Toughness of Generalized Petersen Graphs

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Abstract

Upper and lower bounds are given for the toughness of generalized Petersen graphs. A lower bound of 1 is established for $t(G(n,k))$ for all n and k. This bound of 1 is shown to be sharp if $n = 2k$ or if n is even and k is odd. The upper bounds depend on the parity of k. For k odd, the upper bound $\frac{n}{n-\frac{k+1}{2}}$ is established. For k even, the value $\frac{2k}{2k-1}$ is shown to be an asymptotic upper bound. Computer verification shows the reasonableness of these bounds for small values of n and k.

1 Terminology

Generalized Petersen graphs were defined by Watkins [7]. For each $n \geq 3$ and $0 < k < n$, the generalized Petersen graph $G(n,k)$ has vertex set

$$V = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$$

and edge set

$$E = \{(u_i, u_{i+1}) | 1 \leq i \leq n\} \cup \{(u_i, v_i) | 1 \leq i \leq n\} \cup \{(v_i, v_{i+k}) | 1 \leq i \leq n\}.$$

Here, all subscripts are taken modulo n. Watkins excludes the case in which $n = 2k$. However, we adopt the convention of Alspach [1] and include those cases as well. The subgraph of $G(n,k)$ induced by $\{u_1, u_2, \ldots, u_n\}$ is called the outer rim. The subgraph induced by $\{v_1, v_2, \ldots, v_n\}$ is called the inner rim. An edge of the form (u_i, v_i) is called a spoke. Clearly, the automorphism group of $G(n,k)$ contains the dihedral group of order $2n$. In fact, the obvious dihedral action on $G(n,k)$ preserves setwise the inner rim, the outer rim, and the spokes. Given a positive integer $m \leq n$, an m-section of $G(n,k)$ is a subgraph induced by a subset of V of the form

$$\{u_i, u_{i+1}, \ldots, u_{i+m-1}, v_i, v_{i+1}, \ldots, v_{i+m-1}\}$$

for some i. If $n = \Sigma m_j$, then $G(n,k)$ can be built from a union of m_j-sections in the obvious way.

The toughness of a connected non-complete graph $G = (V, E)$ is defined by Chvátal [3] as

$$t(G) = min\{\frac{|S|}{\omega(G-S)} \mid S \subseteq V, \ S \text{ disconnects } G\}, \qquad (1.1)$$

where $\omega(G-S)$ is the number of components of the graph obtained from G by removing the vertices of S. A subset $S \subseteq V$ that achieves the minimum in (1.1) is called a tough set for G. Two tough sets S_1 and S_2 for $G(n,k)$ are said to be dihedral equivalent if there is an automorphism ϕ in the dihedral group such that $\phi(S_1) = S_2$. A dihedral tough class is a dihedral equivalence class of tough sets. Note that there are some pairs (n,k) such that the automorphism group of $G(n,k)$ is bigger than the dihedral group [5]. However, we do not consider non-dihedral automorphisms when identifying tough classes.

Our main interest is in determining bounds for $t(G(n,k))$. Related to this is the notion of asymptotic bounds, which we now define. For a fixed value of k, we say that b is an asymptotic upper bound for the toughness of the class of graphs $G(n,k)$ if

$$\lim_{n\to\infty} t(G(n,k)) \le b.$$

Of course, asymptotic lower bounds are defined similarly.

2 Introduction

The toughness of generalized Petersen graphs was first explored in [6]. There, the case in which $k = 1$ is completely settled.

Theorem 2.1 **([6]).** *For $n \ge 3$,*

$$t(G(n,1)) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n}{n-1} & \text{if } n \text{ is odd} \end{cases}$$

Note that

$$\lim_{n\to\infty} t(G(n,1)) = 1.$$

Hence, 1 is an asymptotic upper bound for the toughness of the class of graphs $G(n,1)$.

The case in which $k = 2$ is considered in [4]. The value $\frac{5}{4}$ is shown to be the critical value for $t(G(n,2))$. Namely, $\frac{5}{4}$ is both a lower bound and an asymptotic upper bound.

Theorem 2.2 ([4]). *For $n \geq 5$,*

$$t(G(n,2)) \leq \frac{5}{4} + \epsilon_2(n),$$

where

(i) $\epsilon_2(n) \geq 0$ *for* $n \neq 8$,

(ii) $\epsilon_2(n) \longrightarrow 0$ *as* $n \longrightarrow \infty$, *and*

(iii) $\epsilon_2(n) = 0$ *if* $n \equiv 0 \bmod 7$.

Remark 2.3. *For $n > 8$, the values for ϵ_2 given in [4] are:*

$$\epsilon_2(n) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 7, \\ \frac{7}{8n+20} & \text{if } n \equiv 1 \bmod 7, \\ \frac{7}{8n+12} & \text{if } n \equiv 2 \bmod 7, \\ \frac{7}{8n+4} & \text{if } n \equiv 3 \bmod 7, \\ \frac{7}{16n+20} & \text{if } n \equiv 4 \bmod 7, \\ \frac{7}{16n+4} & \text{if } n \equiv 5 \bmod 7, \\ \frac{7}{16n-12} & \text{if } n \equiv 6 \bmod 7. \end{cases}$$

Note that $\epsilon_2(n) \leq \frac{7}{8n} \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Theorem 2.4 ([4]). *For $n \geq 5$ and $n \neq 8$,*

$$t(G(n,2)) \geq \frac{5}{4}.$$

Also, $t(G(3,2)) = \frac{3}{2}$, $t(G(4,2)) = 1$, *and* $t(G(8,2)) = \frac{6}{5}$.

Corollary 2.5 ([4]). *For $n \equiv 0 \bmod 7$, $t(G(n,2)) = \frac{5}{4}$.*

3 Main Results

We present results like Theorems 2.1 and 2.2, but for general $G(n,k)$.

Theorem 3.1. *For $n \geq 3$ and $0 < k < n$,*

(a) $t(G(n,k)) \geq 1$.
Moreover, if n is odd and $gcd(n,k) = 1$, then $t(G(n,k)) > 1$.

(b) *If k is odd and $n \geq 2k+1$, then*

$$t(G(n,k)) \leq \begin{cases} 1 & \text{if } n \text{ is even}, \\ \frac{n}{n-(\frac{k+1}{2})} & \text{if } n \text{ is odd}. \end{cases}$$

(c) *If $k \geq 4$, k is even, and $n \geq 2k+1$, then*

$$t(G(n,k)) \leq \frac{2k}{2k-1} + \epsilon_k(n),$$

where $\epsilon_k(n) < \frac{4k}{n}$. Moreover, $\epsilon_k(n) = 0$ if $2k \mid n$.

(d) *$t(G(n,k)) = 1$ if $n = 2k$ or if n is even and k is odd.*

Note that Theorem 3.1(b) with $k = 1$ yields the exact value given in Theorem 2.1. However, the asymptotic upper bound of $\frac{5}{4}$ given in Theorem 2.2 is not obtained by plugging the value $k = 2$ into the asymptotic upper bound $\frac{2k}{2k-1}$ given in Theorem 3.1(c). There is nonetheless a strong relationship between the case in which $k = 2$ and the other cases in which k is even. This is evident from the similarity in the proofs of Theorems 2.2 and 3.1(c).

The proof of Theorem 3.1 follows from the sequence of results in the subsequent sections.

4 Lower Bounds

Chvátal's primary interest in considering toughness [3] is its relationship with Hamiltonicity. One such relationship is our main tool for establishing lower bounds.

Theorem 4.1 **([3]).** *If G is a Hamiltonian graph, then $t(G) \geq 1$.*

Theorem 4.1 applies to all Hamiltonian generalized Petersen graphs. A complete characterization of all such graphs is given by Alspach [1].

Theorem 4.2 **([1]).** *All generalized Petersen graphs are Hamiltonian except for*

(i) *$G(n,2) \cong G(n,n-2) \cong G(n,\frac{n-1}{2}) \cong G(n,\frac{n+1}{2})$ for $n \equiv 5$ mod 6,*

(ii) *$G(n,\frac{n}{2})$ for $n \equiv 0$ mod 4, $n \geq 8$.*

Theorem 4.2 leaves only two exceptional cases in which the lower bound of 1 must be established directly. Since Theorem 2.4 takes care of the exceptions in Theorem 4.2(i), it only remains to consider the case in which $n = 2k$.

Theorem 4.3. *For n even, $t(G(n,\frac{n}{2})) = 1$.*

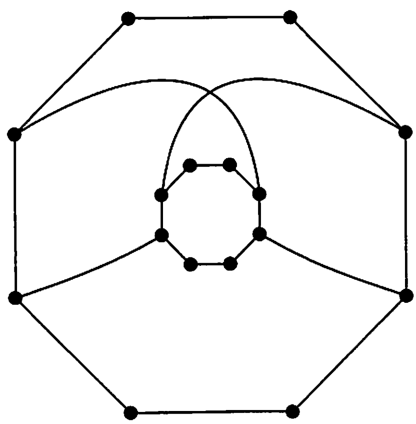

Figure 1: A nonstandard picture of $G(8,4)$

Proof. Let $k = \frac{n}{2}$. The disconnecting set $S = \{u_1, v_{k+1}\}$ shows that

$$t(G(n,k)) \leq 1.$$

If $k = 2$ or k is odd, then the lower bound of 1 follows from Theorem 4.1. For $k = 4$, the graph $G(8,4)$ can be drawn as pictured in Figure 1. Let C_1 and C_2 be the pictured inner and outer cycles, respectively. Let S be a tough set for $G(8,4)$, and let $S_i = S \cap C_i$, for $i = 1, 2$. If either S_1 or S_2 is empty, then it is easy to see that $\omega(G(8,4) - S) \leq |S|$. Hence, it suffices to assume that both S_1 and S_2 are nonempty. Since $t(C_1) = t(C_2) = 1$, it follows that $\omega(C_i - S_i) \leq |S_i|$, for $i = 1, 2$. Hence,

$$\omega(G(8,4) - S) \leq \omega(C_1 - S_1) + \omega(C_2 - S_2) \leq |S_1| + |S_2| = |S|.$$

That is, $t(G(8,4)) \geq 1$.

It remains to establish the lower bound of 1 for the case in which k is even and $k > 4$. Let S be a nonempty subset of the vertices of $G(n,k)$. To show that

$$\omega(G(n,k) - S) \leq |S| \tag{4.1}$$

it suffices to show that $\omega(H - S) \leq |S|$, where H is a subgraph of $G(n,k)$ obtained by removing some of the edges of $G(n,k)$. The subgraph H that we consider is a subgraph induced by a cycle C of order $2n - 3$ in $G(n,k)$ plus the 3 remaining vertices of $G(n,k)$. The cycle C is described as a union of paths in $G(n,k)$. For each integer $1 \leq j \leq \frac{k}{2} - 1$, let

$$P_j = v_{2j-1}v_{k+2j-1}u_{k+2j-1}u_{k+2j}v_{k+2j}v_{2j}u_{2j}u_{2j+1}v_{2j+1}$$

Also let $P_{\frac{k}{2}} = v_{k-1}v_{-1}u_{-1}u_nu_1v_1$. The cycle C is obtained as

$$C = P_1 \cup P_2 \cup \cdots \cup P_{\frac{k}{2}-1} \cup P_{\frac{k}{2}}.$$

Observe that C contains all of the vertices of $G(n,k)$ except $\{u_k, v_k, v_n\}$. The subgraph H is pictured in Figure 2, where C is the outer cycle.

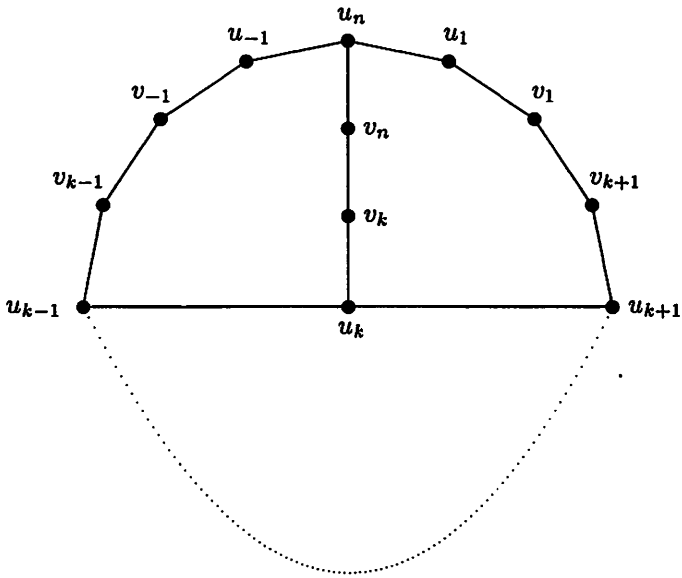

Figure 2: Subgraph H of $G(n, \frac{n}{2})$

Let $T = \{u_{k-1}, u_k, u_{k+1}, u_n\}$. Note that T consists of all the vertices of degree 3 in H. The remaining $2n-4$ vertices have degree 2 in H and can therefore be adjacent to at most 2 components of $H - S$.

Case 1: $S \cap T$ is empty.

Let $S_2 = S \cap \{v_k, v_n\}$ and $S_1 = S - S_2$. If S_2 is empty, then inequality (4.1) follows easily from the fact that $t(C) = 1$. If S_2 is nonempty, consider removing the vertices of S by first removing the vertices of S_1 and then removing the vertices of S_2 one at a time. Clearly, $\omega(H - S_1) \leq |S_1|$. Since $S \cap T$ is empty, the removal of a vertex of S_2 can not increase the number of components of $H - S_1$. Hence, inequality (4.1) must hold.

Case 2: $S \subseteq T$.

Let ϕ be the automorphism of $G(n,k)$ defined by

$$\phi(u_i) = u_{i+3} \quad \text{and} \quad \phi(v_i) = v_{i+3}$$

for each $1 \leq i \leq n$. Note that $\phi(T) = \{u_{k+2}, u_{k+3}, u_{k+4}, u_3\}$. Since $k > 4$, $T \cap \phi(T)$ is empty. Hence, $S \cap \phi(T)$ is empty. Of course, $\phi(H)$ is a subgraph

of $G(n,k)$ which is isomorphic to H. By replacing H by $\phi(H)$ and T by $\phi(T)$, the argument in Case 1 shows that inequality (4.1) holds.
Case 3: $S \cap T$ is nonempty.
Let $S_1 = S \cap T$ and $S_2 = S - S_1$. Consider removing the vertices of S_1 and then removing the vertices of S_2 one at a time. By Case 2, $\omega(G(n,k) - S_1) \leq |S_1|$. Since each removal of a vertex of S_2 can increase the number of components by at most one, inequality (4.1) must hold. □

Combining the results of Theorems 4.1, 4.2, 2.4, and 4.3 gives the first halves of parts (a) and (d) of Theorem 3.1. The second half of Theorem 3.1(d) is justified in Section 5. The second half of Theorem 3.1(a) is a consequence of the following results from [2] and [5].

Theorem 4.4 ([2]). *If G is a connected vertex-transitive graph such that $t(G) = 1$, then G is bipartite or an odd cycle.*

Theorem 4.5 ([5]). *If $gcd(n,k) = 1$, then $G(n,k) \cong G(n,k')$ is vertex transitive, where k' is the multiplicative inverse for k modulo n.*

5 Upper Bounds

Theorem 5.1. *For k odd and $n \geq 2k+1$,*

$$t(G(n,k)) \leq \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n}{n-(\frac{k+1}{2})} & \text{if } n \text{ is odd} \end{cases}$$

Proof. It suffices to give an example of a disconnecting subset $S \subseteq V$ such that the fraction

$$\frac{|S|}{\omega(G(n,k) - S)}$$

equals the asserted upper bound. The set S is specified by partitioning the vertices of $G(n,k)$ into sections and specifying which vertices of each section belong to S by circling them. The section that we use the most is the 2-section pictured in Figure 3. Our convention is to plot the outer rim

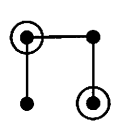

Figure 3: Key 2-section

vertices on the top row and the inner rim vertices on the bottom row. The outer rim edges and the spokes are always drawn but the inner rim edges

are only sometimes drawn.
Case 1: n is even.
In this case, $G(n,k)$ can be built entirely with $\frac{n}{2}$ copies of the 2-sections from Figure 3. The set S is simply obtained by choosing two vertices from each of those 2-sections. Since k is odd, the vertices in $G(n,k) - S$ form an independent set.
Case 2: n is odd.
In this case, $G(n,k)$ can not be built entirely with the 2-sections from Figure 3. Our aim is to mostly use that 2-section. However, we also need a starter section of odd length. The starter section that is used is the $(2k-1)$-section generated by $\{u_1, u_2, \ldots, u_{2k-1}, v_1, v_2, \ldots, v_{2k-1}\}$. The subset of vertices to be included in S is

$$\{u_{2j+1} | 0 \le j < \tfrac{k-1}{2}\} \cup \{u_{k+2j+1} | 0 \le j < \tfrac{k-1}{2}\} \cup$$
$$\{v_{2j+2} | 0 \le j < \tfrac{k-1}{2}\} \cup \{v_{k+2j} | 0 \le j \le \tfrac{k-1}{2}\}.$$

The starter $(2k-1)$-section for the case in which $k = 7$ is pictured in Figure 4. Note that if the $2k-1$ vertices from S are removed from the starter

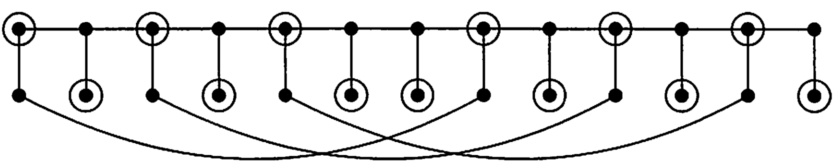

Figure 4: Key $(2k-1)$-section for $k = 7$

$(2k-1)$-section, then $\frac{3(k-1)}{2}$ components remain. The graph $G(n,k)$ can be built from the starter $(2k-1)$-section and $\frac{n-2k+1}{2}$ copies of the 2-section in Figure 3. Clearly $|S| = n$. It is also easy to see that

$$\omega(G(n,k) - S) = \frac{3(k-1)}{2} + n - 2k + 1 = n - \frac{k+1}{2}.$$

Here, components from the $(2k-1)$-section do not get attached to components from the 2-sections. □

Note that Theorem 5.1 combined with Theorem 3.1(a) gives the second half of Theorem 3.1(d).

Theorem 5.2. *For k even, $k \ge 4$, and $n \ge 2k+1$,*

$$t(G(n,k)) \le \frac{2k}{2k-1} + \epsilon_k(n),$$

where

(i) $\epsilon_k(n) \longrightarrow 0$ *as* $n \longrightarrow \infty$, *and*

(ii) $\epsilon_k(n) = 0$ *if* $n \equiv 0$ *mod* $2k$.

Proof. Our aim is to use a particular $2k$-section as much as possible. If the set of vertices in the $2k$-section is $\{u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_{2k}\}$, then the subset to be included in S is

$$\{u_1\} \cup \{u_{2j} | 0 < j \leq \tfrac{k}{2}\} \cup \{u_{k+2j+1} | 0 < j < \tfrac{k}{2}\} \cup$$
$$\{v_{2j+1} | 0 < j \leq \tfrac{k}{2}\} \cup \{v_{k+2j} | 0 < j \leq \tfrac{k}{2}\}.$$

The $2k$-section for the case in which $k = 6$ is pictured in Figure 5. Note

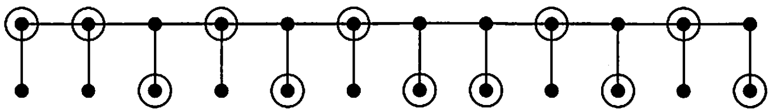

Figure 5: Key $2k$-section for $k = 6$

that if the $2k$ vertices from S are removed from the $2k$-section, then $2k-1$ components remain. If $2k \mid n$, then $G(n,k)$ can be built entirely with copies of the $2k$-section represented in Figure 5. Moreover,

$$\frac{|S|}{\omega(G(n,k) - S)} = \frac{2k}{2k-1}.$$

That is, $\epsilon_k(n) = 0$ and $\frac{2k}{2k-1}$ is the associated upper bound. If $2k \nmid n$, then a starter m_n-section is needed, where $m_n \equiv n$ modulo $2k$. The graph $G(n,k)$ can then be built with one copy of the m_n-section and $\frac{n-m_n}{2k}$ copies of the $2k$-section represented in Figure 5. An upper bound of $\frac{4k}{n}$ for $\epsilon_k(n)$ is obtained by taking m_n to be the unique integer such that

$$0 \leq m_n < 2k \quad \text{and} \quad m_n \equiv n \text{ modulo } 2k$$

and specifying a starter m_n-section with all of its vertices in S. □

Remark 5.3. *The method of computing the error term* $\epsilon_k(n)$ *in the proof of Theorem 5.2 is certainly not the best possible if* $2k \nmid n$. *A better approach for* $n > 4k$ *is to use a starter* m_n*-section where*

$$2k < m_n < 4k \quad \textit{and} \quad m_n \equiv n \textit{ modulo } 2k.$$

Moreover, the best starter section is likely to come from one of the possible tough sets for $G(m_n, k)$. *More details and evidence regarding this point are given in Subsection 6.2.*

6 Conjectures and Computer Evidence

This section is devoted to showing the reasonableness of the upper bounds given in parts (b) and (c) of Theorem 3.1. In fact, modulo finding the smallest possible values of the error terms $\epsilon_k(n)$ in Theorem 3.1(c), we believe that our upper bounds in parts (b) and (c) of Theorem 3.1 are sharp. Specifically, we conjecture that the following lower bounds complement our upper bounds.

Conjecture 6.1. *If n and k are odd and $n \geq 3k$, then*

$$t(G(n,k)) \geq \frac{n}{n - \frac{k+1}{2}}.$$

Conjecture 6.2. *If k is even, $k \geq 4$, and $n \geq 2k+1$, then*

$$t(G(n,k)) \geq \frac{2k}{2k-1}.$$

To give evidence that our bounds are the best possible, we use a computer to determine $t(G(n,k))$ for all $3 \leq k \leq 7$ and $2k+1 \leq n \leq 18$. Additionally, all of the dihedral tough classes have been determined in those cases. However, the complete list of dihedral tough classes is only presented here for $k = 4$. That case is given special attention in Subsection 6.2.

Recall that Theorem 3.1(d) tells us that $t(G(n,k)) = 1$ if $n = 2k$ or if n is even and k is odd. Table 1 lists the values of $t(G(n,k))$ for all $3 \leq k \leq 7$ and $5 \leq n \leq 18$. The value X signifies that $G(n,k)$ is not defined for those values of n and k. Since $G(n,k) \cong G(n,n-k)$, the table is completely determined by the portion for which $n \geq 2k$.

$k \backslash n$	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	$\frac{4}{3}$	1	$\frac{5}{4}$	1	$\frac{9}{7}$	1	$\frac{11}{9}$	1	$\frac{13}{11}$	1	$\frac{15}{13}$	1	$\frac{17}{15}$	1
4	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	1	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{11}{9}$	$\frac{5}{4}$	$\frac{13}{11}$	$\frac{14}{11}$	$\frac{5}{4}$	$\frac{8}{7}$	$\frac{16}{13}$	$\frac{6}{5}$
5	X	1	$\frac{5}{4}$	1	$\frac{4}{3}$	1	$\frac{9}{7}$	1	$\frac{13}{10}$	1	$\frac{5}{4}$	1	$\frac{17}{14}$	1
6	X	X	$\frac{7}{6}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{5}{4}$	$\frac{9}{7}$	1	$\frac{9}{7}$	$\frac{7}{6}$	$\frac{5}{4}$	$\frac{16}{13}$	$\frac{17}{15}$	$\frac{5}{4}$
7	X	X	X	1	$\frac{4}{3}$	1	$\frac{11}{9}$	1	$\frac{9}{7}$	1	$\frac{13}{10}$	1	$\frac{17}{14}$	1

Table 1: Toughness Values

6.1 Odd k and n

For n and k odd, the proof of Theorem 5.1 gives a disconnecting set S such that

$$\frac{|S|}{\omega(G(n,k)-S)} = \frac{n}{n-\frac{k+1}{2}}. \tag{6.1}$$

However, Table 1 shows that $t(G(n,k))$ is not always given by (6.1). If $k=3$ and $n=7$, then $t(G(7,3)) = \frac{5}{4} < \frac{n}{n-2}$. If $k=5$ and $n=11$, then $t(G(11,5)) = \frac{9}{7} < \frac{n}{n-3}$. If $k=7$ and $n \in \{15,17\}$, then $t(G(n,7)) < \frac{n}{n-4}$. This shows that some condition like $n \geq 3k$ is needed in Conjecture 6.1.

6.2 Even k

For k even, the proof of Theorem 5.2 uses a key $2k$-section to show that, if $2k \mid n$, then

$$t(G(n,k)) \leq \frac{2k}{2k-1}.$$

In general, the upper bound is bigger than $\frac{2k}{2k-1}$ by an error term $\epsilon_k(n)$. If we write $n = q\cdot 2k + m_n$ for nonnegative q and m_n, then $G(n,k)$ can be built with q copies of the $2k$-section and one copy of some m_n-section. Since the error term comes from the m_n-section, the key to minimizing $\epsilon_k(n)$ is to find the best possible starter m_n-section. The tough sets provided by the computer for $n \leq 18$ help us to accomplish this.

For $k=4$, the key $2k$-section defined in the proof of Theorem 5.2 is the topmost 8-section in Figure 6. Of course, shifted versions of this 8-section can alternatively be used in the same way. Two such shifted versions are included in Figure 6.

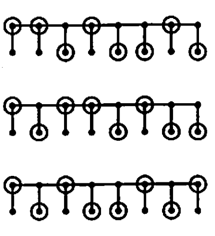

Figure 6: Shifted versions of key $2k$-section for $k=4$

Figures 7 through 16 show all of the dihedral tough classes for $G(n,4)$ for $9 \leq n \leq 18$. Note that the unique dihedral tough class for $G(16,4)$ is obtained from two copies of the key 8-section. The unique dihedral tough class for $G(17,4)$ is obtained by joining the key 8-section with the first tough set for $G(9,4)$ in Figure 7. Also, each of the dihedral tough classes for $G(18,4)$ comes from joining a key 8-section from Figure 6 with a tough set for $G(10,4)$ in Figure 8.

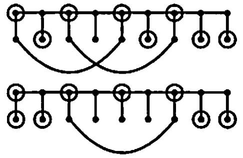

Figure 7: Dihedral tough classes for $G(9,4)$

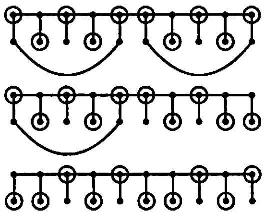

Figure 8: Dihedral tough classes for $G(10,4)$

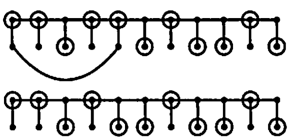

Figure 9: Dihedral tough classes for $G(11,4)$

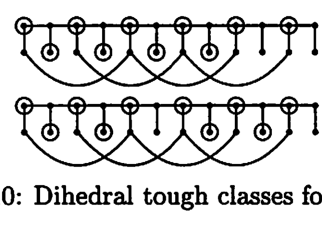

Figure 10: Dihedral tough classes for $G(12,4)$.

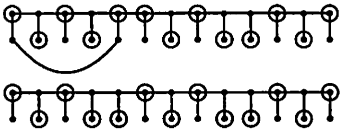

Figure 11: Dihedral tough classes for $G(13,4)$

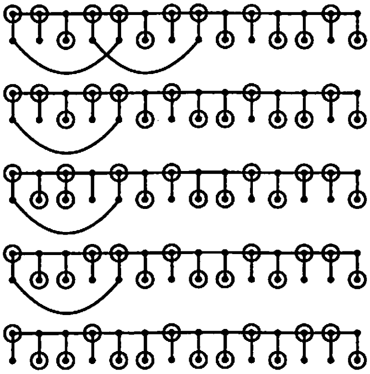

Figure 12: Dihedral tough classes for $G(14,4)$

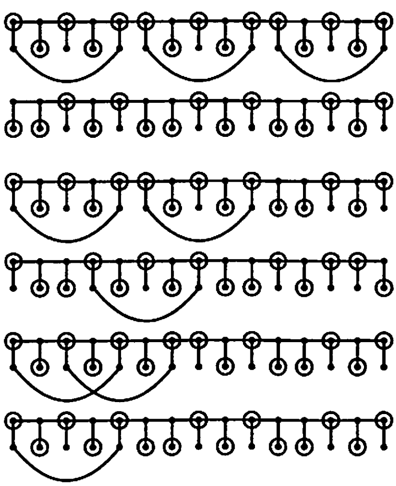

Figure 13: Dihedral tough classes for $G(15,4)$

Figure 14: Unique dihedral tough class for $G(16,4)$

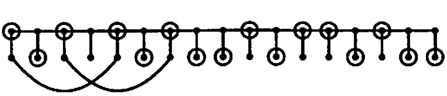

Figure 15: Unique dihedral tough class for $G(17,4)$

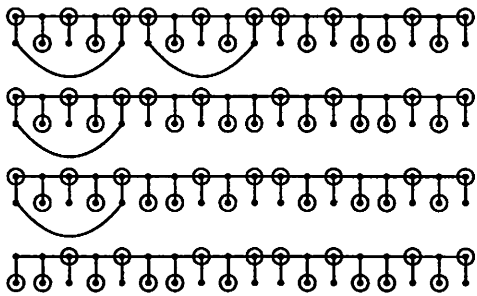

Figure 16: Dihedral tough classes for $G(18,4)$

In light of the evidence in Figures 7 through 16, we propose for starter sections the first tough sets listed for $G(n,4)$ for $n \in \{9,10,11,13,14,15\}$. For $n \equiv 4$ modulo 8, we propose the starter section shown in Figure 17. Note that this 12-section does not appear in Figure 10. Those 12-sections

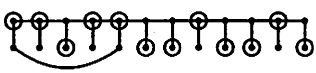

Figure 17: Starter section for $G(n,4)$ if $n \equiv 4 \bmod 8$

can not be used as starter sections due to the cyclic components which appear in them. This occurs since 12 can be factored as $12 = 3 \cdot 4$. A similar exception appears in Theorem 2.2 for $G(8,2)$ since $8 = 4 \cdot 2$.

Remark 6.3. *The values for ϵ_4 given by our proposed starter sections are:*

$$\epsilon_4(n) = \begin{cases} 0 & \textit{if } n \equiv 0 \textit{ mod } 8 \\ \frac{8}{7} \cdot \frac{15}{7n-15} & \textit{if } n \equiv 1 \textit{ mod } 8 \\ \frac{8}{7} \cdot \frac{6}{7n-6} & \textit{if } n \equiv 2 \textit{ mod } 8 \\ \frac{8}{7} \cdot \frac{5}{7n-5} & \textit{if } n \equiv 3 \textit{ mod } 8 \\ \frac{8}{7} \cdot \frac{12}{7n-12} & \textit{if } n \equiv 4 \textit{ mod } 8 \\ \frac{8}{7} \cdot \frac{3}{7n-3} & \textit{if } n \equiv 5 \textit{ mod } 8 \\ \frac{8}{7} \cdot \frac{10}{7n-10} & \textit{if } n \equiv 6 \textit{ mod } 8 \\ \frac{8}{7} \cdot \frac{9}{7n-9} & \textit{if } n \equiv 7 \textit{ mod } 8 \end{cases}$$

Note that, for our choice of starter sections for $k = 4$, the appropriate key section from Figure 6 depends on the starter section. For $n \equiv_8 3,4,6$, the first key section in Figure 6 should be used. For $n \equiv_8 1$, the second key section is required. The third key section from Figure 6 is needed for

$n \equiv_8 2, 5, 7$. It is possible to use the same key section for each of the congruence classes modulo 8. However, longer starter sections are needed. In this case, we would need to use starter m_n-sections where

$$16 < m_n < 24 \quad \text{and} \quad m_n \equiv n \text{ modulo } 8.$$

References

[1] B. Alspach, *The classification of Hamiltonian generalized Petersen graphs*, Journal of Combinatorial Theory, Series B **34** (1983), 293–312.

[2] K. J. Bagga and M. J. Lipman, *A characterization of vertex transitive graphs of toughness one*, Bulletin of the ICA **10** (1994), 39–49.

[3] V. Chvátal, *Tough graphs and Hamiltonian circuits*, Discrete Math. **5** (1973), 215–228.

[4] K. K. Ferland, *On the toughness of some generalized Petersen graphs*, Ars Combinatoria **36** (1993), 65–88.

[5] R. Frucht, J. E. Graver, and M. E. Watkins, *The groups of the generalized Petersen graphs*, Proc. Camb. Phil. Soc. **70** (1971), 211–218.

[6] B. Piazza, R. Ringeisen, and S. Stueckle, *On the vulnerability of cycle permutation graphs*, Ars Combinatoria **29** (1990), 289–296.

[7] M. E. Watkins, *A theorem on Tait colorings with an application to the generalized Petersen graphs*, Journal of Combinatorial Theory **6** (1969), 152–164.