GEOMETRICALLY DISTRIBUTED RANDOM VARIABLES AND PERMUTATIONS AVOIDING CONSECUTIVE 3-LETTER PATTERNS

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ABSTRACT. For words of length n, generated by independent geometric random variables, we consider the probability that these words avoid a given consecutive 3-letter pattern. As a consequence we count permutations in S_n avoiding consecutive 3-letter patterns.

1. Introduction

Let X denote a geometrically distributed random variable, i.e. $\mathbb{P}\{X = k\} = pq^{k-1}$ for $k \in \mathbb{N}$ and $p = 1 - q$. We assume that we have n independent random variables $X_1, X_2, \cdots, X_n$ from this distribution.

In this paper we derive recursive formulae for the probabilities of words of length n avoiding consecutive 3-letter patterns (subwords).

Definition 1. *A word is a sequence of characters or letters drawn from a fixed alphabet. That is, an ordered n-tuple of symbols is an n-word. The empty word is denoted by ε.*

Definition 2. *An n-word w, say $w = a_1a_2 \cdots a_n$, contains a consecutive* 123 *pattern (subword) if and only if there exists $1 \leq i \leq n-2$ such that $a_i < a_{i+1} < a_{i+2}$. That is, if there is a 3-letter block $a_i\ a_{i+1}\ a_{i+2}$ satisfying $a_i < a_{i+1} < a_{i+2}$. Otherwise w is said to avoid a consecutive* 123 *pattern.*

The other five consecutive 3-letter patterns, namely 132, 231, 213, 321, and 312, are defined in the same manner.

Definition 3. *A 3-letter block $a_i\ a_{i+1}\ a_{i+2}$ is said to satisfy*
(i) an up-down pattern if $a_i < a_{i+1} > a_{i+2}$;
(ii) a down-up pattern if $a_i > a_{i+1} < a_{i+2}$.

The other cases, namely up-up and down-down are defined similarly.

Definition 4. *A 3-letter block $a_i\ a_{i+1}\ a_{i+2}$ is said to satisfy an up-down pattern in the sense of* 132 (231) *if $a_i < a_{i+2} < a_{i+1}$ ($a_{i+2} < a_i < a_{i+1}$).*

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We can also define down-up patterns in the sense of 312 or 213 in the same manner.

The generating function related to an up, given that the previous element was i is given by

$$\sum_{i<j} pq^{j-1}x^j = \frac{px}{1-qx}(qx)^i \tag{1}$$

where the indices $i < j$ show that the pattern of the last two letters is an up. The factor $(qx)^i$ of the term on the right side of (1) means that we substitute x by qx. The generating function related to a down, given that the last element was i is given by

$$\sum_{i\geq j} pq^{j-1}x^j = \frac{px}{1-qx} - \frac{px}{1-qx}(qx)^i, \tag{2}$$

where the indices $i \geq j$ indicate that the pattern of the last two letters is a down. The first term means that we forget the labelling of the last part ($x := 1$) and the second term means that we replace x by (qx).

In order to find the probabilities of words avoiding 3-letter patterns, we will use a method called ***adding-a-new-slice.*** This method was used successfully by Flajolet and Prodinger in [2] and Knopfmacher and Prodinger in [3] and more recently by Prodinger in [7].

The probability that words of length n avoid consecutive 3-letter pattern, say $\alpha \in S_3$, will be denoted by $c_n^{\langle\alpha\rangle}(q)$.

In order to obtain the recursions for the probabilities we introduce two variables u and v, where v labels the last letter and u labels the second last letter.

In order to count we use the automaton below, where the two states l_1 and l_2 are the n^{th} down and up step, respectively, in the construction of a word of length n.

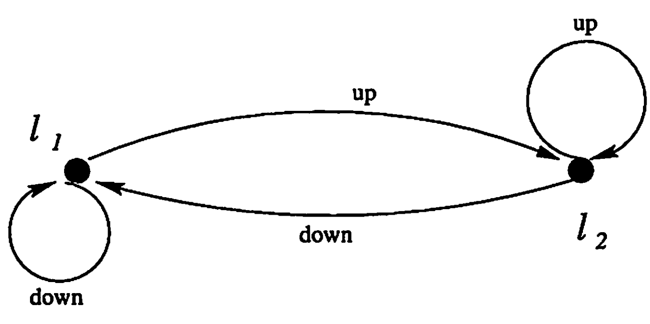

Figure 1.

Let $a_n(u,v)$ ($b_n(u,v)$) be the generating function of words of length n where u marks the second to last letter and v marks the last letter and the last step is a down (up) step. Therefore the generating function related to an up, given that the previous element was i is given by (1) and the generating function related to a down, given that the last element was i is given by (2). Therefore the automaton above gives rise to the following counting functions for $n > 2$

$$\begin{aligned} a_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv) \\ &\quad + \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(1,quv) \end{aligned} \tag{3}$$

and

$$b_n(u,v) = \frac{pv}{1-qv}a_{n-1}(1,quv) + \frac{pv}{1-qv}b_{n-1}(1,quv). \tag{4}$$

Note carefully that there are no restrictions involved and every possible word is taken care of. We shall see in Section 2 that $a_n(u,v)$ and $b_n(u,v)$ will be different from the above equations depending on whether they are restricted or not.

We define $a_2(u,v)$ and $b_2(u,v)$ as follows:

$$\begin{aligned} a_2(u,v) &= \sum_{i\geq 1}\sum_{j\leq i} p^2 q^{i+j-2}u^i v^j \\ &= \frac{p^2uv}{(1-qu)(1-qv)} - \frac{p^2quv^2}{(1-qv)(1-q^2uv)} \end{aligned} \tag{5}$$

meaning that the pattern of the first two letters is a down and

$$\begin{aligned} b_2(u,v) &= \sum_{i\geq 1}\sum_{j>i} p^2 q^{i+j-2}u^i v^j \\ &= \frac{p^2quv^2}{(1-qv)(1-q^2uv)}, \end{aligned} \tag{6}$$

meaning that the pattern of the first two letters is an up.

The discussion above shows that the probability that words of length n admit every 3-letter pattern is given by

$$c_n(q) = a_n(1,1) + b_n(1,1) \tag{7}$$

where $a_n(u,v)$ and $b_n(u,v)$ are given by (3) and (4), respectively, for $n > 1$ and $c_1(q) = 1$. Since there are no restrictions, $c_n(q) = 1$ for each $q \in [0,1]$.

2. Words avoiding consecutive 3-letter patterns

In this section we find the probability of words of length n which avoid 3-letter subpatterns. We shall restrict ourselves to consecutive single patterns. Although there are $3! = 6$ different 3-letter patterns, we shall only need to consider 3 cases.

2.1. Consecutive 123 Avoiding Geometrically Distributed Random Variables.

In this case, we consider words avoiding consecutive 123 subpatterns. That is, words avoiding a 3-letter up-up subpatterns such that the leftmost letter is the smallest and the last one is the largest. This leads us to the following theorem:

Theorem 1. *The probability that words of length n avoid consecutive* 123 *patterns is given by* $c_1^{\langle 123\rangle}(q) = 1$ *and for* $n > 1$

$$c_n^{\langle 123\rangle}(q) = a_n(1,1) + b_n(1,1) \tag{8}$$

where $a_2(u,v)$ *and* $b_2(u,v)$ *are given by* (5) *and* (6) *respectively, and for* $n > 2$ *we have*

$$\begin{aligned} a_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv) \\ &\quad + \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(1,quv) \end{aligned} \tag{9}$$

$$b_n(u,v) = \frac{pv}{1-qv}a_{n-1}(1,quv). \tag{10}$$

Proof. In order to prove this theorem, we make use of the following automaton below.

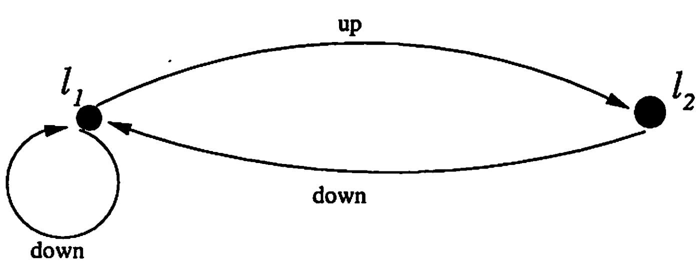

Figure 2: *Consecutive 123 avoiding patterns*

We assume that in trying to construct a word of length n avoiding consecutive 123, we have moved $n-1$ steps. The n^{th} step can either be up or down. To arrive at l_1 and l_2, either the previous step was a down or up.

Case 1. If the n^{th} step is a down, then from (2) we have the generating function

$$\sum_{i\geq j} pq^{j-1}x^j = \frac{px}{1-qx} - \frac{px}{1-qx}(qx)^i. \tag{11}$$

If the $(n-1)^{st}$ step is a down, then adding a new slice means adding a pair (j,k), thereby replacing u^iv^j by u^jv^k. That is, we replace u by 1 and v by u and provide the factor v^k, so that (11) yields

$$z_n(u,v) = \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv), \tag{12}$$

for $n > 2$. If the $(n-1)^{st}$ step is an up, then adding a new slice means adding a pair (j,k), thereby replacing u^iv^j by u^jv^k. That is, replacing u by 1 and v by u and providing the factor v^k, so that (11) yields

$$y_n(u,v) = \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(1,quv), \tag{13}$$

for $n > 2$. Adding (12) and (13) we obtain (9).

Case 2: If the n^{th} step is an up, then from (1) we have the generating function

$$\sum_{i<j} pq^{j-1}x^j = \frac{px}{1-qx}(qx)^i. \tag{14}$$

As seen in the figure above, the $(n-1)^{st}$ step can only be a down, otherwise the word being constructed will not avoid consecutive 123 subpatterns. Adding a new slice means adding a pair (j,k), thereby replacing u^iv^j by u^jv^k, which means that we replace u by 1 and v by u and provide a factor v^k. Hence (14) yields

$$b_n(u,v) = \frac{pv}{1-qv}a_{n-1}(1,quv), \tag{15}$$

for $n > 2$.

The values of $a_2(u,v)$ and $b_2(u,v)$ are given by (5) and (6), respectively. We also define $c_1^{\langle 123\rangle}(q)$ to be equal to 1.

Adding (9) and (10) when both u and v are replaced by 1, we obtain the desired results. □

Let us now consider words of length n avoiding consecutive 321 subpatterns. By left↔right symmetry (i.e. reading a pattern in reverse order) this case is the same as the one for avoiding consecutive 123 patterns. Although the formulae for $a_n(u,v)$ and $b_n(u,v)$ (see theorem below) look different, the values of $c_n^{\langle 321\rangle}(q)$ and $c_n^{\langle 123\rangle}(q)$ coincide when $q \in [0,1]$. This leads us to the following remark:

REMARK. The probability that words of length n avoid consecutive 321 patterns is given by $c_1^{\langle 321\rangle}(q) = 1$ and for $n > 1$

$$c_n^{\langle 321\rangle}(q) = a_n(1,1) + b_n(1,1) \tag{16}$$

where $a_2(u,v)$ and $b_2(u,v)$ are given by (5) and (6) respectively, and for $n > 2$ we have

$$\begin{aligned} a_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{(1-qv)}a_{n-1}(1,quv) \\ &+ \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{(1-qv)}b_{n-1}(1,quv) \end{aligned} \tag{17}$$

$$b_n(u,v) = \frac{pv}{(1-qv)}a_{n-1}(1,quv) + \frac{pv}{(1-qv)}b_{n-1}(1,quv). \tag{18}$$

Below is a graph of values of $c_n^{\langle\alpha\rangle}(q)$ for $q \in [0,1]$, $\alpha \in \{123, 321\}$ and $n = 1, 2, 3, \cdots, 13$.

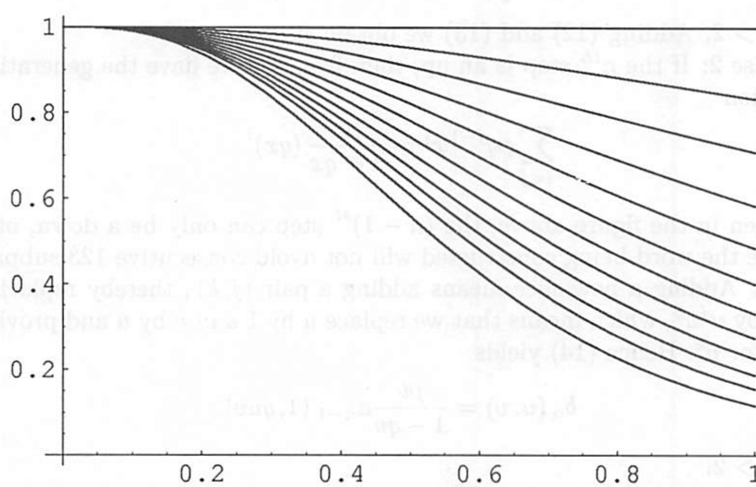

Figure 3: *Values of* $c_n^{\langle\alpha\rangle}(q)$ *for* $q \in [0,1]$, $\alpha \in \{123, 321\}$ *and* $n = 1, 2, 3, \cdots, 13$.

The first line on Figure 3 corresponds to $n = 1, 2$, the second line to $n = 3$, the third line to $n = 4$ and so on.

The cases for the probabilities that words of length n avoid consecutive 123 and 321 patterns can also be done using only one variable, say u.

2.2. Consecutive 132 Avoiding Geometrically Distributed Random Variables.

In this case we look at words avoiding consecutive 132 patterns. This leads us to the following theorem:

Theorem 2. *The probability that words of length n avoid consecutive* 132 *patterns is given by* $c_1^{\langle 132\rangle}(q) = 1$ *and for* $n > 1$

$$c_n^{\langle 132\rangle}(q) = a_n(1,1) + b_n(1,1) \tag{19}$$

where $a_2(u,v)$ *and* $b_2(u,v)$ *are given by* (5) *and* (6) *respectively, and for* $n > 2$ *we have*

$$\begin{aligned} a_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv) \\ &\quad + \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(qv,u) \end{aligned} \tag{20}$$

$$b_n(u,v) = \frac{pv}{1-qv}a_{n-1}(1,quv) + \frac{pv}{1-qv}b_{n-1}(1,quv). \tag{21}$$

Proof. In order to prove this theorem, we make use of the following automaton below.

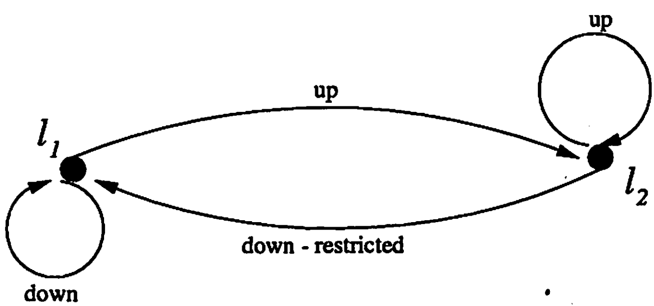

Figure 4: *Consecutive* 132 *avoiding patterns*

□

Also, by left↔right symmetry, the probability that words of length n avoid consecutive 132 subpatterns is the same as the probability that words of length n avoid consecutive 231 subpattern. Unlike in the previous case (consecutive 123 and consecutive 321), where $a_n(u,v)$ and $b_n(u,v)$ were different, both $a_n(u,v)$ and $b_n(u,v)$ coincide.

2.3. Consecutive 213 Avoiding Geometrically Distributed Random Variables. In this case we look at the probabilities of words avoiding consecutive 213 patterns. These are words avoiding down-up pattern in the sense of 213.

Theorem 3. *The probability that words of length n avoid consecutive* 213 *patterns is given by* $c_1^{\langle 213\rangle}(q) = 1$ *and for* $n > 1$

$$c_n^{\langle 213\rangle}(q) = a_n(1,1) + b_n(1,1) \tag{22}$$

where $a_2(u,v)$ *and* $b_2(u,v)$ *are given by* (5) *and* (6) *respectively, and for* $n > 2$ *we have*

$$a_n(u,v) = \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv) + \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(1,quv) \tag{23}$$

$$b_n(u,v) = \frac{pv}{1-qv}a_{n-1}(qv,u) + \frac{pv}{1-qv}b_{n-1}(1,quv) \tag{24}$$

Proof. The proof of this theorem is similar to the proof of Theorem 3, and is omitted. The automaton is given below. newline

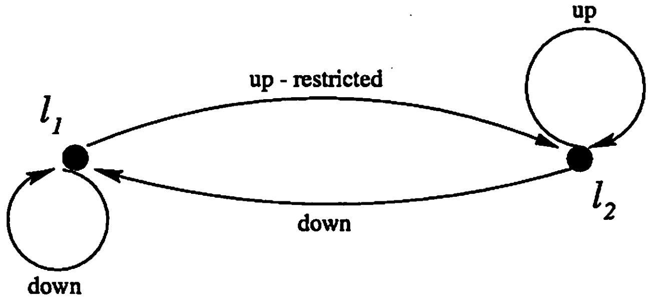

Figure 5: *Consecutive* 213 *avoiding patterns*

□

Also, by left↔right symmetry, the probability that words of length n avoid consecutive 213 subpatterns is the same as the probability that words of length n avoid consecutive 312 subpatterns. Both $c_n^{\langle 312 \rangle}(q)$ and $c_n^{\langle 213 \rangle}(q)$ also coincide.

Below (Fig. 6) is a graph of values of $c_n^{\langle \alpha \rangle}(q)$ for $q \in [0,1]$, $\alpha = 132\,(231)$ and $n = 1,2,3,\cdots,13$. The graph of values of $c_n^{\langle \alpha \rangle}(q)$ for $q \in [0,1]$, $\alpha = 213\,(312)$ and $n = 1,2,3,\cdots,13$ looks similar.

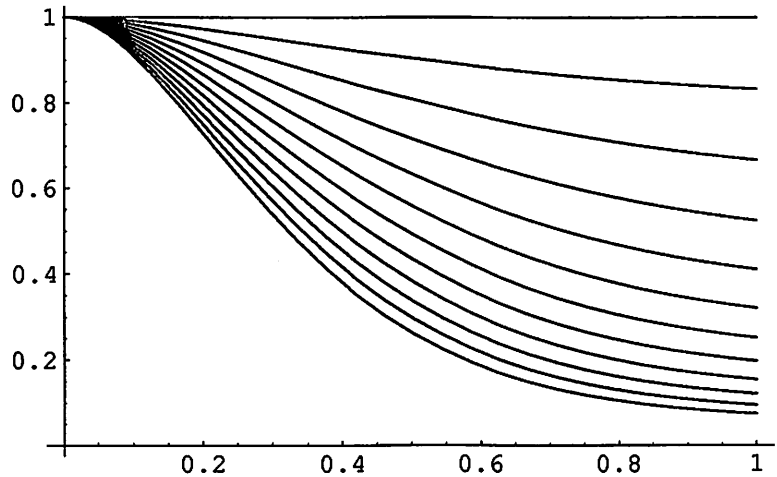

Figure 6: *Values of* $c_n^{\langle\alpha\rangle}(q)$ *for* $q \in [0,1]$, $\alpha = 132\,(231)$*and* $n = 1, 2, 3, \cdots, 13$.

The first line on Figure 6 corresponds to $n = 1, 2$, the second line to $n = 3$, the third line to $n = 4$ and so on.

3. Applications to Permutations

In this section, we count permutations in S_n, the symmetric group on n letters, avoiding consecutive 3-letter patterns. For permutations α and β, we say that α is order-isomorphic to β if the following condition holds: for all $1 \leq i, j \leq n$, $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$.

Definition 5. *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in S_k$ *be a permutation, and let* $k \leq n$. *We say that* $p = (p_1, p_2, \ldots, p_n) \in S_n$ *contains a consecutive subsequence (or pattern) of type* α *if and only if there exists* $1 \leq i \leq n - k + 1$ *such that* $(p_i, p_{i+1}, \ldots, p_{i+k-1})$ *is order-isomorphic to* α. *Otherwise we say that* p *avoids* α.

Let $\alpha \in S_k$ be a permutation, and let $k \leq n$. The set of all permutations in S_n avoiding consecutive patterns which are order-isomorphic to α will be denoted by $S_n(\langle\alpha\rangle)$. For each $\alpha \in S_k$, where $k \leq n$, let $A_n(\langle\alpha\rangle) = |S_n(\langle\alpha\rangle)|$.

Example 1. $S_4(\langle 132\rangle) = \{4123, 4213, 4231, 4312, 4321, 1234, 1342, 3124, 3214, 3241, 3412, 3421, 2314, 2413, 2341, 2134\}$ *and therefore* $A_3(\langle 132\rangle) = 16$.

Let us define the bivariate ordinary generating function (OGF) for the probability that words of length n avoid a given consecutive 3-letter pattern α

$$P(z,q)=\sum_{n\geq 0} c_n^{(\alpha)}(q)\, z^n.$$

Consider now the symmetric group S_n and the exponential generating function (EGF) for permutations in S_n avoiding a given consecutive 3-letter pattern α

$$\hat{P}(z)=\sum_{n\geq 0} A_n(\langle\alpha\rangle)\frac{z^n}{n!}.$$

The following lemma can be found in Flajolet [1].

Lemma 1. *[1] The relation between the sequence (words) model and the permutation model is*

$$\hat{P}(z)=\lim_{q\to 1^-} P(z,q),$$

uniformly for any z such that $|z|<r_0$ with $r_0<1$.

Example 2. *Considering the probability that words of length n admitting every 3-letter patterns we have, from Section 1, that $c_n(q)=1$ for all $q\in[0,1]$, so that*

$$P(z,q)=\sum_{n\geq 0} c_n(q)\, z^n=\sum_{n\geq 0} z^n=\frac{1}{1-z},$$

which agrees with the number of permutations in a symmetric group S_n as

$$n!\,[z^n]\frac{1}{1-z}=n!,$$

as it should be.

We show in the following lemma that there are only two distinct sequences of values that occur.

Lemma 2. *For every symmetric group S_n,*
(a) $A_n(\langle 123\rangle)=A_n(\langle 321\rangle)$;
(b) $A_n(\langle 132\rangle)=A_n(\langle 231\rangle)=A_n(\langle 213\rangle)=A_n(\langle 312\rangle)$.

Proof. (a) We interchange the smallest and the largest letters using the transformation

$$\alpha_i=n+1-\beta_i, \tag{25}$$

for $n\geq 1$ and $1\leq i\leq n$.

(b) This is a consequence of (21) above and the transformation

$$\alpha_i = \beta_{n+1-i},$$

where $n \geq 1$ and $1 \leq i \leq n$. □

Table 1

Values of $A_n(\langle\alpha\rangle)$, for $\alpha \in S_3$ and $n \geq 1$

$A_n(\langle\alpha\rangle)$, $\alpha = 123\,(321)$	1, 2, 5, 17, 70, 349, 2017, 13358, 99377, 822041, 7477162, 74207209, 797771521, 9236662346, ···
$A_n(\langle\alpha\rangle)$, $\alpha \in \{213, 312, 132, 231\}$	1, 2, 5, 16, 63, 296, 1623, 10176, 71793, 562848, 4853949, 45664896, 465403791, ···

One may also look at the number of permutations of n elements which contain no increasing subsequence of length 3 (not necessarily consecutive). Permutations with restrictions of this type can be approached from the Computer Sciences standpoint of sorting problems [4], [8], as well as part of the combinatorial topic of strings with forbidden subwords [5], [6]. The number of permutations of n elements which contain no increasing subsequence of length 3 (not necessarily consecutive) is known [4] to be $C_n = \frac{1}{n+1}\binom{2n}{n}$, where C_n denotes the n^{th} Catalan number. It is also known [9] that given any pattern of length 3, the number of permutations avoiding that pattern is C_n.

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