Semi-regular Bipartite Graphs and an Extension of the Marriage Lemma

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Abstract

The well-known Marriage Lemma states that a bipartite regular graph has a perfect matching. We define a bipartite graph G with bipartition (X, Y) to be *semi-regular* if both $x \mapsto \deg x, x \in X$ and $y \mapsto \deg y, y \in Y$ are constant. The purpose of this note is to show that if G is bipartite and semi-regular, and if $|X| < |Y|$, then there is a matching which saturates $|X|$. (Actually, we prove this for a condition weaker than semi-regular.) As an application, we show that various subgraphs of a hypercube have saturating matchings. We also exhibit classes of bipartite graphs, some of them semi-regular, whose vertices are the vertices of various weights in the hypercube Q_n but which are not subgraphs of Q_n.

1 Introduction

A graph G is regular of degree k (or k-regular) if each vertex has degree k. From now on, G will be a bipartite graph with bipartition (X, Y).

Definition 1 *G is semi-regular of bi-degree (k, m) if every vertex in one member of the bipartition has degree k and every vertex in the other has degree m.*

The Marriage Lemma says that if a bipartite graph is regular then $|X| = |Y|$ and G has a perfect matching. If G is *semi-regular* of bi-degree (k, m) with $m < k$ then, as we shall show, $|X| < |Y|$, so that G has no perfect matching. However, G *does* have a matching which saturates X.

We then use this to show that the subgraph of the hypercube Q_n created by deleting all vertices of weight $\leq i$ and all vertices of weight $\geq n-j$ has a saturating matching.

2 The Marriage Lemma for semi-regular bipartite graphs

Lemma 1 *Let G be a bipartite graph with bipartition (X,Y). Assume that for all $x \in X$ and for all $y \in Y$, $\deg x \geq \deg y$. Then $|X| \leq |Y|$ and there exists a matching in G which saturates X.*

Proof. $\sum_{x\in X} \deg x = e(G) = \sum_{y\in Y} \deg y$. Let $\delta_X = \min\{\deg x \,|\, x \in X\}$ and let $\Delta_Y = \min\{\deg y \,|\, y \in Y\}$. Then the first sum is $\geq \delta_X|X|$, while the second sum is $\leq \Delta_Y|Y|$. So $\delta_X|X| \leq \Delta_Y|Y|$. Thus $|X| \leq \frac{\Delta_Y}{\delta_X}|Y|$. It follows from the hypothesis that $\Delta_Y \leq \delta_X$, and so $|X| \leq |Y|$.

Now to show the existence of a matching which saturates X we show that Hall's condition is satisfied. So let S be any subset of X. We must show that $|S| \leq |N(S)|$. Suppose instead that $|S| > |N(S)|$. Let H denote the subgraph of G induced by $S \cup N(S)$.

$$\sum_{y\in N(S)} \deg_H(y) = \sum_{x\in S} \deg_H(x) = \sum_{x\in S} \deg_G(x)$$

Thus the average degree, relative to H, of $y \in N(S)$ is

$$\frac{\sum_{x\in S} \deg x}{|N(S)|}.$$

Now if $|N(S)| < |S|$, then

$$\frac{\sum_{x\in S} \deg x}{|N(S)|} > \frac{\sum_{x\in S} \deg x}{|S|}.$$

Thus the average degree,in H of the y's in $N(S)$, is greater than the average degree, in G, of the x's in S. But from the hypothesis, the maximum of $\{\deg y \,|\, y \in N(S)\}$ is less than or equal to the minimum of $\{\deg x \,|\, x \in S\}$. Thus the *average* of $\{\deg y \,|\, y \in N(S)\}$ is less than or equal to the *average* of $\{\deg x \,|\, x \in S\}$, contradicting the strict inequality obtained above. This contradiction shows that Hall's condition is satisfied, and hence the desired matching saturating X exists. □

Remark. With the hypotheses of the previous lemma, if for at least one pair x, y with $x \in X$ and $y \in Y$, $\deg x > \deg y$, then $|X| < |Y|$.

The next result is an immediate consequence of the preceding lemma and the remark.

Corollary 1 *Let G be a semi-regular bipartite graph of bi-degree (k,m), (X,Y) a bipartition of G. Assume that for all $x \in X$, $\deg x = k$ and for all $y \in Y$, $\deg y = m$, with $m < k$. Then $|X| < |Y|$ and there exists a matching in G which saturates X.*

3 An Application to Subgraphs of Hypercubes

Denote the n-dimensional hypercube by Q_n. We think of the vertices as the subsets of $[n] = \{1, 2, \ldots, n\}$. Two subsets are considered adjacent if one is a subset of the other and their cardinalities differ by 1. By the *weight* of a vertex we mean the cardinality of the corresponding subset. We call the set of vertices of weight j the j^{th} level of Q_n, and denote it by L_j. The bipartition of Q_n is given by (X,Y) where $X =$ the set of vertices of odd weight and $Y =$ the set of vertices of even weight. Thus X is the union of the odd levels and Y is the union of the even ones. For $i < j$ we denote by $L_{(i,j)}$ the subgraph of Q_n induced by all vertices z such that $i \leq \text{weight}\,(z) \leq j$.

Lemma 2 *Let $1 \leq j \leq n-1$, and let x belong to level j. Then x has j neighbors in level $j-1$ and $n-j$ neighbors in level $j+1$.*

Proposition 1 *Let $n = 2k$ and G be the subgraph of Q_n obtained by deleting all vertices in levels $\leq i$ and all vertices in levels $\geq 2k - i$, where $i < k$, i.e. $G = L_{(i+1, 2k-(i+1))}$. Then G has a matching which saturates all vertices whose weight is $\equiv i \pmod 2$.*

Proof. By Lemma 2, it is easy to see that every vertex in G whose weight is greater than $i+1$ and less than $2k-(i+1)$ has degree $2k$ in G. The vertices of weight $i+1$ and the vertices of weight $2k-(i+1)$ have degree $2k-(i+1)$. Hence by Lemma 1 there is a matching of G which saturates all the vertices of that member of the bipartition of G of smaller cardinality. Again by Lemma 1, that member is the one whose vertices have the higher degrees. Since the only vertices of degree less than $2k$ are those of weights $i+1$ and $2k-(i+1)$ and $i+1 \equiv 2k-(i+1) \pmod 2$, it follows that the vertices of weight $\equiv i \pmod 2$ are all saturated by this matching. □

Next we consider the case of odd dimensional hypercubes, and prove a stronger result which does not use Lemma 1.

Proposition 2 *For $1 \leq a \leq k$, $Q_{2k+1}(k-a, k+1+a)$ has a spanning subgraph which is regular of degree $k+1+a$. Hence this graph has a perfect matching.*

Proof. By induction on a. First, let $a = 1$. Note that if $\mathrm{wt}(x) = k$ or $k+1$, then $\deg(x) = 2k+1$, while if $\mathrm{wt}(x) = k-1$ or $k+2$, then $\deg(x) = k+2$. Now the subgraph $Q_{2k+1}(k, k+1)$ is $k+1$-regular. By the Marriage Lemma, this subgraph has $k-1$ pairwise disjoint perfect matchings $M_1, \ldots, M_{k-1}$. Deleting these $k-1$ perfect matchings reduces the degree of each x of weight k or $k+1$ by $k-1$, so that its degree in $Q_{2k+1}(k-1, k+2) - \bigcup_{i=1}^{k-1} M_i$ is $k+2$. Thus we obtain a $k+2$-regular spanning subgraph of $Q_{2k+1}(k-1, k+1)$.

Now suppose that $2 \leq a \leq k-1$ and suppose that $Q_{2k+1}(k-a, k+1+a)$ has a $(k+1+a)$-regular spanning subgraph. Let $b = a+1$. In $Q_{2k+1}(k-b, k+1+b)$ all vertices, except those of weight $k-b$ or weight $k+1+b$, have degree $2k+1$. The vertices of weight $k-b$ and those of weight $k+1+b$ have degree $k+1+b$. By the Marriage Lemma, the $(k+1+a)$-regular spanning subgraph of $Q_{2k+1}(k-a, k+1+a)$ has $k-a-1 = k-b$ pairwise disjoint perfect matchings. Deleting these $k-b$ perfect matchings from $Q_{2k+1}(k-b, k+1+b)$ reduces the degree of each vertex x with $k-b < \mathrm{wt}(x) < k+1+b$ by $k-b$. Thus in this spanning subgraph of $Q_{2k+1}(k-b, k+1+b)$, the degree of the vertices x with non-extreme weight equals $2k+1-(k-b) = k+1+b$. Thus this subgraph is $(k+1+b)$-regular. The last statement follows from Hall's Marriage Lemma. □

4 Some Other Semi-regular Bipartite Graphs

There are a number of graphs which can be defined on the node set $V = \{x | x \subseteq [n]\}$.

Example 1. For $0 \leq t \leq n-1$ define x and y to be adjacent $\iff |x \cap y| = t$. To guarantee a bipartite graph, we can simply restrict adjacency to those pairs (x, y) with $|x \cap y| = t$ and with weights of opposite parity. Suppose $\mathrm{weight}(v) = a$ and $\mathrm{weight}(u) = b$. Then

$$\deg(v) = \binom{a}{t}\binom{b}{b-t} = \binom{a}{t}\binom{b}{t} = \deg(u).$$

Thus the graph is regular.

Example 2. Same as Example 1, except that we replace the condition $|x \cap y| = t$ with $|x \cap y| \leq t$. If again we restrict to the nodes of weights a

and b, where a and b have opposite parity, then it follows from Example 1 that

$$\deg(v) = \sum_{j=0}^{t} \binom{a}{j}\binom{b}{j} = \deg(u).$$

Once again, the graph is regular.

Example 3. Let $0 < b < a \leq \lfloor \frac{n}{2} \rfloor$, let $X = \{x \subset [n] | \text{weight}(x) = a\}$ and let $Y = \{y \subset [n] | \text{weight}(y) = b\}$. Let G be the bipartite graph with bipartition (X, Y), where (x, y) is an edge if and only if $y \subset x$. Then $\deg(x) = \binom{a}{b}$, $\deg(y) = \binom{n-b}{a-b}$ and so G is sem-regular. Note that if $a - b = 1$ then $G = L_{b,a}$ and is therefore a subgraph of the hypercube Q_n. However, if $a - b > 1, G$ is not isomorphic to a subgraph of Q_m for any m. For $K_{2,3}$ is not isomorphic to a subgraph of Q_m for any m, so it suffices to show that G contains a subgraph isomorphic to $K_{2,3}$. We illustrate this for the special case $a = b + 2$. Now

$$\{1,2,\ldots,b\} \cup \{2,3,\ldots,b,b+1\}, \cup \{1,3,\ldots,b,b+1\} = \{1,2,3,\ldots,b+1\}$$
$$\subset \{1,2,\ldots,b+1,b+2\} \cup \{1,2,\ldots,b+1,b+3\}.$$

Thus we have three b-sets and two $b+2$-sets such that each b-set is a subset of each of the $b+2$-sets. Hence these five vertices form a copy of $K_{2,3}$ in G.

References

[1] Hall, P. On representatives of subsets, *J. Lond. Math. Soc.* **10** (1935), 26–30.