Some applications of combinatorial designs to extremal graph theory

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ABSTRACT. In this paper, we give a few applications of combinatorial design theory to a few problems in extremal graph theory. Using known results in combinatorial design theory, we have unified, simplified, and extended results on a few problems.

1 Introduction and Definitions

In this short paper, we give a few applications of combinatorial design theory to extremal graph theory. Before we proceed, we need some basic terminology.

Let G be an additively written group of order n. A k-subset D of G is a (n, k, λ)-*difference packing* if every nonzero element of G has at most λ representations as a difference $d - d'$ with elements from D. The difference packing is *abelian*, *cyclic* etc., if the group G has the respective property.

A (m, n, k, λ) *relative difference set (RDS)* R in an abelian group G relative to a normal subgroup N is a k-subset of G with the following property: the list of differences $a - b$ with distinct elements $a, b \in R$ contains each element in $G \setminus N$ exactly λ times. Moreover, no (non-identity) element in N has such a representation. Therefore N is called the *forbidden subgroup*. The meaning of the parameters m and n is as follows: the order of the group G is mn; the order of N is n. Note that each coset of N contains at most one element from R.

The following series are known to exist. For references, see [15].

1. $(p^2 + p + 1, 1, p + 1, 1)$-RDS exists whenever p is a prime power.

2. (p^a, p^b, p^a, p^{a-b})-RDS exists whenever p is a prime and $a \geq b \geq 0$.

3. $(q+1, q-1, q, 1)$-RDS exists for all prime powers q.

4. For any prime power q and any divisor d of $q-1$, $(q+1, \frac{q-1}{d}, q, d)$-RDS exists.

It is easy to see that a (m, n, k, λ)-RDS is a (mn, k, λ) difference packing. Series 1 is the well-known difference set for desarguesian finite projective plane of order p. Series 3 was obtained by Bose [3] in 1942 by proving that the affine plane of order q is 1-rotational. Series 4 can be obtained from Series 3, since the relative difference set $(q+1, q-1, q, 1)$ exists over Z_{q^2-1}. There are many more known series of relative difference sets. We refer the reader to [15] for a survey.

A *transversal design of order* n or *group size* n, *block size* r, and *index* λ, denoted $\mathrm{TD}_\lambda(r, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where:

(1) V is a set of rn elements;

(2) $\mathcal{G}$ is a partition of V into r classes (called *groups*), each of size n;

(3) $\mathcal{B}$ is a collection of r-subsets of V (called *blocks*);

(4) every unordered pair of elements from V is either contained in exactly one group and in no blocks, or is contained in exactly λ blocks, but not both.

It is easy to see that there are λn^2 blocks in a $\mathrm{TD}_\lambda(r, n)$.

A *Steiner system* $S(2, l, v)$ is a pair $(V, \mathcal{B})$, where V is a v-set, whose elements are called *points*, and $\mathcal{B}$ is a family of l-subsets of V called *blocks*, such that any two distinct points belong to exactly one block. An $S(2, l, v)$ is said to be *resolvable* if the blocks can be partitioned into classes so that every point appears once in each class.

2 Turán type problems

Given a family of r-uniform hypergraphs (or r-graphs) $\mathcal{F}$, we say that an r-graph $\mathcal{G}$ is $\mathcal{F}$-free if $\mathcal{G}$ contains no subhypergraph isomorphic to any element in $\mathcal{F}$. Let $\mathrm{ex}(n, F)$ be the maximum number of edges in an n vertex $\mathcal{F}$-free r-graph. If $\mathcal{F} = \{K_k^{(r)}\}$, the complete r-graph on k vertices, then $\mathrm{ex}(n, \mathcal{F})$ is the Turán number $t_r(n, k)$. The determination of $\lim_{n\to\infty} \frac{t_r(n,k)}{\binom{n}{r}}$ is perhaps the most fundamental open problem in extremal hypergraph theory. We consider the related question of determining $\mathrm{ex}(n, \mathcal{F})$ when $\mathcal{F} \neq \{K_k^{(r)}\}$. In this section, we give a unified treatment of some known results using difference packings.

First we consider the case when $r = 2$ and $F = K_{2,t+1}$.

Theorem 2.1. *Suppose there exists a (n, k, t) difference packing, D, over G. Let α be the number of $g \in G$ such that $2g \in D$. Then there exists a graph with $\frac{nk-\alpha}{2}$ edges on n vertices which does not contain any $K_{2,t+1}$.*

Proof: Construct a $|G| \times |G|$ adjacency matrix, A of the graph G, by $A(i, j) = 1$ if and only if $i + j \in D$ where $i, j \in G$ and $i \neq j$. We show that this graph does not contain any $K_{2,t+1}$. Suppose there exists a $K_{2,t+1}$ on $\{x, y, a_1, a_2, \ldots, a_{t+1}\}$. Then there exists $d_1, d_2, \ldots, d_{2t+2} \in D$, such that $x + a_i = d_i$ for $i = 1, 2, \ldots, t + 1$, and $y + a_i = d_{t+1+i}$ for $i = 1, 2, \ldots, t + 1$. Clearly, $d_1 \neq d_2 \neq \cdots \neq d_{t+1}$ and $d_{t+2} \neq d_{t+3} \neq \cdots \neq d_{2t+2}$. Also, we must have $x = d_{t+1+i} - d_i$ for $i = 1, 2, \ldots, t + 1$. Hence, D is not a (v, k, t) difference packing, a contradiction. The number of edges in the graph can be counted in an obvious manner. □

Corollary 2.2. (Erdös, Rényi and Sós [6] and Brown [4]) $ex(K_n, C_4) = \frac{1+o(1)}{2} n^{1.5}$.

Proof: The lower bound can be obtained by applying Theorem 2.1. If p is a prime power, then there exists a $(p^2 + p + 1, 1, p + 1, 1)$-RDS over Z_{p^2+p+1}. Clearly, $\alpha = p + 1$. Simple calculation yields the lower bound. The upper bound is proved by Kövári, Sós and Turán [9]. □

Corollary 2.3. (Füredi [7]) $ex(K_n, K_{2,t+1}) = \frac{1}{2} t^{0.5} n^{1.5} + O(n^{\frac{4}{3}})$.

Proof: If $q \equiv 1 \pmod{t}$, a $(q + 1, \frac{q-1}{t}, q, t)$-RDS exists, the lower bound can be obtained by applying Theorem 2.1. The upper bound is proved by Kövári, Sós and Turán [9]. □

In the remainder of this section, we consider the Turán numbers for r-partite r-graphs. Let r and t be fixed positive integers, $r \geq 2$. Let $\mathcal{H}$ be the complete r-partite r-graph $K^{(r)}(1, 1, \ldots, 1, 2, t + 1)$ consisting of $r - 2$ sets of size 1, one set of size 2, and one set of size $t + 1$.

Theorem 2.4. *Suppose there exists an $(n, k, t + 1)$ difference packing, D, over G. Then there exists an r-uniform hypergraph on n points avoiding $\mathcal{H}$, with the number of hyperedges at least $\frac{k}{n}\binom{n}{r}$.*

Proof: The hypergraph has vertex set G. Any hyperedge of the form $\{x_1, x_2, \ldots, x_r\}$ is in the hypergraph if and only if $x_1 + x_2 \cdots + x_r \in D$ and $x_1 \neq x_2 \neq \cdots \neq x_r$. It is easy to see that the hypergraph does not contain any $K^{(r)}(1, 1, \ldots, 1, 2, t + 1)$. Suppose there exists a r-uniform r-paritite hypergraph $K^{(r)}(1, 1, \ldots, 12, t + 1)$. Consider the 2-graph induced by the $r - 2$ parts of size 1 in the $K^{(r)}(1, 1, \ldots, 1, 2, t + 1)$. Then, the resulting 2-graph would contain a $K_{2,t+1}$, which is impossible by Theorem 2.1. There are in total $\binom{n}{r}$ hyperedges in the hypergraph. So, we can translate the difference packing from D to $D + i$, which has the maximum number of hypergraphs. Hence, we obtain the required lower bound on the number of hyperedges. □

Theorem 2.5. (Muyabi) [13] *If $q \equiv 1 \pmod{t}$ is a prime power, then there exists a r-uniform hypergraph on $n = \frac{q^1-1}{t}$ points with at least $\frac{q+1}{n}\binom{n}{r}$ hyperedges, such that the hypergraph avoids any r-uniform r-partite graph $K^{(r)}(1,1,\ldots,1,2,t+1)$.*

Proof: There exists a D, $(\frac{q^2-1}{t}, q, t)$-difference packing, by the existence of RDS. The results follow by Theorem 2.4 □

3 Multicolor Ramsey theory

The multicolor Ramsey number $r_k(G)$ is the smallest integer n for which any k-coloring of the edges of the complete graph K_n must produce a monochromatic 4-cycle. It was proved by Chung and Graham [5] that $r_k(C_k) \geq k^2 - k + 2$ whenever $k-1$ is a prime power. It has recently been shown by Lazebnik and Woldar [10] that $r_k(C_k) \geq k^2 + 2$ when k is an odd prime power. In this section, we give a uniform treatment to both constructions. As a by-product, we give an extension of Lazebnik and Woldar's construction.

Theorem 3.1. *Let G be an abelian group of order n. Suppose $G = D_1 \cup D_2 \cup \cdots \cup D_k$ such that each D_i is a (n, k_i, t) difference packing over G where $|G| = n$. Then, $r_k(K_{2,t+1}) \geq n+1$.*

Proof: Since D_i is a $(n, k, t+1)$-difference packing, we can construct a graph on vertex set G with edges $\{x, y\}$ if and only if $x + y \in D_i$. It was shown in Theorem 2.1 that the graph obtained in this manner does not contain any $K_{2,t+1}$. Since D_i partitions G, if we obtain k graphs using $D_1, D_2, \ldots, D_k$ in turn, we can partition the edge of K_n. Hence, $r_k(K_{2,t+1}) \geq n+1$. □

The generalization of the above lemma to hypergraphs is immediate, and thus omitted.

The following is immediate from the partition of Z_{k^2-k+1} into difference packings.

Corollary 3.2. [5] *$r_k(C_4) \geq k^2 - k + 2$ when $k-1$ is a prime power.*

Corollary 3.3. *$r_k(C_4) \geq k^2 + 1$ for all prime powers k.*

Proof: Let D be a $(k, k, k, 1)$-RDS over G. Let N be the normal subgroup in the RDS. It is clear that $\{\{D + i\} : i \in N\}$ partitions all elements in G. The result follows from Theorem 3.1. □

In a certain situations, we can further increase the bound by 1.

Theorem 3.4. *$r_k(C_4) \geq q^2 + 2$ when q is a prime power.*

Proof: It suffices to exhibit a q-coloring of the edges of K_{q^2+1} in which there is no monochromatic C_4. Fix a vertex v of K_{q^2+1} and denote the subgraph induced by its set of neighbors by G. As G is isomorphic K_{q^2}, we can q-color its edges in Corollary 3.3. Thus, it remains only to color the edges $\{u, v\}$ for $u \in V(G)$, and we do this by simply assigning color i to the edges of the form $\{v, x\}$ where x is in the ith coset of the normal subgroup in the $(q, q, q, 1)$-RDS. Suppose, by way of contradiction, that a monochromatic 4-cycle exists, say of color j. Clearly v must be one of its vertices, so denote the consecutive vertices of this cycle by v, w, x, y. Then w and y are in the same coset of the normal subgroup. Also, $w+x = d_1+j$ and $y+x = d_2+j$, where $d_1, d_2 \in D$, $d_1 \neq d_2$ and $j \in N$, where N is the normal subgroup. Then, we have $w - y = d_1 - d_2$. Since w and y are in the same coset, $w - y$ must be in the normal subgroup. Hence, D is not a relative difference set. □

The proof of the above theorem when p is odd was first proved by Lazebnik and Woldar. We have extended the result to the case in which p is an even prime power.

4 Splittable colorings of graphs and hypergraphs

Let $\mathcal{K}_n^k$ denote the complete k-uniform hypergraph on n vertices; i.e. we have a ground set of n elements and we take all the k-sets to be edges. Often, we will denote the ground set $\{1, 2, \ldots, n\}$ by $[n]$. An r-coloring of $\mathcal{E}(\mathcal{K}_n^k)$ is a coloring of the k-sets with r colors.

Given a coloring of the vertices and edges of $\mathcal{K}_n^k$, a *totally monochromatic m-clique*, for $k \leq m \leq n$, is a $\mathcal{K}_m^k$ whose edges (k-sets) all get the same color.

An r-coloring of $\mathcal{E}(\mathcal{K}_n^k)$ is *(r, m)-splittable* if there is a coloring of the ground set with r colors so that no totally monochromatic m-clique is produced.

Let $f_r^k(m)$ be the minimum n for which there is an r-colorring of $\mathcal{E}(\mathcal{K}_n^k)$ which is not (r, m)-splittable. This means that we can find an r-colorring of the k-sets of $\mathcal{K}_n^k$ with the property that every r-vertex coloring produces a totally monochromatic m-clique. An r-colorring of $\mathcal{E}(\mathcal{K}_n^k)$ where every set of size $\lceil \frac{n}{r} \rceil$ has a monochromatic m-clique in every color is such a non-(r, m)-splittable coloring and will be called an *(r, m)-balanced* coloring. $g_n^k(m)$ is then defined as the minimum n for which there is an r-colorring $\mathcal{E}(\mathcal{K}_n^k)$ is defined as the minimum n for which there is an r-colorring of $\mathcal{E}(\mathcal{K}_n^k)$ which is (r, m)-balanced.

Füredi and Ramamurthi [8] have defined and obtained various bounds on $f_r^k(m)$ and $g_r^k(m)$. The upper bound construction in [8] is different when $k = 2$ and when $k \geq 3$. The construction when $k \geq 3$ is fairly complicated. The purpose of this note is to give a unified construction for all $k \geq 2$.

Furthermore, the upper bound that is given in the construction is slightly better when $k \geq 3$.

Using an algebraic construction, Füredi and Ramamurthi [8] proved the following theorem.

Theorem 4.1. *$f_r^k(m) \leq g_r^k(m) \leq \frac{q^2-1}{t}$ where $q \equiv 1 \pmod{t}$ is a prime power such that $q \geq r(m+1) - 1$ and $t < k$.*

Theorem 4.2. *If there exists a $TD_{k-1}(r, v)$ and $v > \frac{r(m-1)}{k-1}$, then $f_r^k(m) \leq g_r^k(m) \leq v^2(k-1)$.*

Proof: Let $n = v^2(k-1)$. We will show that $\mathcal{K}_n^k$ has an (r, m)-balanced coloring. The vertices of $\mathcal{K}_n^k$ is the block in the transversal designs. For any point in group j, x in the transversal designs, let $A_1, A_2, \ldots, A_{(k-1)v}$ be all blocks such that $x \in A_i$ for $i = 1, 2, \ldots, (k-1)v$. We color all k subsets of $A_1, A_2, \ldots, A_{(k-1)v}$ in color i. We color all remaining hyperedges arbitrarily. First, we need to show that any k subsets receive at most one color. Suppose $B_1, B_2, \ldots, B_k$ are k blocks in the transversal design which receive two colors. Then there must be a pair of point y and z such that y, z are all in of the k blocks $B_1, B_2, \ldots, B_k$. Hence, it contradicts the fact that any pair of points are on at most $k-1$ blocks.

Let S be a set of $|S|$ points in $\mathcal{K}_n^k$. By interchanging the roles of points and blocks, these $|S|$ points become $|S|$ blocks in the transversal designs. Consider color class i. Since every block intersects group i, if $|S| \geq \lceil \frac{n}{r} \rceil > (m-1)v$, some point in group i must be on at least m blocks. Then these m blocks intersecting in a point in group i define a totally monochromatic clique of size m of color class i. □

By taking multiple copies of an affine plane of order q, $TD_{k-1}(q, q)$ exists for all prime powers q. Hence, we have the following corollary.

Corollary 4.3. *$f_r^k(m) \leq g_r^k(m) \leq (k-1)q^2$ where q is a prime of a power such that $q > \frac{r(m-1)}{k-1}$.*

$TD_{k-1}(r, v)$ is known to exist when v is sufficiently large (see [2]). Hence, we have the following.

Corollary 4.4. *For fixed r and a sufficiently large m, $f_r^k(m) \leq g_r^k(m) \leq (\lfloor \frac{r(m-1)}{k-1} \rfloor + 1)^2(k-1)$.*

5 Certain coloring of $K_{n,n}$ avoiding monochromatic C_4

In this section, we study a problem considered by Axenovich, Füredi, and Mubayi [1] on the generalized Ramsey theory.

Given graphs G and H, a coloring of $E(G)$ is called (H, q)-coloring if the edges of every copy of $H \subset G$ together receive at least q colors. Let $r(G, H, q)$ denote the minimum number of colors in an (H, q)-coloring of G.

Axenovich, Füredi, and Mubayi proved the following.

Theorem 5.1. *If n is odd, then $r(K_{n,n}, C_4, 3) \leq n$. If n is even, then $r(K_{n,n}, C_4, 3) \leq n+1$.*

They commented that improving these upper bound is very difficult. Eichhorn improved it by one when $n = 4, 12, 20, 36, 60$ by exhibiting $(C_4, 3)$-colorings of $K_{n,n}$ with n colors. We apply a known Latin square result to improve the result.

Theorem 5.2. *If $n \geq 3$, $r(K_{n,n}, C_4, 3) \leq n$.*

Proof: Eichhorn has proved the theorem for instances in which $n = 4$. Suppose $n \geq 3$, $n \neq 4$. There exists a Latin square of order n which does not contain a subsquare of order 2 [12]. The (i,j)th entry in the Latin square represents the color of $x_i y_j$, where the partite sets of G are $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. □

Axenovich, Füredi and Mubayi also proved the following lower bound on $r(K_{n.n}, C_4, 3)$.

Theorem 5.3. $r(K_{n,n}, C_4, 3) > \lfloor \frac{2n}{3} \rfloor$.

In proving the lower bound, they did not make use of the fact that the C_4 can contain an alternating C_4, a 2-colored C_4 whose edges alternate between its two colors when viewed cyclically.

Hence, Axenovich, Füredi, and Mubayi defined the following: a *weak* $(C_4, 3)$-coloring of $K_{n,n}$ is a coloring of the edges of $K_{n,n}$ in which every copy of C_4 has at least three colors or is alternately 2-colored. Let $r'(K_{n,n}, C_4, 3)$ denote the minimum number of colors in a weak $(C_4, 3)$-coloring of $K_{n,n}$. Using a sophisticated theorem of Pippenger and Spencer [14] and the probabilistic method, they proved the following.

Theorem 5.4. *As $n \to \infty$, $r'(K_{n,n}, C_4, 3) \leq \frac{3n}{4}(1 + o(1))$.*

A simple use of resolvable designs can lead to the following construction.

Theorem 5.5. *If there exists a weak C_4 coloring on $K_{n,n}$ with r color classes and a resolvable $S(2, n, v)$, then there exists a weak C_4 coloring on $K_{v,v}$ with $\frac{r(v-1)}{n-1} + 1$ color classes.*

Proof: Let C be the coloring on $K_{n,n}$ with r colors. By taking edges from other color classes, we can obtain a coloring on $K_{n,n}$ with $r+1$ colors such that one of the color class is a 1-factor on $K_{n,n}$. Taking a resolvable $S(2, n, v)$ on V, we will color the edges of $V \times \{0, 1\}$. Our method is to give r colors to each parallel class in the Steiner system. For every parallel class with blocks $B_1, B_2, \ldots, B_{\frac{v}{n}}$, we color the edges on $B_i \times \{0, 1\}$ by the color class of $K_{n,n}$. Repeat the same procedure for every parallel class. Since B_i in each parallel class is disjoint, each color class is clearly a star. Finally, we add a new parallel class corresponding to the 1-factor. In total, we have

r color classes from each of the $\frac{v-1}{n-1}$ parallel classes and 1 color class for a 1-factor. Hence, we obtain the required number of color classes. It is not too difficult to check that the required conditions in [1] are satisfied. For details, we refer the reader to [1]. □

The above construction basically shows that if one can find a good coloring with few colors when n is small, one can then obtain a good upper bound asymptotically. Due to the complexity of the proof, we refer the reader to [11]. In fact, using other techniques from combinatorial design theory, it is indeed possible the prove the following.

Theorem 5.6. [11] *There exists a constant C such that $r'(K_{n,n}, C_4, 3) \leq \frac{2n}{3} + C$ for all $n \geq 1$.*

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