Total Domination Edge Critical Graphs with Minimum Diameter

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Abstract

Denote the total domination number of a graph G by $\gamma_t(G)$. A graph G is said to be total domination edge critical, or simply γ_t-critical, if $\gamma_t(G+e) < \gamma_t(G)$ for each edge $e \in E(\overline{G})$. For 3_t-critical graphs G, that is, γ_t-critical graphs with $\gamma_t(G) = 3$, the diameter of G is either 2 or 3. We study the 3_t-critical graphs G with *diam* $G = 2$.

1 Introduction

Let $G = (V, E)$ be a graph with order $|V(G)| = n$. The *open neighbourhood* of a vertex v is the set of vertices adjacent to v, that is, $N(v) = \{w \mid vw \in E(G)\}$, and the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$ we define the *open* and *closed neighbourhoods* $N(S)$ and $N[S]$ of S by $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$ respectively. For sets $S, X \subseteq V(G)$, if $N[S] = X$ ($N(S) = X$, respectively), we say that S *dominates* X, written $S \succ X$ (S *totally dominates* X, respectively, written $S \succ_t X$). If $S = \{s\}$ or $X = \{x\}$, we also write $s \succ X$, $S \succ_t x$, etc. If $S \succ V(G)$ ($S \succ_t V(G)$, respectively), we say that S is a *dominating set* (*total dominating set*) of G. The cardinality of a minimum dominating (minimum total dominating) set of G is called the *domination number* (*total domination number*) of G and is denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively); if S is a minimum dominating (minimum total dominating) set, we also call S a γ-*set* (γ_t-*set*) of G. We note that the parameter $\gamma_t(G)$ is only defined for graphs G without isolated vertices. Domination-related concepts not defined here can be found in [3].

The addition of an edge to a graph can change the domination number by at most one. Sumner and Blitch [9] studied *domination edge critical graphs* G, that is, graphs G for which $\gamma(G+e) = \gamma(G) - 1$ for each $e \in E(\overline{G})$. The problem of characterising such graphs proved to be difficult and they were able to characterise only those domination edge critical graphs G for which $\gamma(G) = 1$ or $\gamma(G) = 2$. Although the domination edge critical graphs G with $\gamma(G) \geq 3$ have not been characterised, many interesting properties of these graphs found and many are still under investigation. (See [7] and [9] for recent surveys.)

In this paper, we consider the same concept for total domination. A graph G is *total domination edge critical* or just *γ_t-critical* if $\gamma_t(G+e) < \gamma_t(G)$ for any edge $e \in E(\overline{G}) \neq \emptyset$. It is shown in [4] that the addition of an edge to a graph can change the total domination number by at most two.

Proposition 1 [4] *For any edge $e \in E(\overline{G})$,*

$$\gamma_t(G) - 2 \leq \gamma_t(G+e) \leq \gamma_t(G).$$

Graphs G with the property $\gamma_t(G+e) = \gamma_t(G) - 2$ for any $e \in E(\overline{G})$ are called *supercritical* and are characterised in [5].

As with domination edge critical graphs the problems associated with γ_t-critical graphs also appear to be difficult, even when restricted to $\gamma_t(G) = 3$. For the remainder of this paper, we restrict our attention to 3_t-critical graphs G, that is, γ_t-critical graphs G with $\gamma_t(G) = 3$. Note that since $\gamma_t(G) \geq 2$ for any graph G, the addition of an edge to a 3_t-critical graph reduces the total domination number by exactly one. Also, observe that any graph G with $\gamma_t(G) = 3$ is connected. Sharp bounds on the diameter of a 3_t-critical graph are determined in [4].

Proposition 2 [4] *If G is a 3_t-critical graph, then*

$$2 \leq diam\ G \leq 3.$$

The graphs in Figures 1 and 2 illustrate sharpness of these bounds. In [6] we characterised the 3_t-critical graphs G with *diam* $G = 3$. Here our goal is to investigate the 3_t-critical graphs with diameter two.

Cockayne, Dawes and Hedetniemi [1] showed that if a graph G is connected and $\Delta(G) < n-1$, then $\gamma_t(G) \leq n - \Delta(G)$. We thus formulate the following observation.

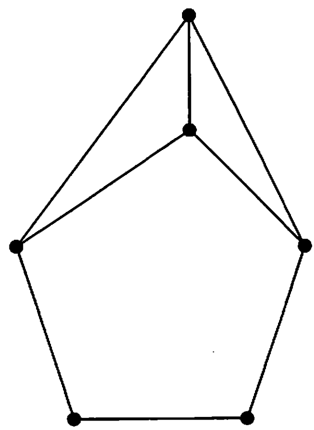

Figure 1: A 3_t-critical graph G with *diam* $G = 2$.

Observation 3 *Any graph G with $\gamma_t(G) = 3$ has $\Delta(G) \le n - 3$, or more generally, if $\gamma_t(G) \ge 3$, then $\Delta(G) \le n - \gamma_t(G)$.*

We make a distinction between two types of 3_t-critical graphs with diameter two. For such a graph G, we say that G is of

Type 1 if every pair of nonadjacent vertices dominates G, and of

Type 2 otherwise.

In [4] the authors showed that any 3_t-critical graph G with a cutvertex has exactly one cutvertex and it is adjacent to an endvertex. Moreover, they proved that such graphs G have *diam* $G = 3$ and are the only 3_t-critical graphs with an endvertex. Thus, the 3_t-critical graphs G with diameter two have $\delta(G) \ge 2$ and are 2-connected.

In Section 2 we characterise the 3_t-critical graphs G with *diam* $G = 2$ and $\delta(G) = 2$. In Section 3 the Type 1 graphs are characterised and results concerning the Type 2 graphs are presented in Section 4.

Throughout we make use of the following observation.

Observation 4 *For any 3_t-critical graph G and non-adjacent vertices u and v, either*

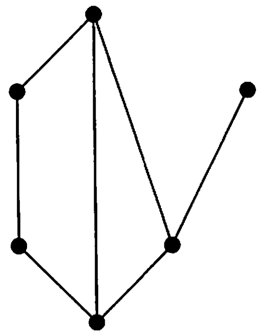

Figure 2: A 3_t-critical graph G with an *diam* $G = 3$.

(1) $\{u,v\}$ *dominates* G

or

(2) *(without loss of generality)* $\{u,w\}$ *dominates* $G - v$*, but not* v*, for some* $w \in N(u)$*. In this case, we write* $uw \mapsto v$*.*

2 3_t-Critical Graphs G with $\delta(G) = diam\ G = 2$

To characterise the 3_t-critical graphs G having both diameter and minimum degree equal to two, we use the following notation. Let v be a vertex of minimum degree with $N(v) = \{x, y\}$ and define

$$\begin{aligned} A &= N(x) \cap N(y) - \{v\}, \\ B &= N(x) - N(y), \\ C &= N(y) - N(x). \end{aligned}$$

Note that since *diam* $G = 2$, $\{N[v], A, B, C\}$ is a partition of $V(G)$.

Theorem 5 *A graph* G *with diam* $G = 2$ *and* $\delta(G) = 2$ *is* 3_t*-critical if and only if for each vertex* v *of degree two and* $N(v) = \{x, y\}$*,*

(1) $xy \notin E(G)$,

(2) *either* $A = \emptyset$ *or* $\langle A \rangle$ *is complete,*

(3) $\langle B \cup C \rangle$ *is complete, and*

(4) *if* $A \neq \emptyset$, *then every vertex in* A *is adjacent to* $|B| - 1$ *vertices in* B *and to* $|C| - 1$ *vertices in* C, *and* $|B| \geq 2$ *and* $|C| \geq 2$. *If* $A = \emptyset$, *then* $|B| \geq 1$ *and* $|C| \geq 1$.

Proof. Let G be a 3_t-critical graph with *diam* $G = 2$ and $\delta(G) = 2$. Let v be a vertex of degree two with $N(v) = \{x, y\}$. Since *diam* $G = 2$, the set $\{x, y\}$ dominates G. But $\gamma_t(G) = 3$, so x and y are not adjacent. Assume that $\langle B \cup C \rangle$ is not complete and let $d, e \in B \cup C$ with $de \notin E(G)$. Then $\{d, e\} \not\succ G$ since neither d nor e is adjacent to v. Thus, we can assume without loss of generality that $dx \mapsto e$, implying that $d \in A$ (to dominate y), which contradicts the fact that $d \in B \cup C$. Therefore, $\langle B \cup C \rangle$ is complete.

Now we show that if $A \neq \emptyset$, then $\langle A \rangle$ is complete. Let $a, d \in A$ and suppose that $ad \notin E(G)$. Since neither a nor d dominates v, we may assume, without loss of generality, that $ax \mapsto d$. But $d \in N(x)$, a contradiction.

Next we prove that (4) holds. If $A = \emptyset$, then it is easy to see that $|B| \geq 1$ and $|C| \geq 1$. On the other hand, let $a \in A$ and suppose there are two vertices $b, d \in B$ not adjacent to a. Neither a nor b dominates v, so $\{a, b\}$ does not dominate G. Since $\{b, x\}$ does not dominate y, it follows that $ax \mapsto b$ or $ay \mapsto b$ to dominate v. But $ax \succ b$ and ay does not dominate d, a contradiction in both cases. Therefore a is adjacent to at least $|B| - 1$ vertices in B. If $a \succ B$, then ay is a dominating edge of G, contradicting that $\gamma_t(G) = 3$. Thus, a is adjacent to exactly $|B| - 1$ vertices in B. Similarly, a is adjacent to exactly $|C| - 1$ vertices in C.

We also show that if $A \neq \emptyset$, then, without losing generality, there is at least one edge from A to B. Suppose there is no edge from A to B. Since $a \in A$ is adjacent to $|B| - 1$ vertices in B, we know that $|B| = 1$. If $|C| = 1$, then $\gamma_t(G + va) = 3$, contradicting the fact that G is 3_t-critical. Hence $|C| \geq 2$. Thus there is a vertex in C that is not in $N(a)$. It is easy to check that $\gamma_t(G + va) = 3$, again a contradiction. Thus there is at least one edge from A to B and similarly at least one edge from A to C and we conclude that if $A \neq \emptyset$, then $|B| \geq 2$ and $|C| \geq 2$.

Conversely, by the construction, G is a 3_t-critical graph. □

We note that the graphs characterised by Theorem 5 are of Type 1 if $A = \emptyset$ and of Type 2 otherwise.

3 Characterisation of Type 1 Graphs

Our next lemma follows directly from the definition of Type 1 graphs.

Lemma 6 *If G is a Type 1 graph, then*

(a) $G - N[v] = K_t$, $t \geq 2$, *for any* $v \in V(G)$;

(b) *for each edge* $uv \in E(G)$, $X = V(G) - (N[u] \cup N[v]) \neq \emptyset$ *and* $\langle X \rangle$ *is complete.*

Lemma 7 *If G is a Type 1 graph, then G has no vertex v such that $N[v]$ is complete.*

Proof. Let G be a Type 1 graph with $v \in V(G)$ such that $N[v]$ is complete. By Lemma 6, $G - N[v]$ is complete, and so the connectedness of G implies that $\gamma_t(G) = 2$, a contradiction. □

We now characterise the graphs of Type 1.

Theorem 8 *A graph G is of Type 1 if and only if*

(1) $\langle V(G) - N[v] \rangle = K_t$, $t \geq 2$, *for every* $v \in V(G)$,

and

(2) $X = V(G) - (N[u] \cup N[v]) \neq \emptyset$ *and* $\langle X \rangle$ *is complete for every* $uv \in E(G)$.

Proof. The sufficiency follows from Lemma 6. Conversely, let G be a graph such that (1) and (2) hold. We first show that *diam* $G = 2$ and $\gamma_t(G) = 3$. Let $v \in V(G)$. By Lemma 7, $N(v)$ is not complete. Let $a, b \in N(v)$ be nonadjacent vertices. By (1), $\langle V(G) - N[a] \rangle$ is complete, implying that $b \succ V(G) - N[a]$. Hence $\{a, b, v\}$ is a total dominating set of G and so $\gamma_t(G) \leq 3$. By (2), no edge dominates G. Thus $\gamma_t(G) = 3$. Furthermore, $d(v, x) \leq 2$ for any $x \in V(G)$. Since v is an arbitrary vertex, *diam* $G \leq 2$. Since G has no dominating edge, *diam* $G = 2$. That G is γ_t-critical and of Type 1 follows from (1). □

We note that if G is a Type 1 graph, then we may increase the order of G by successively adding a vertex adjacent to every vertex of $G - H$, where

$\langle H \rangle$ is any maximal clique of order $k \geq 2$. The resulting graph is a Type 1 graph.

We conclude this section with properties of Type 1 graphs. The following theorem is due to Fraisse [2, p. 109].

Theorem 9 *If G is a 2-connected graph of order n such that for all distinct nonadjacent vertices u and v,*

$$|N(u) \cup N(v)| \geq \frac{2n-1}{3},$$

then G is hamiltonian.

A simple corollary to Theorem 9 affirms the hamiltonicity of Type 1 graphs.

Corollary 10 *If G is a Type 1 graph, then G is hamiltonian.*

Proof. Since G is 2-connected and every pair of nonadjacent vertices $u, v \in V(G)$ dominates G, the result follows. □

Recall that the (vertex) independence number of G is denoted by $\beta_0(G)$.

Observation 11 *If G is a graph of Type 1, then $\beta_0(G) = 2$.*

A *claw* is an induced $K_{1,3}$. Obviously, Observation 11 implies that a Type 1 graph is claw-free.

4 Type 2 Graphs

Although we have not been able to characterise the Type 2 graphs, in this section we present characterisations of several subclasses of this family. We begin with some properties of Type 2 graphs.

Since a Type 2 graph G has a pair of nonadjacent vertices that does not dominate G, the following result is immediate.

Observation 12 *If G is a graph of Type 2, then $\beta_0(G) \geq 3$.*

We note that a vertex in a Type 2 graph may or may not have a neighbourhood that induces a complete subgraph. For example, there is no vertex v in the graph G in Figure 1 such that $\langle N(v)\rangle$ is complete. On the other hand, the graph in Figure 3 is an example of a Type 2 graph with a vertex v such that $\langle N(v)\rangle$ is complete.

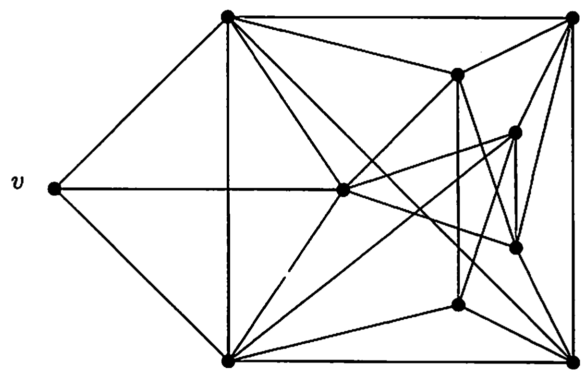

Figure 3: A Type 2 graph with a vertex v such that $\langle N(v)\rangle$ complete.

4.1 Characterisation where $\langle N(v)\rangle$ is Complete

The first family of Type 2 graphs that we characterise are the graphs with a vertex v such that $\langle N(v)\rangle$ is complete. Let $A = N(v)$ and $B = V(G) - N[v]$. It is obvious that $deg(v) \geq 3$, otherwise $\gamma_t(G) = 2$.

Theorem 13 *A graph G with a vertex v such that $\langle A\rangle$ is complete is a Type 2 graph if and only if the following conditions hold:*

(1) *No pair of adjacent vertices dominates G.*

(2) *There is no $b \in B$ such that $b \succ B$ and for every $b_i \in B$ there exists $b_j \in B$ such that $b_i b_j \mapsto v$.*

(3) *For every pair of nonadjacent vertices $b, b' \in B$, without loss of gen-*

erality, there exists $a \in A$ such that $ab \mapsto b'$.

(4) *For every pair of nonadjacent vertices $a \in A$ and $b \in B$, $\{a, b\} \succ G$ or there exists $w \in A \cup B$ such that $aw \mapsto b$.*

(5) *Every vertex in A (B, respectively) has a neighbour in B (A, respectively).*

Proof. Let G be a Type 2 graph with a vertex v such that $\langle N(v) \rangle$ is complete. Condition (1) follows directly from the fact that $\gamma_t(G) = 3$. To prove (2), we first prove part of (5). If $b \in B$ has no neighbour in A, then $d(v, b) \geq 3$, contradicting *diam* $G = 2$. Thus each $b \in B$ is adjacent to some vertex in A. Now if $b \succ B$ for some $b \in B$, then $\{a, b\} \succ_t G$, where $a \in A$ is a vertex adjacent to b. This contradicts $\gamma_t(G) = 3$. Hence no vertex in B dominates B. Suppose there exists $a \in A$ and $b \in B$ such that $va \longmapsto b$. Then $a \succ B - \{b\}$, hence $\deg(a) = n - 2$, contradicting Observation 3. Condition (2) now follows from the criticality of G.

Condition (3) follows from the fact that no vertex in B dominates v. For $a \in A$ and $b \in B$, if $\{a, b\} \succ G$, then Condition (4) holds. If $\{a, b\} \not\succ G$, then since G is 3_t-critical, there exists a vertex w such that $aw \mapsto b$ or $bw \mapsto a$. If $bw \mapsto a$, then $w \in A$ to dominate v. But since $\langle A \rangle$ is complete, $w \in N(a)$, contradicting that $bw \mapsto a$. Hence $aw \mapsto b$ for some $w \in A \cup B$ and Condition (4) holds.

To prove the remainder of (5), suppose $a \in A$ is not adjacent to any vertex in B. Then for any $b \in B$, $\{a, b\} \not\succ G$, for $b \not\succ B$ and a is not adjacent to any vertex in B. Thus by (4) there exists $w \in A \cup B$ such that $aw \longmapsto b$. Since a is not adjacent to any vertex in B, $w \in A$ and $w \succ B - \{b\}$, hence $\deg(w) = n - 2$, contradicting Observation 3.

Conversely, let G be a graph such that the stated properties hold. We first show that *diam* $G = 2$. By Condition (5) every $b \in B$ is adjacent to some $a \in A$. Therefore $d(v, u) \leq 2$ for every $u \in A \cup B$ and $d(a, b) \leq 2$ for every $a \in A$ and $b \in B$. By Condition (2) every $b_i \in B$ is adjacent to some $b_j \in B$ such that $\{b_i, b_j\} \succ B$, so $d(b, b') \leq 2$ for every pair $b, b' \in B$. Obviously, every pair of vertices in A are adjacent and hence *diam* $G = 2$.

Condition (1) implies that $\gamma_t(G) \geq 3$. Consider nonadjacent vertices $b, b' \in B$. Let $a \in A$ be a vertex (which exists by Condition (3)) such that $ab \longmapsto b'$, and let $a' \in A$ be adjacent to b' (see (5)). Then $\{a, a', b\} \succ_t G$ and thus $\gamma_t(G) = 3$.

Next we show that there is a pair of nonadjacent vertices that does not dominate G. From Condition (2) we know that no $b_i \in B$ dominates B. Thus if b_i and b_j are nonadjacent vertices in B, then $\{b_i, b_j, v\}$ is an independent set of vertices, that is, $\{b_i, b_j\} \not\succ G$. Finally, the fact that G is 3_t-critical follows from Conditions (3)-(5). $\Box$

4.2 Type 2 Crown Graphs

We say that a graph G is a Type 2 *crown graph* if for every pair of nonadjacent vertices $u, v \in V(G)$, there exist vertices x and y such that $ux \mapsto v$ and $vy \mapsto u$. Figure 4 gives an example of a Type 2 crown graph. Note that not all Type 2 graphs are crown graphs. For example, the graph G in Figure 5 is a Type 2 graph where $\{u, v\} \not\succ G$ but $uw \mapsto v$ and there is no $z \in N(v)$ such that $vz \mapsto u$. In this section we characterise the Type 2 crown graphs.

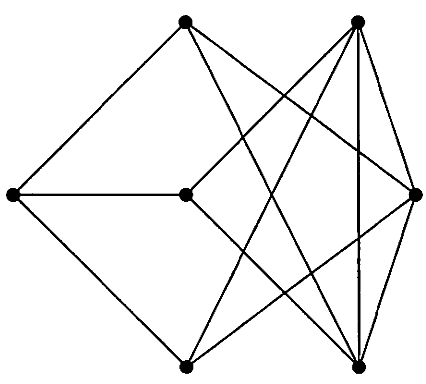

Figure 4: A Type 2 crown graph.

First we determine some properties of Type 2 crown graphs.

Proposition 14 *If G is a Type 2 crown graph, then every pair of nonadjacent vertices of G lies on an induced C_5.*

Proof. Let G be a Type 2 crown graph and consider nonadjacent vertices

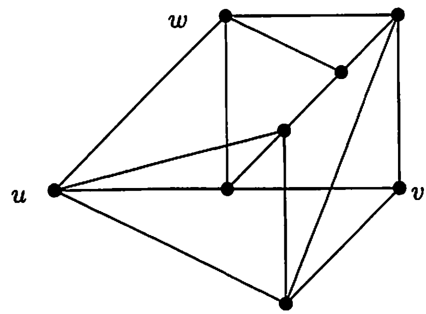

Figure 5: A Type 2 graph G with $uw \mapsto v$.

u and v. Then $ux \mapsto v$ and $vy \mapsto u$ for some $x, y \in V(G)$, where necessarily $xy \in E(G)$. Furthermore, there is at least one vertex z not adjacent to x and y. Clearly, z is adjacent to u and v. Hence $\{u, z, v, y, x\}$ induces a C_5. □

Proposition 15 *Every Type 2 crown graph contains a claw.*

Proof. Let G be a Type 2 crown graph with an independent set $S = \{x_1, x_2, x_3\}$. Let y_2 be a vertex such that $x_2y_2 \mapsto x_1$ and consider $\{x_1, y_2\}$. There exists a vertex z such that $x_1z \mapsto y_2$. Then $z \succ S$ and the result follows. □

The proof of Proposition 15 also shows that every independent set S of vertices of a Type 2 crown graph satisfies $S \subseteq N(v)$ for some $v \in V(G)$. We state this as a corollary.

Corollary 16 *If G is a Type 2 crown graph with an independent set S, then $S \subseteq N(v)$ for some $v \in V(G)$.*

The following corollary is now also immediate.

Corollary 17 *If G is a Type 2 crown graph, then $3 \leq \beta_0(G) \leq \Delta(G)$.*

We now characterise the Type 2 crown graphs. Let G be a Type 2 crown graph and let $v \in V(G)$ be an arbitrary vertex with $A = N(v)$ and $B = V(G) - N[v]$.

Theorem 18 *A graph G is a Type 2 crown graph if and only if the following conditions hold for every vertex v:*

(1) $\beta_0(G) \geq 3$ *and for every independent set of vertices S, $S \subseteq N(u)$ for some $u \in V(G)$.*

(2) *Every vertex in A (B, respectively) has at least one neighbour in B (A, respectively). Furthermore, no pair of adjacent vertices dominates G.*

(3) *For every $b \in B$ there exists $a \in A$ such that $va \mapsto b$ and $b' \in B$ such that $bb' \mapsto v$.*

Proof. Let G be a graph such that the stated properties hold. For any $y, z \in V(G)$, if $yz \notin E(G)$, then by (1), $\{y, z\} \subseteq N(u)$ for some vertex u and so $d(y, z) = 2$. Hence *diam* $G = 2$. Condition (2) implies that $\gamma_t(G) \geq 3$. Let S be a maximum independent set of vertices. Since $\beta_0(G) \geq 3$, we may assume that $\{x_1, x_2, x_3\} \subseteq S$. From Condition (1) we have that $S \subseteq N(u)$ for some $u \in V(G)$. Consider $\{x_1, x_3\}$. By Condition (3) there exists $w \in N(x_1)$ such that $x_1 w \mapsto x_3$. Thus $\{u, x_1, w\} \succ_t G$ and hence $\gamma_t(G) = 3$. Since the vertex v of the theorem is arbitrary, it also follows from Condition (3) that G is 3_t-critical and of Type 2.

Conversely, let G be a Type 2 crown graph and consider arbitrary $v \in V(G)$. That $\beta_0(G) \geq 3$ follows from Observation 12. By Corollary 16, for every independent set of vertices S, the vertices of S share a common neighbour in G. Hence Condition (1) holds. Every $a \in A$ dominates at most $|B| - 1$ vertices in B, otherwise $\gamma_t(G) = 2$. Consider $\{v, b_i\}$, $b_i \in B$. Since $b_i b_j \mapsto v$ for some $b_j \in B$, every $a \in A$ is adjacent to some $b \in B$. Also, every $b \in B$ is adjacent to some $a \in A$ since *diam* $G = 2$. Furthermore, since $\gamma_t(G) = 3$, there is no edge of G that dominates. Hence Condition (2) holds. Condition (3) follows from the fact that G is a Type 2 crown graph. □

Although Theorem 18 characterises the Type 2 crown graphs, we are able to give more descriptive characterisations for several subclasses of this family. First we determine more properties of these graphs beginning with two propositions that give lower and upper bounds on their order.

Proposition 19 *If G is a Type 2 crown graph, then G has order $n \geq 7$ and this bound is sharp.*

Proof. Let G be a Type 2 crown graph. By Corollary 16, G has a vertex v such that $N(v)$ contains an independent set of cardinality at least three. Let $B = V(G) - N[v]$. If there exists a vertex $a \in N(v)$ such that $a \succ B$ (as is the case if $|B| = 1$), then $\{a, v\} \succ_t G$, a contradiction. This implies that $|B| \geq 2$, *i.e.*, G has order $n \geq 7$ unless $|N(v)| = 3$ and $|B| = 2$. Let $N(v) = \{v_1, v_2, v_3\}$ and $B = \{u_1, u_2\}$. Recall that no vertex in $N(v)$ dominates both u_1 and u_2. Hence, in order to ensure that $d(u_1, u_2) \leq 2$, $u_1u_2 \in E(G)$. Also, to ensure that $d(v_i, u_j) \leq 2$ for each $1 \leq i \leq 3$ and $1 \leq j \leq 2$, every v_i is adjacent to some u_j. Thus we may assume that u_1 is adjacent to v_1 and v_2, and u_2 is adjacent to v_3. Then $\{v_1, v_2\} \not\succ G$ and so, without loss of generality, $v_1x \mapsto v_2$ for some vertex x. But $x \in N(v_2)$ for all $x \in N(v_1)$, a contradiction.

The graph in Figure 4 has order 7. □

An interesting upper bound on the order of G is given in terms of the minimum degree $\delta(G)$.

Proposition 20 *If G is a Type 2 crown graph of order n, then $n \leq 2\delta(G) + 1$ and this bound is sharp.*

Proof. Let G be a Type 2 crown graph. For a vertex v with $\deg(v) = \delta(G)$, let $D = V(G) - N[v]$. Then, for every $d \in D$, there is a vertex $u \in N(v)$ such that $vu \mapsto d$. Hence $|N(v)| \geq |D|$ and it follows that

$$\begin{aligned} n &= |N(v)| + |D| + 1 \\ &\leq 2|N(v)| + 1 \\ &= 2\delta(G) + 1. \end{aligned}$$

Hence $n \leq 2\delta(G) + 1$.

That the bound is sharp can be seen by a family of graphs G generalizing the graph in Figure 4 described as follows. Let $v \in V(G)$, $A = N(v)$ and $B = V(G) - N[v]$ such that A is independent, $|A| = |B|$, $\langle B \rangle$ is complete and G contains all edges between A and B except for a 1-factor. □

Since $n \geq 7$ and $n \leq 2\delta(G) + 1$, we have the following corollary.

Corollary 21 *If G is a Type 2 crown graph, then $\delta(G) \geq 3$.*

Not all Type 2 crown graphs G have order $n = 2\delta(G) + 1$. For instance, strict inequality is achieved by the graph in Figure 6. We characterise the Type 2 crown graphs G having this maximum order. Let G be a graph with $n = 2\delta(G) + 1$, let v be a vertex of minimum degree and $A = N(v) = \{a_1, ..., a_k\}$, $B = V(G) - N[v] = \{b_1, ..., b_k\}$. There is no $xy \in E(\langle A \rangle)$ such that $\{x, y\} \succ A$. Furthermore, $\langle A \rangle \cong \langle \overline{B} \rangle$ under the isomorphism $a_i \longmapsto b_i$, $i = 1, ..., k$, and G contains all possible edges from A to B except for the 1-factor $\{a_i b_i : i = 1, ..., k\}$. Let $\mathcal{G}$ be the family of all such graphs G.

Theorem 22 *A graph G with $n = 2\delta(G) + 1$ is a Type 2 crown graph if and only if $G \in \mathcal{G}$.*

Proof. Let G be a Type 2 crown graph with a vertex v of minimum degree and $n = 2\delta(G) + 1$. Let $A = N(v)$ and $B = V(G) - N[v]$, say $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. First consider $\{v, b_i\}$, for $1 \leq i \leq k$. Without loss of generality we assume that $va_i \mapsto b_i$. Thus, G contains all edges between A and B except for the 1-factor $\{a_i b_i : i = 1, ..., k\}$. Furthermore, there is no edge $xy \in \langle A \rangle$ such that $\{x, y\} \succ A$, for otherwise $\gamma_t(G) = 2$.

We next show that $\langle A \rangle \cong \langle \overline{B} \rangle$ under the isomorphism $a_i \longmapsto b_i$, $i = 1, ..., k$. Without losing any generality, suppose $a_1 a_2 \in E(G)$ and $b_1 b_2 \in E(G)$. But then $\{a_2, b_1\} \succ_t G$ and $\{a_1, b_2\} \succ_t G$, contradicting $\gamma_t(G) = 3$. On the other hand, suppose $\{a_1 a_2, b_1 b_2\} \cap E(G) = \emptyset$. Consider $\{b_1, b_2\}$ and let x be a vertex such that $b_1 x \mapsto b_2$. Then $x \in A$ to dominate v. The only vertex in A not adjacent to b_2 is a_2, but since $va_1 \mapsto b_1$, neither b_1 nor a_2 dominates a_1, a contradiction.

Conversely, it is a simple exercise to check that G is a Type 2 crown graph. □

From Corollary 21 and Observation 3 we see that Type 2 crown graphs G have $3 \leq \delta(G) \leq \Delta(G) \leq n - 3$. The final two subfamilies that we characterise are the ones obtaining these lower and upper bounds. The following lemma will be used to characterise the Type 2 crown graphs with minimum degree 3.

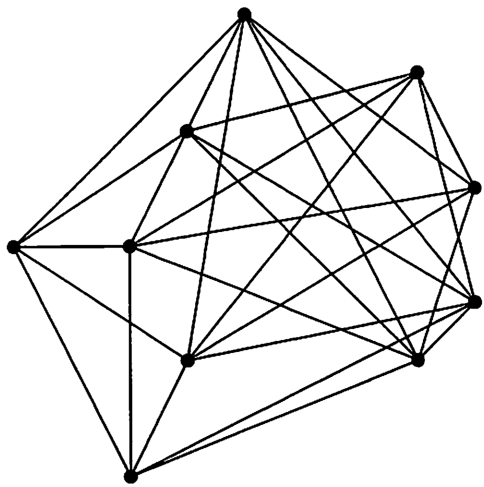

Figure 6: A Type 2 crown graph G with $n < 2\delta(G) + 1$.

Lemma 23 *For each vertex v of a Type 2 crown graph, $\gamma(\langle N(v) \rangle) \geq 2$.*

Proof. Let G be a Type 2 crown graph and suppose G has a vertex v such that $\gamma \langle N(v) \rangle = 1$. Let $x \in N(v)$ be a vertex that dominates $N(v)$. Let $y \in V(G) - N[v]$ be a vertex that is not dominated by x and consider $\{x, y\}$. It is easy to see that there is no vertex w such that $wy \mapsto x$. □

Theorem 24 *A graph G with $\delta(G) = 3$ is a Type 2 crown graph if and only if G is the graph in Figure 4.*

Proof. Let G be a Type 2 crown graph and let v be a vertex of G with degree three. Let $B = V(G) - N[v]$. We know by the proof of Proposition 20 that $|B| \leq 3$ and Proposition 19 implies that $|B| = 3$. Let $B = \{b_1, b_2, b_3\}$ and $N(v) = \{v_1, v_2, v_3\}$. Since we know from Lemma 23 that no vertex in $N(v)$ dominates $N(v)$, it follows that $N(v)$ is either independent or induces a graph with a single edge, say $v_1 v_2$. First assume that $N(v)$ is independent and consider $\{v, b_1\}$. Without loss of generality, $vv_1 \mapsto b_1$ and v_1 is adjacent to b_2 and b_3 (and $v_1 b_1 \notin E(G)$). Similarly, $vv_2 \mapsto b_2$ and $vb_3 \mapsto b_3$.

Suppose that $\langle B \rangle$ is not complete. We may assume that $b_1 b_2 \in E(\overline{G})$. Then $b_1 x \mapsto b_2$ for some vertex x. However, $N(b_1) \subseteq \{v_2, v_3, b_3\}$, but

$\{b_1, v_2\} \not\succ v_1$, $\{b_1, v_3\} \succ b_2$ and $\{b_1, b_3\} \not\succ v$, a contradiction. Hence $\langle B \rangle$ is complete.

Next, let $\langle N(v) \rangle$ have a single edge $v_1 v_2$. Again when we consider $\{v, b_j\}$, $1 \le j \le 3$, each v_j dominates $B - \{b_j\}$. If $b_1 b_2 \in E(G)$, then $\{v_1, b_2\} \succ_t G$, a contradiction. It is now easily checked that $\langle B \rangle = K_3 - b_1 b_2$ and that the graph thus obtained is isomorphic to the graph G in Figure 4.

Conversely, it is a simple exercise to see that G is a Type 2 crown graph with $\delta(G) = 3$. □

We next consider the Type 2 crown graphs G with $\Delta(G) = n - 3$.

Lemma 25 *If G is a Type 2 crown graph with a vertex v such that* $\deg(v) = n - 3$, *then* $G - N[v] = K_2$.

Proof. Let $V(G) - N[v] = \{x, y\}$. If x and y have a common neighbour in $N(v)$, say z, then $\{v, z\} \succ_t G$, contradicting the fact that $\gamma_t(G) = 3$. Hence x and y have no common neighbour in $N(v)$, and since *diam* $G = 2$ it follows that $xy \in E(G)$. □

Let $\mathcal{F}$ be the family of graphs G with $\Delta(G) = n - 3$ described as follows. Let v be a vertex with $\deg(v) = n - 3$ and $V(G) - N[v] = \{x, y\}$, where $xy \in E(G)$, and let $N(v) = A \cup B$ (disjoint union), where $A = N(x) \cap N(v) = \{a_1, ..., a_k\}$ and $B = N(y) \cap N(v) = \{b_1, ..., b_k\}$ such that $\langle A \rangle \cong \langle \overline{B} \rangle$ under the isomorphism $a_i \longmapsto b_i$, $i = 1, ..., k$. Also, G contains all edges from A to B except for the 1-factor $\{a_i b_i : i = 1, ..., k\}$.

Theorem 26 *A graph G with $\Delta(G) = n - 3$ is a Type 2 crown graph if and only if $G \in \mathcal{F}$.*

Proof. Let G be a Type 2 crown graph with $\deg(v) = n - 3$. By Lemma 25, $V(G) - N[v] = \{x, y\}$ where $xy \in E(G)$. First consider $\{v, x\}$. Since G is a Type 2 crown graph, $xy \mapsto v$, implying that every vertex in $N(v)$ is adjacent to x or y. We also know that no vertex in $N(v)$ is adjacent to both x and y. Hence $\{A, B\}$ is a partition of $N(v)$. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$. Next consider $\{a_i, y\}$, for every $1 \le i \le k$. Note that $|B| \ge |A|$ since for every a_i there is a distinct $b_i \in B$ such that $yb_i \mapsto a_i$. Similarly, by considering $\{b_j, x\}$ for every $1 \le j \le l$, we see that for every b_j there exists a distinct vertex a_j such that $xa_j \mapsto b_j$, so that $|A| \ge |B|$.

Hence $|A| = |B|$ with all the edges between A and B minus a 1-factor; say $a_i b_j \in E(G)$ for all $i \neq j$, $1 \leq i, j \leq k$.

By Proposition 19, $k \geq 2$. Now suppose without losing generality that $a_1 a_2 \in E(G)$ and $b_1 b_2 \in E(G)$. Here both $a_1 b_2$ and $b_1 a_2$ totally dominate G, contradicting $\gamma_t(G) = 3$. Therefore at most one of $a_1 a_2$ and $b_1 b_2$ is an edge of G. Since $a_1 a_2$ and $b_1 b_2$ are arbitrary, in general at most one of $a_i a_j$ and $b_i b_j$ is an edge of G. Assume that neither of $a_1 a_2$ and $b_1 b_2$ is an edge of G. Consider $\{a_1, a_2\}$ and note that $\{a_1, b_2\} \not\succ b_1$, $\{a_1, x\} \not\succ b_1$ and $\{a_1, v\} \not\succ y$. Therefore, without losing generality, let $a_1 b_3 \mapsto a_2$. But then $a_1 a_3$ and $b_1 b_3$ are edges of G, contradicting that at most one of $a_1 a_3$ and $b_1 b_3$ is an edge of G. Thus, exactly one of $a_i a_j$ and $b_i b_j$ is an edge of G, implying that $\langle A \rangle = \langle \overline{B} \rangle$.

Conversely, it is easily checked that G is a Type 2 crown graph. □

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