A note on integral sum crowns

G. Suresh Singh
Department of Mathematics
University of Kerala
Kariavattom - 695 581
Trivandrum, Kerala, India

G. Santhosh
Department of Mathematics
T.K. Madhava Memorial College
Nangiarkulangara - 690 513
Alleppey (Dist.), Kerala, India
email: santhosheraviperoor@eudoramail.com

ABSTRACT. The *sum graph* of a set S of positive integers is the graph $G^+(S)$ having S as its vertex set, with two vertices adjacent if and only if their sum is in S. A graph G is called a *sum graph* if it is isomorphic to the sum graph $G^+(S)$ of some finite subset S of N. An *integral sum graph* is defined just as the sum graph, the difference being that S is a subset of Z instead of N. The *sum number* of a graph G is defined as the smallest number of isolated vertices when added to G results in a sum graph. The *integral sum number* of G is defined analogously. In this paper we study some classes of integral sum graphs.

Keywords: Integral sum graph, Integral sum number, Crown.

1. Introduction

All graphs considered here are finite simple graphs. We follow Harary[3] for various graph theoretic terminology and notation not explained here.

Let N denote the set of positive integers. The *sum graph* $G^+(S)$ of a finite subset S of N is the graph with vertex set S and edge set E such that for $u, v \in S$, $uv \in E$ if and only if $u+v \in S$. A graph G is called a *sum graph* if it is isomorphic to the sum graph $G^+(S)$ of some finite subset S of N. The *sum number* $\sigma(G)$ of a graph G is defined as the smallest nonnegative integer m for which $G \cup m K_1$ is a sum graph.

An equivalent definition of a sum graph is given below. A graph G is called a *sum graph* if there exists a labeling of the vertices of G by distinct positive integers such that the vertices labeled u and v are adjacent if and only if there exists a vertex labeled $u+v$.

The notions of sum graph and sum number were first introduced by Harary [4]. Later in 1990 Harary [5] also introduced integral sum graphs and integral sum number as given below. An *integral sum graph* $G^{+}(S)$ is the sum graph with $S \subset Z$ instead of $S \subset N$. The *integral sum number* $\zeta(G)$ of a graph G is the smallest nonnegative integer s such that $G \cup s K_1$ is an integral sum graph. By definition it is clear that G is an integral sum graph if and only if $\zeta(G)=0$.

In general it is very difficult to determine $\sigma(G)$ and $\zeta(G)$ for a given graph G and integral sum numbers seem more difficult to compute than sum numbers. Harary [5] proved that paths and matchings $m K_2$ are integral sum graphs. Chen [2] showed that any tree obtained from a star by extending each edge to a path is an integral sum graph. Chang et al. [1] proved that caterpillars, cycles and wheels are integral sum graphs.

In this paper we show that the crowns, $C_n \odot K_1$ are integral sum graphs for $n \geq 4$.

2. On cycle graphs

If G_1 and G_2 are graphs and G_1 has n vertices then $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. In this section we show that the crowns $C_n \odot K_1$ are integral sum graphs for $n \geq 4$. We first consider crowns $C_n \odot K_1$ when n is odd.

In the graph $C_n \odot K_1$ where n is odd and $n > 7$, let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of the cycle C_n and u_{n-i} be the vertex attached to the vertex v_{1+i}, $0 \leq i \leq (n-1)$.

Set $v_1 = 1,\ v_2 = 4,\ v_3 = 5,\ v_4 = -4$ and $v_i = v_{i-2} - v_{i-1}$, for $5 \leq i \leq n$.

$u_1 = v_{n-1} - v_n\ ,\ u_2 = v_n - v_{n-1}$

$u_{3+i} = u_{1+i} - v_{n-2-i}$ for $0 \leq i \leq (n-7)$

$u_{n-3} = v_n + v_1$

$u_{n-2} = u_{n-4} - v_3 = u_{n-4} - 5$

$u_{n-1} = u_{n-3} - v_2 = u_{n-3} + v_4 = u_{n-3} - 4$

$u_n = u_{n-2} - v_1 = u_{n-2} - 1$

Let $S = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ where v_i's and u_i's are as defined above. The labeling is illustrated in Figure 1.

In order to prove the main result of this section we require the following observation and lemmas.

Observation 1. The vertices v_i and u_i of S defined above satisfy

$$0 < |v_1| < |v_2| = |v_4| < |v_3| < |v_5| < |v_6| < |v_7| < \ldots < |v_{n-1}| < |u_{n-1}|$$

$< |v_n| < |u_{n-3}| < |u_1| = |u_2| < |u_4| < |u_6| < \ldots < |u_{n-5}| < |u_3| < |u_5| < \ldots < |u_n|$

Lemma 2.1. (Corollary 2 of [2])

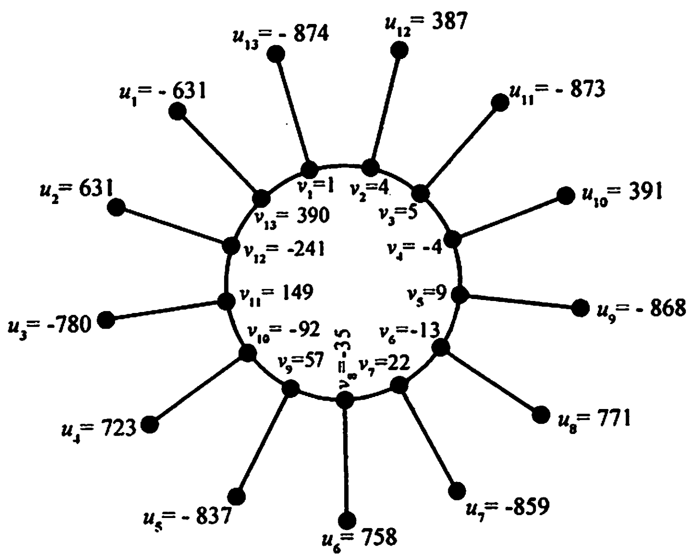
Let $P_n = b_1\ b_2 \ldots b_n$ be a path with n vertices b_i, $1 \le i \le n$. Define a labeling f of the vertices of P_n as follows:

$f(b_1) = 1, f(b_2) = t, f(b_3) = 1 + t, f(b_4) = -t$ *and*

$f(b_k) = f(b_{k-2}) - f(b_{k-1})$ *for* $5 \le k \le n$, *where t is an integer greater than* 1.

Then (1) $|f(b_i)| > |f(b_{i-1})|$ *and* $f(b_i)\, f(b_{i-1}) < 0$ *for any* $5 \le i \le n$.

(2) *f gives an integral sum labeling for P_n ($n \ge 5$).* □



$C_{13} \odot K_1$

Figure 1

In the following lemmas we prove some properties of the labeling of $C_n \odot K_1$ (n is odd, $n > 7$) defined above.

Lemma 2.2. *Suppose n is odd and $n > 7$. There exists no i, j, k, $1 \le i, j, k \le n$ such that $u_i + u_j = u_k$.*

Proof. Suppose that $u_i + u_j = u_k$ for some i, j, k, $1 \le i, j, k \le n$. We consider two cases.

Case 1. k is even

Then $u_k > 0$.

(1) If i and j are both even then min $\{ u_i + u_j \mid i$ and j are even, $1 \le i, j \le n \} > u_k$ for all even k, $1 \le k \le n$. Hence $u_i + u_j = u_k$ is impossible.

(2) If i and j are both odd then $u_i < 0, u_j < 0$ so that $u_i + u_j < 0$. Since $u_k > 0$, $u_i + u_j = u_k$ is impossible.

(3) If i is even and j is odd then by Observation 1, $u_i + u_j < 0$ when $j \neq 1$, $2 \leq j \leq n$, $1 \leq i \leq n$. Since $u_k > 0$ for even k, $1 \leq k \leq n$, $u_i + u_j = u_k$ is impossible for even i and k, $1 \leq i, k \leq n$ and odd j, $2 \leq j \leq n$. Also $u_2 + u_1 = 0 \notin S$. $u_{n-1} + u_1 < 0$ and $u_{n-3} + u_1 < 0$. Since $u_k > 0$, $u_i + u_1 = u_k$ is impossible for $i \in \{2, (n-1), (n-3)\}$. For even $i \notin \{2, (n-1), (n-3)\}$, $1 \leq i \leq n$, $\max\{u_i + u_1\} < u_k$ for all even k, $1 \leq k \leq n$. Hence $u_i + u_1 = u_k$ is impossible for even $i \notin \{2, (n-1), (n-3)\}$, $1 \leq i \leq n$ and for all even k, $1 \leq k \leq n$.

(4) The case when i is odd and j is even is similar to (3)

Case 2. k is odd

Then $u_k < 0$.

(1) If i and j are both even then $u_i + u_j > 0$ and since $u_k < 0$, $u_i + u_j = u_k$ is impossible.

(2) If i and j are both odd then $u_i < 0$, $u_j < 0$. Also $u_k < 0$. But for all odd k, $1 \leq k \leq n$, $u_k > \max\{u_i + u_j \mid i, j \text{ odd}, 1 \leq i, j \leq n\}$. Hence $u_i + u_j = u_k$ is impossible in this case.

(3) If i is even and j is odd then $u_i > 0$ and $u_j < 0$. Also $u_k < 0$. But for odd k, $1 \leq k \leq n$, $u_k < \min\{u_i + u_j \mid i \text{ even}, j \text{ odd}, 1 \leq i, j \leq n\}$. Hence $u_i + u_j = u_k$ is impossible in this case also.

(4) The case when i is odd and j is even is similar to (3). This completes the proof of the lemma. □

Lemma 2.3. *Suppose that* n *is odd and* $n > 7$. *Then there exists no* i, j, k, $1 \leq i, j, k \leq n$ *such that* $u_i + u_j = v_k$.

Proof. Suppose that $u_i + u_j = v_k$ for some i, j, k, $1 \leq i, j, k \leq n$. We consider two cases.

Case 1. k is even.

Clearly $u_i + u_j \neq v_2$, $1 \leq i, j \leq n$. We consider even $k \neq 2$, $1 \leq k \leq n$. Then $v_k < 0$.

(1) If i and j are both even the $u_i + u_j > 0$ and since $v_k < 0$ for even $k \neq 2$, $u_i + u_j = v_k$ is impossible.

(2) If i and j are both odd then $u_i < 0$ and $u_j < 0$ so that $u_i + u_j < 0$. Then $u_i + u_j = v_k$ implies $|u_i + u_j| = |v_k|$, a contradiction since $|v_k| < |u_i + u_j|$ for all even k and i, j odd, $1 \leq i, j, k \leq n$.

(3) If i is even and j is odd then $u_i > 0$ and $u_j < 0$. By definition $u_1 = v_{n-1} - v_n$, $u_{n-3} = v_n + 1$, $u_{n-1} = v_n - 3$. Hence $u_{n-3} + u_1 = v_{n-1} + 1 \notin S$, $u_{n-1} + u_1 = v_{n-1} - 3 \notin S$. Also $u_2 + u_1 = 0 \notin S$.

For all even $i \notin \{2, (n-3), (n-1)\}$, $1 \le i \le n$, $u_i + u_1 > 0$ by Observation 1. Since $v_k < 0$, $u_i + u_1 = v_k$ is impossible in this case. When $j \neq 1$, using Observation 1 we get $u_i + u_j < 0$. In this case $u_i + u_j = v_k$ implies $(-1)^i a_i + (-1)^j a_j = (-1)^{k+1} w_k$ where $a_i = |u_i|$ and $w_k = |v_k|$. Since i and k are even and j is odd we get $|a_i - a_j| = w_k$ which is impossible.

(4) If i is odd and j is even then we proceed as in (3).

Case 2. k is odd

Then $v_k > 0$

(1) If i and j are both even then $u_i + u_j > 0$. But $u_i + u_j = v_k$ is impossible since $v_k < u_i + u_j$.

(2) If i and j are both odd then $u_i + u_j < 0$ and since $v_k > 0$, $u_i + u_j = v_k$ is impossible.

(3) If i is even and j is odd then $u_i < 0$ and $u_j > 0$. By definition $u_2 + u_1 = 0 \notin S$. By Observation 1, $u_i + u_1 < 0$ for $i = (n-1)$ and $i = (n-3)$. Since $v_k > 0$, $u_i + u_1 = v_k$ is impossible for $i = (n-1)$ and $i = (n-3)$. For even $i \notin \{2, (n-1), (n-3)\}$, $1 \le i \le n$, $u_i + u_1 = v_k$ implies $(-1)^i a_i - a_1 = (-1)^{k+1} w_k$ so that $|a_i - a_1| = w_k$ which is impossible for odd k, $1 \le k \le n$. By Observation 1, for odd $j \neq 1$, $u_i + u_j < 0$ and since $v_k > 0$, $u_i + u_j = v_k$ is impossible.

(4) If i is odd and j is even then we have a similar argument as in (3).

This completes the proof of the lemma. □

Lemma 2.4. *Suppose n is odd and $n > 7$. For $1 \le i, j, k \le n$, $v_i + v_j = u_k$ holds only when $v_n + v_1 = u_{n-3}$.*

Proof. By definition, $v_n + v_1 = u_{n-3}$, $v_n > 0$, $v_1 = 1$ and since $v_i + v_j < v_n$ for all $i, j \in \{2, 3, \ldots, (n-1)\}$ there exist no $i, j \in \{2, 3, \ldots, (n-1)\}$ such that $v_i + v_j = u_{n-3}$. By definition, $u_{n-1} = u_{n-3} - v_2 = u_{n-3} + v_4$. Also $u_{n-1} = v_n - 3$, $-3 \notin S$. If $i, j \in \{1, 2, 3, \ldots, (n-1)\}$ then $v_i + v_j < u_{n-1}$. Since $v_n > u_{n-1}$, $v_n + v_j > u_{n-1}$, if $v_j > 0$. If $v_j < 0$, max $\{v_n + v_j, v_j < 0, v_j \in S\} = v_n - 4 < v_n - 3 = u_{n-1}$. Hence $v_n + v_j < u_{n-1}$ if $v_j < 0$. Hence $v_i + v_j = u_{n-1}$ is impossible for $i, j \in \{1, 2, 3, \ldots, n\}$. For all i, j, k, $1 \le i, j, k \le n$, $k \notin \{(n-1), (n-3)\}$, $|v_i + v_j| < |u_k|$. Hence $v_i + v_j = u_k$ is impossible in this case also. This completes the proof of the lemma. □

Lemma 2.5. *Suppose n is odd and $n > 7$. For $1 \le i, j, k \le n$, $u_i + v_j = v_k$ holds only in the cases $u_1 + v_n = v_{n-1}$ and $u_2 + v_{n-1} = v_n$.*

Proof. By definition, $u_1 + v_n = v_{n-1}$ and $u_2 + v_{n-1} = v_n$. Also for $i \notin \{1, 2\}$ if u_i is the vertex attached to v_j, by definition, $u_i + v_j = u_k$, $3 \le i \le n$, $1 \le j, k \le n$. Suppose that $u_i + v_j = v_k$, for some other i, j, k, $1 \le i, j, k \le n$. We consider two cases.

Case 1. k is even.

Then $v_k < 0$ for even $k \neq 2$, $v_2 = 4$. Clearly $u_i + v_j \neq v_2$. We consider even $k \neq 2, 1 \leq k \leq n$.

(1) If i and j are even then $u_i > 0$ and $v_j < 0$. By Observation 1, $u_i + v_j > 0$. Since $v_k < 0, u_i + v_j = v_k$ is impossible.

(2) If i is even and j is odd then $u_i > 0$ and $v_j > 0$ so that $u_i + v_j > 0$. Since $v_k < 0$, $u_i + v_j = v_k$ is impossible.

(3) If i is odd and j is even then $u_i < 0$ and $v_j < 0$ ($j \neq 2$) so that $u_i + v_j < 0$. Since $| u_i + v_j | > | v_k |, u_i + v_j = v_k$ is impossible.

(4) If i and j are both odd then $u_i < 0$, $v_j > 0$ and $u_i + v_j < 0$. By Observation 1, for odd $j \neq n$, $u_1 + v_j < v_{n-1}$ and for odd $i \neq 1$, $u_i + v_n < v_{n-1}$. Also for odd $i \neq 1$, odd $j \neq n$, $1 \leq i, j \leq n$, it is easy to see that $u_i + v_j < v_{n-1}$. Since $v_{n-1} = \min \{ v_r \mid r \text{ even}, 1 \leq r \leq n \}$ it follows that for odd i, j and even k, $1 \leq i, j, k \leq n$, $u_i + v_j = v_k$ holds only in the case $u_1 + v_n = v_{n-1}$.

Case 2. k is odd.

Then $v_k > 0$.

(1) If i and j are even then $u_i > 0$ and $v_j < 0 (j \neq 2)$, $v_2 = 4$. Then $u_i + v_j > 0$. By Observation 1, for even $j \neq (n - 1)$, $1 \leq j \leq (n - 3)$, $u_2 + v_j > v_k$ for all k, $1 \leq k \leq n$. For even $i \neq 2$, $4 \leq i \leq (n - 5)$, $u_i + v_{n-1} > v_k$ for all k, $1 \leq k \leq n$. It is easy to verify that for even $j \neq 4$, $2 \leq j \leq n$, $u_{n-3} + v_j \notin S$ and for even j, $4 \leq j \leq n$, $u_{n-1} + v_j \notin S$. For even $i \neq 2$, for even $j \neq (n - 1)$, $4 \leq i \leq (n - 5), 1 \leq j \leq (n - 3)$, $u_i + v_j > v_k$ for all k, $1 \leq k \leq n$. It follows that for even i, j and odd k, $1 \leq i, j, k \leq n$, $u_i + v_j = v_k$ holds only in the case $u_2 + v_{n-1} = v_n$.

(2) If i and j are both odd then $u_i < 0$, $v_j > 0$ and $u_i + v_j < 0$. Since $v_k > 0$, $u_i + v_j = v_k$ is impossible.

(3) If i is even and j is odd then $u_i > 0$, $v_j > 0$ so that $u_i + v_j > 0$. Since $u_i + v_j > v_k$, $u_i + v_j = v_k$ is impossible.

(4) If i is odd and j is even then $u_i < 0$, $v_j < 0 (j \neq 2)$, $v_2 = 4$ so that $u_i + v_j < 0$. Since $v_k > 0$, $u_i + v_j = v_k$ is impossible.

This completes the proof of the lemma. □

Lemma 2.6. *Suppose $n > 7$ and n is odd. For $1 \leq i, j, k \leq n$, $u_i + v_j = u_k$ holds only in the following cases.*

(i) $u_r = u_{r+2} + v_{n-(r+1)}, 1 \leq r \leq (n - 6)$

(ii) $u_{n-1} = u_{n-3} + v_4$ (iii) $u_{n-2} = u_n + v_1$

(iv) $u_{n-3} = u_{n-1} + v_2$ (v) $u_{n-4} = u_{n-2} + v_3$

Proof. By definition we have (i), (ii), (iii), (iv) and (v). Suppose that $u_i + v_j = u_k$ holds in cases other than (i) to (v) also. We consider two cases.

Case 1. k is even.

Then $u_k > 0$.

(1) If i is odd and j is even then $u_i < 0$, $v_j < 0\,(j \neq 2)$, $v_2 = 4$ and $u_i + v_j < 0$. Since $u_k > 0$, $u_i + v_j = u_k$ is impossible.

(2) If i and j are both odd then $u_i < 0$, $v_j > 0$ and since $|u_i| > |v_j|$ we get $u_i + v_j < 0$. Since $u_k > 0$, $u_i + v_j = u_k$ is impossible.

(3) If i and j are even then $u_i > 0$ and $v_j < 0\,(j \neq 2)$, $v_2 = 4$ and since $|u_i| > |v_j|$ we get $u_i + v_j > 0$. By definition, $u_{n-1} = u_{n-3} + v_4$. Note that $u_{n-3} + v_2 = u_{n-3} + 4 \notin S$. Also $u_{n-1} = \min\{u_r \mid r \text{ even}, 1 \le r \le n\}$. For even j, $4 < j \le (n-1)$, $v_j < 0$, $|v_j| > |v_4| = 4$. Hence $u_{n-3} + v_j < u_{n-1}$ for even j, $4 < j \le (n-1)$. Thus $u_{n-3} + v_j = u_k$ is impossible for even j, $4 < j \le (n-1)$ and even k, $1 \le k \le n$.

Now by definition, $u_{n-3} = u_{n-1} + v_2$. For even j, $2 < j \le (n-1)$, $u_{n-1} + v_j < u_{n-1}$ since $u_{n-1} > 0$ and $v_j < 0$. Hence $u_{n-1} + v_j = u_k$ is impossible for even j, $2 < j \le (n-1)$ and for even k, $1 \le k \le n$.

By definition, $u_r = u_{r+2} + v_{n-(r+1)}$, for even r, $2 \le r \le (n-7)$. Since $v_2 > 0$, $u_i < u_i + v_2 < u_{i+2}$ for all even i, $2 \le i \le (n-7)$ and $u_{n-5} + v_2 > u_{n-5} = \max\{u_i \mid i \text{ even}, 1 \le i \le n\}$. Hence by Observation 1, $u_i + v_2 = u_k$ is impossible for even i, $2 \le i \le (n-5)$ and even k, $1 \le k \le n$. Also $u_{n-3} < u_2 + v_4 < u_2$ and $u_{i-2} < u_i + v_4 < u_i$ for all even i, $4 \le i \le (n-5)$. So by Observation 1, $u_i + v_4 = u_k$ is impossible for even i, $2 \le i \le (n-5)$ and even k, $1 \le k \le n$. Note that $u_2 < u_2 + v_2 < u_4$ and for all even j, $4 \le j \le (n-3)$, $u_{n-3} < u_2 + v_j < u_2$. Hence Observation 1 implies that $u_2 + v_j = u_k$ is impossible for all even j, $2 \le j \le (n-3)$, and even k, $1 \le k \le n$. Now consider $u_i + v_j = u_k$ for even i, j, k, $4 \le i \le (n-5)$, $6 \le j \le (n-3)$, $1 \le k \le n$. Suppose that the vertex u_i is attached to the vertex v_s in $C_n \odot K_1$ where s is even, $6 \le s \le (n-3)$. If j is even and $6 \le j < s \le (n-3)$, then $u_{i-2} < u_i + v_j < u_i$ for all even i, $4 \le i \le (n-7)$. If j is even and $(n-3) \ge j > s \ge 6$ then $u_{t(j+2)} < u_i + v_j < u_{t(j)}$ where $u_{t(j)}$ is the vertex attached to v_j, $6 \le i \le (n-5)$. Finally, $u_{n-3} < u_i + v_{n-1} < u_2$, for all even i, $4 \le i \le (n-5)$. Hence by Observation 1, $u_i + v_j = u_k$ is possible for even i, j, k, $1 \le i, j, k \le n$ only in the cases (i), (ii) and (iv).

(4) If i is even and j is odd then $u_i > 0$ and $v_j > 0$. By definition, $u_{n-1} = u_{n-3} + v_4$ where $u_{n-3} > 0$ and $v_4 = -4$.

$u_{n-3} + v_n > \max\{u_i \mid i \text{ even}, 1 \le i \le n\}$. Hence $u_{n-3} + v_n = u_k$ is impossible for even k, $1 \le k \le n$. Now $u_{n-3} + v_{n-2} > u_{n-1}$.

Also $u_{n-3} + v_{n-2} < \min\{u_i \mid i \text{ even}, 1 \le i \le n, i \neq (n-3), (n-1)\}$ where $v_{n-2} = \max\{v_j \mid j \text{ odd}, 1 \le j \le (n-2)\}$. Hence $u_{n-3} + v_j = u_k$ is impossible for even k and odd j, $1 \le j, k \le n$.

Now, by definition $u_{n-3} = u_{n-1} + v_2$ where $u_{n-1} > 0$ and $v_2 = 4$. Also $u_{n-1} + v_n > \max\{u_i \mid i \text{ even}, 1 \le i \le n\}$. Hence $u_{n-1} + v_n = u_k$ is impossible for even k, $1 \le k \le n$.

Since $v_{n-2} = \max\{v_j \mid j \text{ odd}, 1 \le j \le (n-2)\}$ and $u_{n-1} + v_{n-2} < u_k$, for all even $k \notin \{(n-3), (n-1)\}$, $1 \le k \le n$ and $u_{n-1} + v_{n-2} > u_{n-3}$, $u_{n-1} + v_j = u_k$ is impossible for odd j and even k, $j \ne n$, $k \notin \{(n-3), (n-1)\}$. Hence $u_{n-1} + v_j = u_k$ is impossible for odd j and even k, $1 \le j, k \le n$.

Next we consider $u_{n-5} = \max\{u_i \mid i \text{ even}, 1 \le i \le n\}$. Since $v_j > 0$ for odd j, $u_{n-5} + v_j > u_k$ for all even k, $1 \le k \le n$. Hence $u_{n-5} + v_j = u_k$ is impossible for odd j and even k, $1 \le j, k \le n$.

By definition, $u_{k+2} + v_{n-(k+1)} = u_k$ for $1 \le k \le (n-6)$. We consider $u_i + v_j = u_k$ for even i, $2 \le i \le (n-7)$, even k and odd j, $1 \le j, k \le n$. Suppose that the vertex u_i is attached to the vertex v_s in $C_n \odot K_1$. Note that s is even and $u_i + v_{s-1} > u_{n-5} = \max\{u_i \mid i \text{ even}, 1 \le i \le n\}$ for $2 \le i \le (n-7)$. If j is odd, and $j > (s-1)$ then $v_j > v_{s-1}$. Hence $u_i + v_j > u_{n-5}$ for all odd j, $(s-1) < j \le n$. Also for odd j, $1 \le j < (s-1)$, $u_i < u_i + v_j < u_{i+2}$, $2 \le i \le (n-7)$. By Observation 1, $u_i + v_j = u_k$ is impossible for even i, $2 \le i \le (n-7)$ and odd j, $1 \le j \le n$.

Case 2. k is odd

Then $u_k < 0$.

(1) If i is even and j is odd then $u_i > 0$, $v_j > 0$ and $u_i + v_j > 0$. Since $u_k < 0$, $u_i + v_j = u_k$ is impossible.

(2) If i and j are even then also $u_i + v_j > 0$ and since $u_k < 0$, $u_i + v_j = u_k$ is impossible.

(3) If i and j are both odd then $u_i < 0$, $v_j > 0$ and $u_i + v_j < 0$. $u_1 = \max\{u_i \mid i \text{ odd}, 1 \le i \le n\}$. Since $v_j > 0$, $u_1 + v_j > u_1$ for all odd j, $1 \le j \le n$. Hence $u_1 + v_j = u_k$ is impossible for all j, k odd, $1 \le j, k \le n$. $u_i + v_n > u_1$ for all odd i, $1 \le i \le n$. Hence $u_i + v_n = u_k$ is impossible for odd i, k, $1 \le i, k \le n$. Suppose that u_i is the vertex attached to v_s where i, s odd. As in case 1 (3) it is easy to see that $|u_p| < |u_i + v_j| < |u_{p+2}|$ for all odd i, j, s, p, $2 \le i \le n$, $1 \le j \le (n-2)$, $1 \le s \le n$, $1 \le p \le (n-4)$, $j \ne s$. Hence Observation 1 implies $u_i + v_j = u_k$ is possible only when $i = k+2$ and $j = n-(k+1)$ for odd k.

(4) If i is odd and j is even then $u_i < 0$, $v_j < 0$ $(j \ne 2)$, $v_2 = 4$ so that $u_i + v_j < 0$. Since $v_2 > 0$, $u_1 + v_2 > u_k$ for all odd k, $1 \le k \le n$. Also for odd i, $3 \le i \le (n-2)$, $u_i < u_i + v_2 < u_{i-2}$ and $u_{n-2} < u_n + v_2 < u_{n-4}$. Hence $u_i + v_2 = u_k$ is impossible for odd i and k. Suppose that the vertex u_i is attached to the vertex v_s in $C_n \odot K_1$. It is easy to see that for odd i, k, p, s, even j, $1 \le i, k, s \le n$, $4 \le j \le (n-1)$, $1 \le p \le (n-2)$, $u_{p+2} < u_i + v_j < u_p$

if $j < s$ and $u_i + v_j > u_k$ if $j > s$. Hence $u_i + v_j = u_k$ is impossible for odd i, k and even j, $1 \le i, j, k \le n$. This completes the proof of the lemma. □

Now we prove the following theorem.

Theorem 2.1. *The crowns $C_n \odot K_1$ are integral sum graphs for all odd $n \ge 5$.*

Proof. It is easy to verify that $C_5 \odot K_1 \cong G^+\{1, 5, 6, -5, 11, 18, 34, 12, 29, 33\}$ and $C_7 \odot K_1 \cong G^+\{1,4, 5, -4, 9, -13, 22, -35, 35, -44, 23, -49, 19, -50\}$. For $n > 7$ we claim that $C_n \odot K_1 \cong G^+(S)$ where $S = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$ with the v_i's and u_i's as defined in the beginning of this section. By Lemma 2.1. the v_i's in S make the path $v_1 v_2 \ldots v_n$ an integral sum graph.

Since $v_n + v_1 = u_{n-3} \in S$, $v_n v_1$ is also an edge of $G^+\{v_1, v_2, ..., v_n, u_{n-3}\}$. By Lemma 2.3. there exists no i, j, k, $1 \le i, j, k \le n$ such that $u_i + u_j = v_k$. By Lemma 2.2. there exists no i, j, k, $1 \le i, j, k \le n$ such that $u_i + u_j = u_k$. Hence $u_i + u_j \notin S$ for all i, j, $1 \le i, j \le n$. By Lemma 2.4., $v_i + v_j = u_k$ only in the case $v_n + v_1 = u_{n-3}$. For $1 \le i, j, k \le n$, Lemma 2.5. shows that $u_i + v_j = v_k$ only in the cases $u_1 + v_n = v_{n-1}$ and $u_2 + v_{n-1} = v_n$. Finally by Lemma 2.6., for $1 \le i, j, k \le n$, $u_i + v_j = u_k$ holds only in the following cases:

(i) $u_r = u_{r+2} + v_{n-(r+1)}$, $1 \le r \le (n-6)$

(ii) $u_{n-1} = u_{n-3} + v_4$ (iii) $u_{n-2} = u_n + v_1$

(iv) $u_{n-3} = u_{n-1} + v_2$ (v) $u_{n-4} = u_{n-2} + v_3$

This completes the proof of the theorem. □

Now we consider the crowns $C_n \odot K_1$ for even $n \ge 4$. First we consider even $n \ge 10$. For even $n \ge 10$ we denote the consecutive vertices of the cycle in $C_n \odot K_1$ by $v_1, v_2, ..., v_n$ and let $u_1, u_2, ..., u_n$ be the vertices attached to $v_4, v_2, v_n, v_{n-1}, v_{n-2}, v_{n-4}, v_{n-6}, ..., v_6, v_1, v_3, v_5, ..., v_{n-3}$ such that the vertex attached to v_1 is $u_{(n/2)+2}$.

Set $v_1 = 1$, $v_2 = n$, $v_3 = n+1$, $v_4 = -n$, $v_k = v_{k-2} - v_{k-1}$ for $5 \le k \le n$

$u_1 = v_n + v_1$, $u_2 = u_1 + v_4$, $u_3 = v_{n-1} - v_n$, $u_4 = v_n - v_{n-1}$

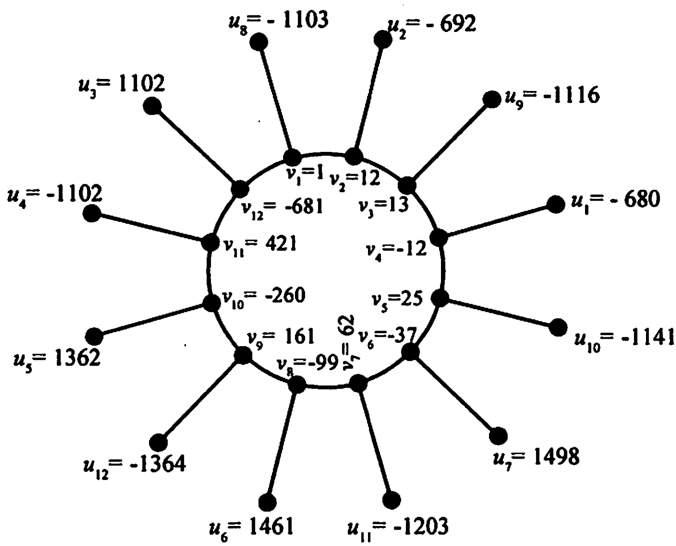
$u_5 = u_3 - v_{n-2}$, $u_{6+i} = u_{5+i} - v_{n-(2i+4)}$ for $0 \le i \le ((n/2)-5)$

$u_{(n/2)+2} = u_4 - v_1$, $u_{(n/2)+3+i} = u_{(n/2)+2+i} - v_{2i+3}$

for $0 \le i \le ((n/2)-3)$.

Let $S_1 = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$ where the v_i's and u_i's are as defined above.

The labelling is illustrated in Figure 2



$C_{12} \odot K_1$

Figure 2

Observation 2. The vertices v_i and u_i of S_1 defined above satisfy

$$|v_1| < |v_2| = |v_4| < |v_3| < |v_5| < |v_6| < |v_7| < \ldots < |v_{n-1}| < |u_1| < |v_n| < |u_2| < |u_3| = |u_4| < |u_{(n/2)+2}| < |u_{(n/2)+3}| < \ldots < |u_{n-1}| < |u_5| < |u_n| < |u_6| < |u_7| < \ldots < |u_{(n/2)+1}|.$$

Lemma 2.7. *Suppose n is even and $n \geq 10$. Then there exists no i, j, k, $1 \leq i, j, k \leq n$ such that $u_i + u_j = u_k$.*

Proof. The u_i's in S_1 satisfy the following inequalities.

$\max\{u_i + u_j \mid u_i < 0, u_j < 0\} = u_1 + u_2 < u_k$ for all $k, 1 \leq k \leq n$.

$\min\{u_i + u_j \mid u_i > 0, u_j > 0\} = u_3 + u_5 > u_k$ for all $k, 1 \leq k \leq n$.

$\max\{u_i + u_j \mid u_i < 0, u_j > 0, u_i + u_j > 0\} = u_{(n/2)+1} + u_1 < u_k$ for all $u_k > 0$.

$\min\{u_i + u_j \mid u_i < 0, u_j > 0, u_i + u_j < 0\} = u_n + u_3 > u_k$ for all $u_k < 0$.

Hence $u_i + u_j = u_k$ is impossible for $i, j, k, 1 \leq i, j, k \leq n$. □

Lemma 2.8. *Suppose n is even and $n \geq 10$. Then there exists no i, j, k, $1 \leq i, j, k \leq n$ such that $u_i + u_j = v_k$.*

Proof. Suppose that $u_i + u_j = v_k$ for some i, j, k, $1 \leq i, j, k \leq n$.
We consider two cases.

Case 1. k is odd

Then $v_k > 0$. If $u_i < 0$ and $u_j < 0$ then $u_i + u_j < 0$. Since $v_k > 0$, $u_i + u_j = v_k$ is impossible. If $u_i > 0$ and $u_j > 0$ then $u_i + u_j > v_k$. If $u_i < 0$ and $u_j > 0$ then either $u_i + u_j < 0$ or $u_i + u_j > 0$. If $u_i + u_j < 0$ then since $v_k > 0$, $u_i + u_j = v_k$ is impossible. If $u_i + u_j > 0$ then $u_i + u_j = v_k$ implies $-a_i + a_j = w_k$ which is impossible for odd k where $|u_i| = a_i$ and $|v_k| = w_k$, $1 \le i, k \le n$.

The case $u_i > 0$ and $u_j < 0$ is similar to the above case.

Case 2. k is even

Then $v_k < 0$ for even $k \ne 2$ and $v_2 = n > 0$. Clearly $u_i + u_j \ne v_2$. We consider the case when k is even and $k \ne 2$. If $u_i > 0$ and $u_j > 0$ then $u_i + u_j > 0$ and since $v_k < 0$, $u_i + u_j = v_k$ is impossible. If $u_i < 0$ and $u_j < 0$ then since $|u_i + u_j| > |v_k|$, $u_i + u_j = v_k$ is impossible. If $u_i < 0$ and $u_j > 0$ then either $u_i + u_j > 0$ or $u_i + u_j < 0$. If $u_i + u_j > 0$ then since $v_k < 0$, $u_i + u_j = v_k$ is impossible. If $u_i + u_j < 0$ then $u_i + u_j = v_k$ implies $|-a_i + a_j| = w_k$ which is impossible. A similar argument holds if $u_i > 0$ and $u_j < 0$. This completes the proof of the lemma. □

Lemma 2.9. *Suppose n is even and $n \ge 10$. For $1 \le i, j, k \le n$, $v_i + v_j = u_k$ holds only when $v_n + v_1 = u_1$.*

Proof. By definition, $v_n + v_1 = u_1$, $v_n < 0$, $v_1 = 1$. Since $v_i + v_j > v_n + 1 = u_1$ for all $i, j \in \{2, 3, \ldots, (n-1)\}$ there exists no $i, j \in \{2, 3, \ldots, (n-1)\}$ such that $v_i + v_j = u_1$. By definition, $u_2 = u_1 + v_4 = u_1 - v_2$. Also $u_2 = v_n - (n-1)$, $-(n-1) \notin S_1$. If $i, j \in \{1, 2, 3, \ldots, (n-1)\}$, $v_i + v_j > u_2$. If $v_j < 0$ then since $v_n < 0$, $u_2 < 0$, $|v_j| > (n-1)$ we get $v_n + v_j < u_2$. If $v_j > 0$, since $\min\{v_n + v_j \mid v_j > 0, v_j \in S_1\} = v_n + v_1 = u_1 > u_2$, we get $v_n + v_j > u_2$ when $v_j > 0$. So $v_i + v_j = u_2$ is impossible for $i, j \in \{1, 2, 3, \ldots, n\}$. For all i, j, k, $1 \le i, j, k \le n$, $k \notin \{1, 2\}$, $|v_i + v_j| < |u_k|$. Hence $v_i + v_j = u_k$ is impossible in this case also. This completes the proof of the lemma. □

Lemma 2.10. *Suppose n is even and $n \ge 10$. For $1 \le i, j, k \le n$, $u_i + v_j = v_k$ holds only in the cases $u_3 + v_n = v_{n-1}$ and $u_4 + v_{n-1} = v_n$.*

Proof. From the labelling of $C_n \odot K_1$ it is easy to see that $u_i + v_j > v_k$ for all k if $u_i > 0, v_j > 0$ and $u_i + v_j < v_k$ for all k if $u_i < 0, v_j < 0$, $1 \le i, j, k \le n$. Now consider the case when u_i and v_j are of opposite signs. Suppose that $u_i > 0$ and $v_j < 0$. Then, by definition, $u_3 + v_n = v_{n-1}$. Except in this case it is easy to see that $u_i + v_j > v_k$ for all k when $u_i > 0$ and $v_j < 0$, $1 \le i, j, k \le n$. If $u_i < 0$ and $v_j > 0$ then by definition, $u_4 + v_{n-1} = v_n$. Except in this case it can be verified that $u_i + v_j < v_k$ for all k when $u_i < 0$ and $v_j > 0$, $i \notin \{1, 2\}$, $1 \le i, j, k \le n$. Also $v_n < u_1 + v_j < v_{n-2}$ for $v_j > 0$, $1 \le j \le (n-3)$ and $v_{n-2} < u_1 + v_{n-1} < v_{n-4}$. Again $u_2 + v_1 < v_k$ for all k and $v_n < u_2 + v_j < v_{n-2}$ for all j, $2 \le j \le (n-1)$. By Observation 2, both $u_1 + v_j = v_k$ and $u_2 + v_j = v_k$ are impossible. This completes the proof of the lemma. □

Lemma 2.11. *Suppose n is even and $n \ge 10$. For $1 \le i, j, k \le n$, $u_i + v_j = u_k$ holds only in the following cases:*

$u_1 + v_4 = u_2$, $u_2 + v_2 = u_1$, $u_5 + v_{n-2} = u_3$,

$u_{6+i} + v_{n-(2i+4)} = u_{5+i}$ for $0 \le i \le ((n/2)-5)$
$u_{(n/2)+2} + v_1 = u_4$, $u_{(n/2)+3+i} + v_{2i+3} = u_{(n/2)+2+i}$ for $0 \le i \le ((n/2)-3)$

Proof. We consider the set S_1 from the labeling of $C_n \odot K_1$.
Let $S_1 = U^+ \cup U^- \cup V^+ \cup V^-$ where
$U^+ = \{ u_i | u_i \in S_1, u_i > 0 \} = \{ u_3, u_5, u_6, u_7, ..., u_{(n/2)+1} \}$
$U^- = \{ u_i | u_i \in S_1, u_i < 0 \} = \{ u_1, u_2, u_4, u_{(n/2)+2}, u_{(n/2)+3}, ..., u_n \}$
$V^+ = \{ v_i | v_i \in S_1, v_i > 0 \} = \{ v_1, v_2, v_3, v_5, v_7, v_9, ..., v_{n-1} \}$
$V^- = \{ v_i | v_i \in S_1, v_i < 0 \} = \{ v_4, v_6, v_8, ..., v_n \}$.
We consider $u_i + v_j$ when $u_i > 0$ and $v_j > 0$. Note that $\min \{ u_i | u_i \in U^+ \} = u_3$ and $\max \{ u_i | u_i \in U^+ \} = u_{(n/2)+1} = \max \{ x | x \in S_1 \}$. Since $v_j > 0$, $u_{(n/2)+1} + v_j \notin S_1$. For every $u_i > 0$, $u_i + v_{n-1} > u_{(n/2)+1}$ and so $u_i + v_{n-1} \notin S_1$. Also $u_3 < u_3 + v_j < u_5$ for all $v_j \in V^+$, $j \neq (n-1)$. Suppose that the vertex $u_i > 0$ is attached to $v_s < 0$. If $(s-1) \le j \le (n-3)$ then $u_i + v_j > u_{(n/2)+1}$ for all $v_j > 0$ and hence $u_i + v_j \notin S_1$ in this case. Also if $1 \le j < (s-1)$ then $u_i < u_i + v_j < u_{i+1}$ for all $v_j > 0$, $5 \le i \le (n/2)$. Hence $u_i + v_j = u_k$ is impossible if $u_i > 0$ and $v_j > 0$.

Now consider $u_i + v_j$ when $u_i > 0$ and $v_j < 0$. In this case $u_i + v_j > 0$ by Observation 2. By definition, $u_3 + v_n = v_{n-1}$, $u_5 + v_{n-2} = u_3 = \min \{ u_i | u_i \in U^+ \}$. $u_3 + v_j < u_3$ for all $v_j < 0$, $4 \le j \le (n-2)$ and hence $u_3 + v_j \notin S_1$ in this case. For $5 \le i \le ((n/2)+1)$, $u_i + v_n < u_3$ and hence $u_i + v_n \notin S_1$. Suppose that the vertex $u_i > 0$ is attached to $v_s < 0$.
If $4 \le j < s$ then $u_{i-1} < u_i + v_j < u_i$ for $5 < i \le ((n/2)+1)$ and $u_3 < u_5 + v_j < u_5$ since $v_j < 0$. If $s < j \le (n-2)$ then $u_i + v_j$ lies between two consecutive terms of U^+. Hence if $u_i > 0$ and $v_j < 0$ then $u_i + v_j = u_k$ is possible only in the cases listed above.

Next consider $u_i + v_j$ when $u_i < 0$ and $v_j > 0$. In this case $u_i + v_j < 0$ by Observation 2. Note that $u_1 = \max \{ u_i | u_i \in U^- \}$. Since $v_j > 0$, $u_1 + v_j > u_1$ and $u_1 + v_j \notin S_1$. $u_2 < u_2 + v_1 < u_1$. By definition, $u_2 + v_2 = u_1$. Since $v_j > 0$, $u_2 + v_j > u_1$ for $v_j \in V^+$, $j \notin \{1, 2\}$ and hence $u_2 + v_j \notin S_1$. By definition, $u_4 + v_{n-1} = v_n$. Also $u_4 < u_4 + v_j < u_2$ for $v_j \in V^+$, $j \neq (n-1)$ and so $u_4 + v_j \notin S_1$. If the vertex u_i is attached to v_s then $u_4 < u_i + v_j < u_2$ if $s < j \le (n-1)$ and $u_i < u_i + v_j < u_{i-1}$ if $1 \le j < s$, $((n/2)+3) \le i \le n$. Also $u_4 < u_{(n/2)+2} + v_j < u_2$ for $v_j \in V^+$, $j \notin \{1, (n-1)\}$ and $u_2 < u_{(n/2)+2} + v_{n-1} < u_1$. Hence $u_i + v_j \notin S_1$ if $j \neq s$ and $u_i + v_j = u_k$ is possible only in the cases listed above.

Finally we consider $u_i + v_j$ when $u_i < 0$ and $v_j < 0$. $u_n = \min \{ u_i | u_i \in U^- \} = \min \{ x | x \in S_1 \}$. Since $v_j < 0$, $u_n + v_j < u_n$ and $u_n + v_j \notin S_1$. By definition, $u_1 + v_4 = u_2$. $u_4 < u_1 + v_j < u_2$ if $v_j < 0$, $j \notin \{4, n\}$. $u_n < u_1 + v_n < u_{n-1}$,

$u_4 < u_2 + v_j < u_2$ for $v_j \in V^-$, $j \neq n$. $u_i + v_n < u_n$ for $u_i \in U^-$, $i \neq 1$. $u_i + v_{n-2} < u_n$ for $u_i \in U^-$, $i \notin \{1, 2, 4, (n/2)+2\}$.

Also $u_{(n/2)+3+j} < u_i + v_{4+2j} < u_{(n/2)+2+j}$ for $i = 4, ((n/2)+2)$, $0 \leq j \leq ((n/2)-3)$. $u_{(n/2)+4+j} < u_{(n/2)+3+i} + v_{4+2j} < u_{(n/2)+3+j}$ for $0 \leq j \leq ((n/2)-3)$, $0 \leq i \leq ((n/2)-4)$. Hence, for $u_i < 0$ and $v_j < 0$, $u_i + v_j = u_k$ is possible only in the case $u_1 + v_4 = u_2$. This completes the proof of the Lemma. □

Finally we have the theorem.

Theorem 2.2. *$C_n \odot K_1$ are integral sum graphs for all even $n \geq 4$.*

Proof. Since $C_4 \odot K_1 \cong G^+\{1, 4, 5, -4, -10, -3, -9, 9\}$,

$C_6 \odot K_1 \cong G^+\{1, 6, 7, -6, 13, -19, -33, -24, -40, -18, -32, 32\}$,

$C_8 \odot K_1 \cong G^+\{1, 8, 9, -8, 17, -25, 42, -67, -110, -74, -119, -66, -136, 134, -109, 109\}$

we need only consider $C_n \odot K_1$ for even $n \geq 10$. For $n \geq 10$ we claim that $C_n \odot K_1 \cong G^+(S_1)$ where $S_1 = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$ with v_i's and u_i's as defined earlier. By Lemma 2.1, the v_i's in S_1 make the path $v_1 v_2 ... v_n$ an integral sum graph. Since $v_n + v_1 = u_1 \in S_1$, $v_n v_1$ is also an edge of $G^+\{v_1, v_2, ..., v_n, u_1\}$. Our claim follows from lemmas 2.7, 2.8, 2.9, 2.10 and 2.11. This completes the proof of the theorem. □

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