with $|A_i| = 4$, we can see that each A_i is either contained in $\{a_1, a_2, \cdots, a_n\}$ or contained in $\{b_1, b_2, \cdots, b_n\}$. Without loss of generality, we assume $A_1, A_2 \subset \{a_1, a_2, \cdots, a_n\}$. Then there is no edge between A_1 and A_2. This contradiction implies $sa(C_{n,n-2}) \geq (n/2) + 1$.

Thirdly, we show that $sa(C_{n,n-2}) = \lceil n/2 \rceil$ if $n \geq 5$ is odd.

We apply Theorem 3.1. Since $n \geq 5$ is odd, we have $n - 2 \geq 3$ is odd and $n = 2\lceil (n-2)/2 \rceil + 1$. Thus $sa(C_{n,n-2}) = \lceil (n-2)/2 \rceil + 1 = \lceil n/2 \rceil$.

Fourthly, we show that $sa(C_{n,n-3}) = \lceil n/2 \rceil$ if $n \geq 6$ is odd.

Now $n \geq 6$ is odd. Since $sa(C_{n,n-3}) \leq sa(C_{n,n-2}) = \lceil n/2 \rceil$, we only need to show $sa(C_{n,n-3}) \geq \lceil n/2 \rceil$. We apply Corollary 1.4 to $C_{n,n-3}$. We see that $n-3$ is even, $(n-3)/2 \geq 2$ and $(n-3)/2+1 = (n-1)/2$ does not divide $2n$ for $n \geq 6$. Thus $sa(C_{n,n-3}) \geq ((n-3)/2)+2 = (n+1)/2 = \lceil n/2 \rceil$.

Lastly, we show that $sa(C_{n,n-3}) = \lceil n/2 \rceil$ if $n \geq 6$ is even.

We apply Theorem 3.1. Since $n-3 \geq 3$ is odd and $n = 2\lceil (n-3)/2 \rceil + 2$, we have $sa(C_{n,n-3}) = \lceil (n-3)/2 \rceil + 1 = n/2$. □

Due to Theorems 3.2 and 3.3, we propose the following conjecture.

Conjecture A. For $3 \leq k \leq n-1$,

$$sa(C_{n,k}) = \begin{cases} \lceil k/2 \rceil + 1 & \text{if } k \text{ is odd,} \\ \lceil k/2 \rceil + 2 & \text{if } k \text{ is even.} \end{cases}$$

4 The biclique number of $C_{n,k}$

Recall that the biclique number $\tau(G)$ of a graph G is the minimum number of complete bipartite graphs needed to decompose the edges of G. In ths section we investigate the biclique number of the crown.

Lemma 4.1 *If G is a bipartite graph with bipartition (X, Y), then $\tau(G) \leq \min\{|X|, |Y|\}$.*

Proof. Let $|X| = m$ and $X = \{x_1, x_2, \cdots, x_m\}$. It is trivial that G can be decomposed into m complete bipartite graphs, namely, the maximal stars with centers at $x_1, x_2, \cdots, x_m$ respectively. Thus $\tau(G) \leq m = |X|$. Similarly, $\tau(G) \leq |Y|$. □

Suppose $e_1, e_2, \cdots, e_k$ ($k \geq 2$) are edges in a graph G. If e_i and e_j are not in any complete bipartite subgraph of G for every $i \neq j$, then we say that $e_1, e_2, \cdots, e_k$ are *incompatible edges* in G.

Lemma 4.2 *If a graph G contains k incompatible edges, then $\tau(G) \geq k$.*

Proof. Suppose G contains k incompatible edges and suppose G is decomposed into $\tau(G)$ complete bipartite graphs. Since the k incompatible edges belong to distinct complete bipartite subgraphs, $k \leq \tau(G)$. □

Let G be a bipartite graph with bipartition (X, Y) where $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. We use $M(G)$ to denote the $m \times n$ matrix (e_{ij}) where

$$e_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is adjacent to } y_j \text{ in } G, \\ 0 & \text{otherwise.} \end{cases}$$

For a matrix M, we use $r(M)$ to denote the *rank* of M.

Lemma 4.3 [14] *Suppose G is a bipartite graph. Then $\tau(G) \geq r(M(G))$.* □

For a positive integer n, let $\zeta_n = e^{i\frac{2\pi}{n}}$.

Lemma 4.4 [18] *The cyclic determinant*

$$\det \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_0 & \cdots & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_0 \end{bmatrix}$$

$$= \prod_{j=0}^{n-1} (\alpha_0 + \zeta_n^j \alpha_1 + \zeta_n^{2j} \alpha_2 + \cdots + \zeta_n^{(n-1)j} \alpha_{n-1}) \qquad \square$$

Lemma 4.5 *Suppose n, j, k are positive integers such that $j \not\equiv 0 \pmod n$. Then $\zeta_n^j + \zeta_n^{2j} + \zeta_n^{3j} + \cdots + \zeta_n^{kj} = 0$ if and only if $kj \equiv 0 \pmod n$.*

Proof. This follows from the identity $\zeta_n^j + \zeta_n^{2j} + \zeta_n^{3j} + \cdots + \zeta_n^{kj} = \zeta_n^j \dfrac{1 - \zeta_n^{kj}}{1 - \zeta_n^j}$. □

Theorem 4.6 *$\tau(C_{n,k}) = n$ if $k \leq (n+1)/2$ or* $\gcd(k, n) = 1$.

Proof. It follows from Lemma 4.1 that $\tau(C_{n,k}) \leq n$. Now we show $\tau(C_{n,k}) \geq n$.

Suppose $k \leq (n+1)/2$. We show that $a_1b_2, a_2b_3, \cdots, a_{n-1}b_n, a_nb_1$ are incompatible edges. Suppose, to the contrary, that for some $1 \leq i < j \leq n$, a_ib_{i+1} and a_jb_{j+1} are in a complete bipartite subgraph of $C_{n,k}$. Then a_ib_{j+1} and a_jb_{i+1} are edges in $C_{n,k}$, which implies $j+1 \leq i+k$ and $i+1 \leq j+k-n$. Thus $n \leq 2k - 2$. This contradicts that $k \leq (n+1)/2$. This contradiction confirms there are n incompatible edges in $C_{n,k}$. Thus, by Lemma 4.2, $\tau(C_{n,k}) \geq n$.

Suppose $\gcd(k, n) = 1$. The crown $C_{n,k}$ is a bipartite graph with bipartition (A, B) where $A = \{a_1, a_2, \cdots, a_n\}$, $B = \{b_1, b_2, \cdots, b_n\}$. Thus

$$M(C_{n,k}) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_0 & \cdots & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_0 \end{bmatrix}$$

where $\alpha_0 = 0, \alpha_1 = \alpha_2 = \cdots = \alpha_k = 1, \alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_{n-1} = 0$. By Lemma 4.4, $\det(M(C_{n,k})) = \prod_{j=0}^{n-1} (\zeta_n^j + \zeta_n^{2j} + \cdots + \zeta_n^{kj})$. Since $\gcd(k, n) = 1$, we have $kj \not\equiv 0 \pmod n$ for $1 \leq j \leq n-1$. Thus, by Lemma 4.5, $\prod_{j=1}^{n-1} (\zeta_n^j + \zeta_n^{2j} + \cdots + \zeta_n^{kj}) \neq 0$ and hence $\det(M(C_{n,k})) = k \prod_{j=1}^{n-1} (\zeta_n^j + \zeta_n^{2j} + \cdots + \zeta_n^{kj}) \neq 0$ Thus $r(M(C_{n,k})) = n$. By Lemma 4.3, $\tau(C_{n,k}) \geq n$. □

We propose the following conjecture.

Conjecture B. For $1 \leq k \leq n-1$, $\tau(C_{n,k}) = n$.

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