3-Regular Hypergraphs that are Decomposable and Threshold

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In general the class of threshold hypergraphs and decomposable hypergraphs are not equal. In this paper we show however that except for two counter examples, a decomposition hypergraph consisting of five or fewer classes is in fact threshold. In the process of showing this result the paper generates all decomposable quotients with five or fewer classes.

1. Introduction

In 1980, Golumbic [2] challenged the research community by asking if the work of Chvátal and Hammer [1], which showed that a graph was decomposable if and only if it was threshold, could be extended to hypergraphs, a generalization of graphs. Reiterman et al. [4] have since answered this question by showing that the two concepts are not equivalent in hypergraphs.

The counter example constructed by Reiterman et al. [4] to show that decomposable does not imply threshold is a 3-regular hypergraph that can be partitioned into six classes. In this paper we investigate whether or not this was the smallest counter example possible.

A <u>3-regular hypergraph</u> G=(V,E) is a set V of vertices and an edge collection E of subsets of V all of size three. For example, G = (V={1,2,3,4,5}, E={{1,2,3}, {1,2,4}}), illustrated in Figure 1.1, is a 3-regular hypergraph. A subset X of V is <u>stable</u> if no size three subset of X is an edge in G. The subset {1, 3, 4, 5} is a stable subset of the vertex set of the 3-regular hypergraph shown in Figure 1.1.

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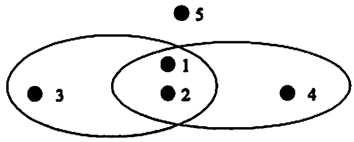

Figure 1.1: The 3-regular hypergraph (V={1,2,3,4,5}, E={{1,2,3}, {1,2,4}).

A 3-regular hypergraph G=(V,E) is said to be threshold if there exists an assignment $[w{:}t]$ consisting of a weighting w of the vertices of V by nonnegative integers and a positive integer threshold t such that a subset X of V is stable if and only if $w(X) \le t$. A 3-regular hypergraph G=(V,E) is said to be edge-threshold if there exists an assignment $[w{:}t]$ consisting of a weighting w of the vertices of V by nonnegative integers and a positive integer threshold t such that a three element subset {x,y,z} of V is an edge of G if and only if $w(x)+w(y)+w(z) > t$. The assignment

$$[w(1)=3,\ w(2)=4,\ w(3)=w(4)=2,\ w(5)=1;\ t=8]$$

shows that the 3-regular hypergraph of Figure 1.1 is edge-threshold.

The edge set of a 3-regular hypergraph provides us with a means of ordering hypergraph vertices. Let G=(V,E) be a 3-regular hypergraph. For $x,y \in V$ define $x \ll y$ if $\{x,z_1,z_2\} \in E$ implies $\{y,z_1,z_2\} \in E$ whenever $z_1, z_2 \in V-\{x,y\}$. For example to illustrate that each of the possible ordering relationships can exist in a single 3-regular hypergraph: let

$$G=(\{1,2,3,4,5,6,7,8\},\{145,245,345,236,467,578,678\}),$$

then (1) $1 \ll 2$ but not $2 \ll 1$, (2) $2 \ll 3$ and $3 \ll 2$, and (3) 4 and 5 are not comparable. (Note that in this example and from here on, in whenever there is not confusion, edge set {x,y,z} will be denoted xyz.) A 3-regular hypergraph G=(V,E) is said to be decomposable if for all $x,y \in V$ $x \ll y$, or $y \ll x$, or both.

The $\ll$ relationship partitions the vertices of a decomposable 3-regular hypergraph G = (V,E) into disjoint classes when [x] is defined as $\{y \in V : x \ll y$ and $y \ll x\}$ for every $x \in V$. The classes of the partition can be ordered by defining $[x] < [y]$ iff $x \ll y$.

When G is decomposable every two classes are comparable. Also, if $[x] < [y]$, there can not exist [z] such that $[z] < [x]$ and $[y] < [z]$. If this were the case $[z] < [x]$ would imply there exists an edge $\{x,v_1,v_2\}$ such that $v_1 \neq z$, $v_2 \neq z$, and $\{z, v_1, v_2\}$ is not an edge. But $[x] < [y]$ implies $\{y,v_1,v_2\}$ is an edge whenever $\{x,v_1,v_2\}$ is an edge and since $[y] < [z]$ it then follows that $\{z,v_1,v_2\}$ is an edge which is a contradiction. Thus the ordering is linear.

Once the vertices are partitioned and ordered as $V= C_1 \cup C_2 \cup \ldots \cup C_n$, the vertex-pairs of V naturally partition into n+1 sets $D_0,D_1,\ldots,D_n$ such that (1) no vertex pair in D_0 is contained in an edge of G, (2) if $i,j \in Z^+$, $\{x,y\} \in D_i$ and $z \in C_j$ where $j \ge i$, then $\{x,y,z\} \in E$, and (3) if $i \in Z^+$, $\{x,y\} \in D_i$, $z \in C_i$, and $w \in C_{i-1}$, then $\{x,y,z\} \in E$ but $\{x,y,w\} \notin E$.

We illustrate the above-described C/D partitioning with an example. The 3-regular hypergraph we use in the example was taken from the threshold hypergraph work of Reiterman et al. [4]. They define the 3-regular hypergraph:
$G = (V=\{1,2,\ldots,9\}, E=\{\{x,y,z\} : \exists\ i,j,k \in Z+ \cup \{0\} \ni (x,y,z) = (2+i,5+j,7+k)$ or $(1+i,4+j,9+k))$.

Table 1.1 is a second representation of G. In the table representation, the entry at cell (i,j) corresponds to the first vertex that appears in an edge with the pair (i,j). For example, the vertex-pair 4,7 appears in the edges 475, 476, 478, 479 since 5 is in cell (4,7). Blank cells in the table indicate that the vertex-pair appears in no edge.

Table1.1: A 3-regular hypergraph.

	1	2	3	4	5	6	7	8	9
1	*			9	9	9	9	9	4
2		*		9	7	7	5	5	4
3			*	9	7	7	5	5	4
4	9	9	9	*	7	7	5	5	1
5	9	7	7	7	*	7	2	2	1
6	9	7	7	7	7	*	2	2	1
7	9	5	5	5	2	2	*	2	1
8	9	5	5	5	2	2	2	*	1
9	4	4	4	1	1	1	1	1	*

Table 1.2: The C/D partitioning of the 3-regular hypergraph of Table 1.1.

	$D_0=\{12,13,23\}$
$C_6=\{9\}$	$D_6=\{14,24,34,15,16,17,18\}$
$C_5=\{7,8\}$	$D_5=\{25,35,45,26,36,46,56\}$
$C_4=\{5,6\}$	$D_4=\{27,37,47,28,38,48\}$
$C_3=\{4\}$	$D_3=\{19,29,39\}$
$C_2=\{2,3\}$	$D_2=\{57,67,58,68,78\}$
$C_1=\{1\}$	$D_1=\{49,59,69,79,89\}$

The table representation of G makes it easy to find the C classes, for it is clear that two vertices i and j are in the same C class iff row i and row j agree except in columns i and j. The table representation also facilities the finding the ordering of the C's. $C_i < Cj$ iff each row i value is less than, with respect to <<, the corresponding row j value. The C/D partition of G is shown in Table 1.2.

It is straightforward to show that for a 3-regular hypergraph: threshold => edge-threshold => decomposable. However, Reiterman et al. [4] have shown that for 3-regular hypergraphs: decomposable ≠> edge-threshold and edge-threshold ≠> threshold. Both negative results use a small 3-regular hypergraph that satisfies one but not the other property. In the case of decomposable not implying edge-threshold, the decomposable 3-regular hypergraph used for the counter example is the one shown in Table 1.1. Recall that when partitioned the example contained six classes.

Was a decomposable hypergraph with six partition classes the smallest such counter example or is it possible to find a smaller counter example? The purpose of the paper is to investigate this question.

2. The Quotient of a 3-Regular Decomposable Hypergraph

Let G=(V,E) be a 3-regular decomposable hypergraph, then G has a C/D decomposition. The vertices of a given C_i in the decomposition are isomorphic in the sense that they are contained in "the same edges". Thus without lose of information about edges we can represent C_i as a single vertex denoted by the partition subscript. Note however since C_i might contain more than a single point i,i needs to be considered as an edge pair. Using this reduced representation, which we shall call the decomposition quotient, the decomposition of the 3-regular hypergraph shown in Table 1.1 is given in Table 2.1.

Table 2.1: The reduced C/D partitioning of the 3-regular hypergraph of Table 2.1.

	D_0={12,22}
C_6={6}	D_6={13,23,14,15}
C_5={5}	D_5={24,34,44}
C_4={4}	D_4={25,35}
C_3={3}	D_3={16,26}
C_2={2}	D_2={45,55}
C_1={1}	D_1={36,46,56}

Although each 3-regular decomposable hypergraph has a unique decomposition quotient, a given decomposition quotient represents an infinite collection of 3-regular decomposable hypergraphs. For example, all complete hypergraphs have C/D partition D_1={11}.

3. Quotients from 3-Regular Hypergraphs with Four or Fewer Classes

In this section we examine decomposition quotients with four or fewer classes. Once constructed, each decomposition quotient is shown to be threshold or not. For those that are threshold a weighting function and threshold value are presented.

If a hypergraph contains singular points, i.e. vertices contained in no hyperedge, these will comprise C_1 and D_1 will be empty, and $11 \in D_0$. Thus from now on, we focus on hypergraphs with no singular points.

In the proofs of the main results several instrumental facts are used. These are immediate consequences from the definitions, and can be grouped into categories that enable the proofs of Theorems 1, 2, 3, 4, and 5 to be tree structured. This organization is helpful since these proofs require an enumeration of all possible partitions. Each proof constructs all possible partitions, and explicitly shows the placement of each ij into the appropriate D_k.

Each proof starts at the root of a decision tree based on the simple observation

Fact 1: $nn \in D_1$

The children of the root will be explored according to the simple facts:

Fact 2: $11 \in D_0$ or D_n
Fact 3: $12 \in D_0$, or D_n, or D_{n-1}
Fact 4: If $D_i = \varnothing$, then $i = 0$.

The three preceding facts yield four children that must be fully explored, namely: $11, 12 \in D_n$, $11 \in D_n$ and $12\ D_{n-1}$, $11 \in D_0$ and $12\ D_n$, and $11, 12 \in D_0$.

As the decision tree is further traversed, it is necessary to discover the children of current and newly recognized vertices. The following two groups of facts provide this information by giving ranges where other pairs of vertices must occur.

Fact 5: If $\{i,j,k\} \in E$, then $jk \in D_k$ where $k \leq i$.
Fact 6: If $\{i,j,k\} \notin E$ then $jk \in D_k$ where $k > i$.
Fact 7: If $i \leq m, j \leq n$ and $ij \in D_r$, then $mn \in D_s$ where $s \leq r$.
Fact 8: If $i \leq m, j \leq n$ and $mn \in D_r$, then $mn \in D_s$ where $s \quad r$.

Fact 9: $n\text{-}1\ n \in D_1$ or D_2
Fact 10: If $1i \in D_j$, then $ij \in D_1$.
Fact 11: If $1i \in D_1$, then, $i = n$.
Fact 12: If $\{i,j,k\} \in E$ and $\{i\text{-}1,j,k\} \notin E$ then $jk \in D_i$.

Theorem 1: There exists a unique 3-regular hypergraph decomposition quotient with one class. It is edge-threshold.

Proof: A decomposition quotient with one class contains a single edge pair, namely: 11. From Fact 4 we know that D_1 cannot be empty, thus $11 \in D_1$, and the decomposition quotient is:

$$\text{(P 1.1)} \qquad \begin{array}{ll} & D_0 = \varnothing \\ C_1 & D_1 = \{11\}. \end{array}$$

As is demonstrated by the weighting function of Table 3.1, the 1-class decomposition quotient is edge-threshold.

Table 3.1: A weighting function for the 1-class decomposition quotient.

Decomposition	P1.1
Threshold	2
Weight(C_1)	1

Theorem 2: There exist two 3-regular hypergraph decomposition quotients with two classes. Both are edge-threshold.

Proof: Assume G can be decomposed into two classes, then G contains three edge pairs, namely: 11, 12, and 22.

Fact 1 tells us that $22 \in D_1$ and Facts 2, 3, and 4 tell us that there exists four ways to position the vertex-pairs 11 and 12 in combination.

If 11, 12 $\in D_2$, then the potential decomposition quotient is:

$$\begin{array}{ll} & D_0 = \varnothing \\ C_2 & D_2 = \{11, 12\} \\ C_1 & D_1 = \{22\}. \end{array}$$

However, an inconsistency exists since $11 \in D_2$ implies $112 \in E$, while $12 \notin D_1$ implies $112 \notin E$.

If $11 \in D_2$, $12 \in D_1$, then the decomposition quotient is:

$$\text{(P 2.1)} \qquad \begin{array}{ll} & D_0 = \varnothing \\ C_2 & D_2 = \{11\} \\ C_1 & D_1 = \{12, 22\}. \end{array}$$

If $11 \in D_0$ and $12 \in D_2$, then the decomposition quotient is:

$$\text{(P 2.2)} \qquad \begin{array}{ll} & D_0 = \{11\}, \\ C_2 & D_2 = \{12\} \\ C_1 & D_1 = \{22\} \end{array}$$

If 11, 12 $\in D_0$, then the potential decomposition quotient is:

$$D_0 = \{11,12\}$$
$$C_2 \quad D_2 = \varnothing$$
$$C_1 \quad D_1 = \{22\}.$$

However, an inconsistency exists since Fact 4 tells us that D_2 cannot be empty.

Figure 3.1 illustrates the decision tree used for finding the 3-regular hypergraph decomposition quotients with two classes. Vertex-pairs represent options that must be explored.

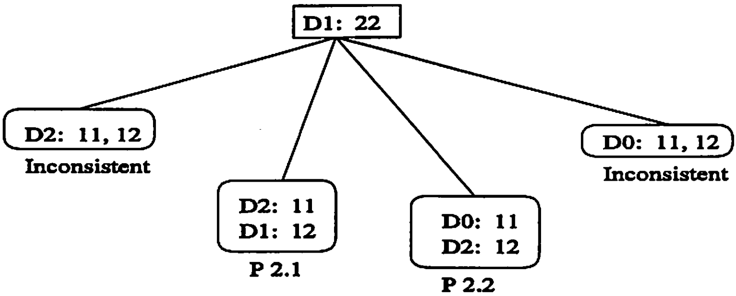

Figure 3.1: Decision tree for the decomposable quotients with two classes.

As is demonstrated by weighting functions of Table 3.2, the two 2-class decomposition quotients are edge-threshold.

Table 3.2: Weighting functions for the 2-class decomposition quotients.

Decomposition	P2.1	P2.2
Threshold	4	3
Weight(C_1)	1	1
Weight(C_2)	2	2

Theorem 3: There exist six distinct 3-regular hypergraph decomposition quotients with three classes. Each is edge-threshold.

Proof: Assume G can be decomposed into three classes, then G contains six edge pairs, namely: 11, 12, 13, 22, 23, and 33.

Fact 1 tells us that $33 \in D_1$ and Facts 2, 3, 4 tell us there exists four ways to position the vertex-pairs 11 and 12 in combination. We examine each of the four cases separately.

CASE 1: Assume 11, 12 $\in D_3$. Since $11 \in D_3$, Fact 10 tells us that $13 \in D_1$. Similarly, $12 \in D_3$ implies by Fact 10 that $23 \in D_1$. After these placements D_2 is empty and 22 is the only vertex-pair not positioned, this implies 22 must be placed in D_2 since Fact 4 tells us that the only D_i that can be empty is D_0. Thus the decomposition quotient is:

(P 3.1) $D_0 = \varnothing$,
C_3 $D_3 = \{11, 12\}$,
C_2 $D_2 = \{22\}$
C_1 $D_1 = \{13, 23, 33\}$.

CASE 2: Assume $11 \in D_3$ and $12 \in D_2$. Since $11 \in D_3$, Fact 10 tells us that $13 \in D_1$. Similarly, $12 \in D_2$ implies by Fact 10 that $22 \in D_1$. Also $2 << 3$ and $122 \in E$ implies $123 \in E$, hence, Fact 10 tells us that $23 \in D_1$. Thus the decomposition quotient is:

(P 3.2) $D_0 = \varnothing$,
C_3 $D_3 = \{11\}$,
C_2 $D_2 = \{12\}$
C_1 $D_1 = \{13, 22, 23, 33\}$.

CASE 3: Assume 11, 12 $\in D_0$. Since $33 \in D_1$, $133 \in E$, hence, Fact 5 tells us that $13 \in D_3$, D_2, or D_1. However, since 11, 12 $\notin D_3$, 113, 123 $\notin$ E, hence, Fact 6 tells us that $13 \notin D_1$ or D_2. Therefore, $13 \in D_3$. Since $13 \in D_3$ and $1 << 2$, Fact 7 tells us that $23 \in D_3$, D_2, or D_1. But $123 \notin E$ implies $23 \notin D_1$, Therefore 23 must be placed in D_3 or D_2. However if 23 were placed in D_3, D_2 would be left empty, since by Fact 8, $22 \notin D_2$ when$23 \in D_3$. This contradicts Fact 4, therefore $23 \in D_2$. Since $23 \in D_2$, $223 \in E$, hence, Fact 5 tells us that $22 \in D_3$, D_2, or D_1. However Fact 7 tells us the case $22 \in D_1$ cannot occur since $2 << 3$ and $23 \in D_2$. Thus we are left with two subcases, which lead to the decomposition quotients:

(P 3.3) $D_0 = \{11,12\}$	(P 3.4) $D_0 = \{11,12\}$
C_3 $D_3 = \{13, 22\}$	C_3 $D_3 = \{13\}$
C_2 $D_2 = \{23\}$	C_2 $D_2 = \{22, 23\}$
C_1 $D_1 = \{33\}$	C_1 $D_1 = \{33\}$

CASE 4: Assume $11 \in D_0$ and $12 \in D_3$. Since $12 \in D_3$, $123 \in E$, hence, Fact 10 tells us that $23 \in D_1$. Also, $123 \in E$ implies using Fact 5 that $13 \in D_2$ or D_1. But since $11 \notin D3$ implies $113 \notin E$, which implies $13 \notin D_1$, therefore $13 \in D_2$. Since $123 \in E$ and $1 << 2$, $223 \in E$, hence, Fact 5 tells us that $22 \in D_3$, D_2, or D_1. However $12 \notin D_2$, implies 122 $\notin$ E so Fact 6 tells us $22 \notin D_1$. Thus we are left with two subcases that lead to the decomposition quotients:

(P 3.5) $D_0 = \{11\}$	(P 3.6) $D_0 = \{11\}$
C_3 $D_3 = \{12, 22\}$	C_3 $D_3 = \{12\}$
C_2 $D_2 = \{13\}$	C_2 $D_2 = \{13, 22\}$
C_1 $D_1 = \{23, 33\}$	C_1 $D_1 = \{23, 33\}$

Figure 3.2 illustrates the decision tree used for finding the 3-regular hypergraph decomposition quotients with three classes. Recall that vertex-pairs represent options that must be explored.

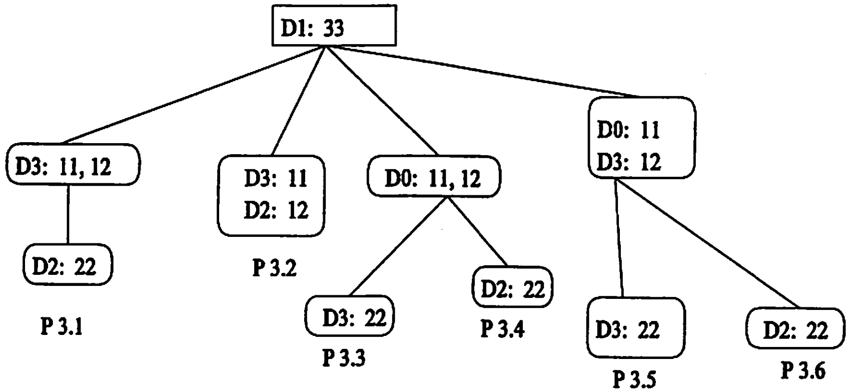

Figure 3.2: Decision tree for the decomposable quotients with three classes.

As is demonstrated by the weighting functions of Table 3.3, the six 3-class decomposition quotients are edge-threshold.

Table 3.3: Weighting functions for the 3-class decomposition quotients.

Decomposition	P3.1	P3.2	P3.3	P3.4	P3.5	P3.6
Threshold	5	4	6	8	6	5
Weight(C_1)	1	1	1	1	1	1
Weight(C_2)	2	2	2	3	2	3
Weight(C_3)	4	3	3	4	4	4

Theorem 4: There exist twenty-five 3-regular hypergraph decomposition quotients with four classes. Each is edge-threshold.

Proof: Assume G can be decomposed into four classes, then G contains ten edge pairs, namely: 11, 12, 13, 14, 22, 23, 24, 33, 34, and 44.

Fact 1 tells us that $44 \in D_1$ and Facts 2, 3, and 4 tell us there exists four ways to position the vertex-pairs 11 and 12 in combination.

CASE 1: Assume 11 and 12 $\in D_4$, then the following placements are implied: 14, 24 $\in D_1$ using Fact 10, $34 \in D_1$ using Fact 7, and $13 \in D_4$ or D_3 using Fact 5 and Fact 6 since $134 \in E$ but $123 \notin E$.

SUBCASE 1.1: Assume $13 \in D_4$, then the following placements are implied: $33 \in D_2$ using Fact 4 and Fact 6, and $23 \in D_3$ or D_2 using Fact 5 and Fact 8 since $233 \in E$ and $33 \in D_2$.

SUBCASE 1.1.1: Assume $23 \in D_3$. Then the following placement is implied: $22 \in D_4$ using Fact 7 and Fact 6. This yields the decomposition quotient:

(P 4.1)

$$
\begin{array}{ll}
 & D_0 = \varnothing \\
C_4 & D_4 = \{11, 12, 13, 22\} \\
C_3 & D_3 = \{23\} \\
C_2 & D_2 = \{33\} \\
C_1 & D_1 = \{14, 24, 34, 44\}.
\end{array}
$$

SUBCASE 1.1.2: Assume $23 \in D_2$, then the following placement is implied: $22 \in D_3$ using Fact 4. This yields the decomposition quotient:

(P 4.2)

$$
\begin{array}{ll}
 & D_0 = \varnothing \\
C_4 & D_4 = \{11, 12, 13\} \\
C_3 & D_3 = \{22\} \\
C_2 & D_2 = \{23, 33\} \\
C_1 & D_1 = \{14, 24, 34, 44\}.
\end{array}
$$

SUBCASE 1.2: Assume $13 \in D_3$, then the following placements are implied: $33 \in D_1$ using Fact 10, $23 \in D_2$ using Fact 4 and Fact 6, and $22 \in D_3$ or D_2 using Fact 5 and Fact 8 since $223 \in E$ and $23 \in D_2$. These two subcases lead to the decomposition quotients:

(P 4.3)

$$
\begin{array}{ll}
 & D_0 = \varnothing \\
C_4 & D_4 = \{11, 12\} \\
C_3 & D_3 = \{13, 22\} \\
C_2 & D_2 = \{23\} \\
C_1 & D_1 = \{14, 24, 33, 34, 44\}
\end{array}
$$

(P 4.4)

$$
\begin{array}{ll}
 & D_0 = \varnothing \\
C_4 & D_4 = \{11, 12\} \\
C_3 & D_3 = \{13\} \\
C_2 & D_2 = \{22, 23\} \\
C_1 & D_1 = \{14, 24, 33, 34, 44\}
\end{array}
$$

Figure 3.3 illustrates the decision tree used for finding the 3-regular hypergraph decomposition quotients with four classes and such that $11, 12 \in D_4$. Only vertex-pairs options that must be explored are shown.

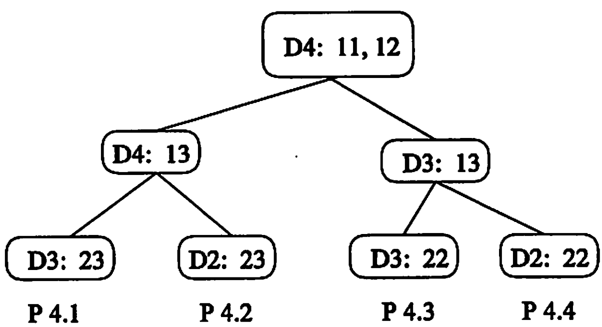

Figure 3.3: Decision tree for the decomposable quotients with three classes and $11, 12 \in D_4$.

CASE 2: Assume $11 \in D_4$ and $12 \in D_3$, then the following placements are implied: 14, 23 $\in D_1$ using Fact 10, 24, 34, 33 $\in D_1$ using Fact 7, $13 \in D_2$ using Fact 5 and Fact 11, and $22 \in D_3$ or D_2 using Fact 5 and Fact 6 since $223 \in E$ but $122 \notin E$. These two subcases lead to the decomposition quotients:

(P 4.5) $D_0 = \varnothing$
C_4 $D_4 = \{11\}$
C_3 $D_3 = \{12, 22\}$
C_2 $D_2 = \{13\}$
C_1 $D_1 = \{14, 23, 24, 33, 34, 44\}$

(P 4.6) $D_0 = \varnothing$
C_4 $D_4 = \{11\}$
C_3 $D_3 = \{12\}$
C_2 $D_2 = \{13, 22\}$
C_1 $D_1 = \{14, 23, 24, 33, 34, 44\}$

Figure 3.4 illustrates the decision tree used for finding the 3-regular hypergraph decomposition quotients with four classes and such that $11 \in D_4$ and $12 \in D_3$. Only vertex-pairs options that must be explored are shown.

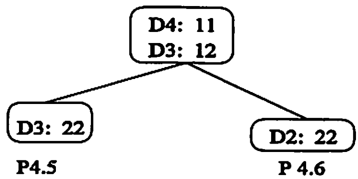

Figure 3.4: Decision tree for the decomposable quotients with three classes and $11 \in D_4$ and $12 \in D_3$.

CASE 3: Assume $11 \in D_0$ and $12 \in D_4$, then the following placements are implied: $24 \in D_1$ using Fact 10, $34 \in D_1$ using Fact 7, $14 \in D_2$ using Fact 5 and Fact 6, and $13 \in D_4$ or D_3 using Fact 5 and Fact 6 since $134 \in E$ but $123 \notin E$.

SUBCASE 3.1: Assume $13 \in D_3$, then the following placements are implied: $33 \in D_1$ using Fact 10, and $23 \in D_3$ or D_2 using Fact 5 and Fact 6 since $13 \in D3$ and $123 \notin E$.

SUBCASE 3.1.1: Assume $23 \in D_3$, then the following placement is implied: $22 \in D_4$ using Fact 5 and Fact 6. This yields the decomposition quotient is:

(P 4.7) $D_0 = \{11\}$
C_4 $D_4 = \{12, 22\}$
C_3 $D_3 = \{13, 23\}$
C_2 $D_2 = \{14\}$
C_1 $D_1 = \{24, 33, 34, 44\}$.

SUBCASE 3.1.2: Assume $23 \in D_2$, then the following placement is implied: $22 \in D_3$ or D_2 using Fact 5 and Fact 8 since $223 \in E$ and $23 \in D_2$. These two subcases lead to the decomposition quotients:

(P 4.8) $D_0 = \{11\}$
C_4 $D_4 = \{12\}$
C_3 $D_3 = \{13, 22\}$
C_2 $D_2 = \{14, 23\}$
C_1 $D_1 = \{24, 33, 34, 44\}$

(P 4.9) $D_0 = \{11\}$
C_4 $D_4 = \{12\}$
C_3 $D_3 = \{13\}$
C_2 $D_2 = \{14, 22, 23\}$
C_1 $D_1 = \{24, 33, 34, 44\}$

SUBCASE 3.2: Assume $13 \in D_4$, then the following placement is implied: $23 \in D_4$, D_3, or D_2 using Fact 5 and Fact 6 since $234 \in E$ but $123 \notin E$.

SUBCASE 3.2.1: Assume $23 \in D_4$. Then the following placements are implied: $22 \in D_4$ using Fact 5, and Fact 8 and $33 \in D_3$ using Fact 4. Thus the decomposition quotient is:

(P 4.10) $D_0 = \{11\}$
C_4 $D_4 = \{12, 13, 22, 23\}$
C_3 $D_3 = \{33\}$
C_2 $D_2 = \{14\}$
C_1 $D_1 = \{24, 34, 44\}$.

SUBCASE 3.2.2: Assume $23 \in D_3$, then the following placements are implied: $33 \in D_2$ using Fact 5 and Fact 6, and $22 \in D_4$ using Fact 7 and Fact 6. Thus the decomposition quotient is:

(P 4.11) $D_0 = \{11\}$
C_4 $D_4 = \{12, 13, 22\}$
C_3 $D_3 = \{23\}$
C_2 $D_2 = \{14, 33\}$
C_1 $D_1 = \{24, 34, 44\}$.

SUBCASE 3.2.3: Assume $23 \in D_2$, then the following placements are implied: $33 \in D_2$ using Fact 7 and Fact 6, and $22 \in D_3$ using Fact 4. Thus the decomposition quotient is:

(P 4.12) $D_0 = \{11\}$
C_4 $D_4 = \{12, 13\}$
C_3 $D_3 = \{22\}$
C_2 $D_2 = \{14, 23, 33\}$
C_1 $D_1 = \{24, 34, 44\}$.

Figure 3.5 illustrates the decision tree used for finding the 3-regular hypergraph decomposition quotients with four classes and such that $11 \in D_0$ and $12 \in D_4$. Only vertex-pairs options that must be explored are shown.

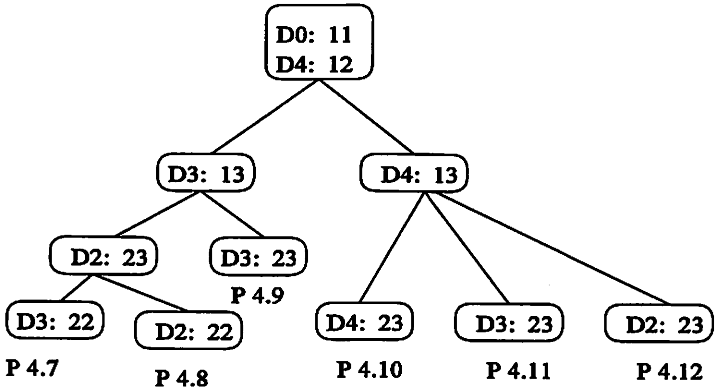

Figure 3.5: Decision tree for the decomposable quotients with three classes and $11 \in D_0$ and $12 \in D_4$.

CASE 4: Assume 11and 12 $\in D_0$, then the following placement is implied: $14 \in D_4$ or D_3 using Fact 5 and Fact 6 since $144 \in E$ but $124 \notin E$.

SUBCASE 4.1: Assume $14 \in D_4$, then the following placements are implied: $13 \in D_0$ using Fact 6, $34 \in D_2$ using Fact 4 and Fact 6, and $24 \in D_3$ or D_2 using Fact 5 and Fact 6 since $234 \in E$ but $124 \notin E$.

SUBCASE 4.1.1: Assume $24 \in D_3$, then the following placements are implied: $22 \in D_0$ using Fact 6, and $23 \in D_4$ or D_3 using Fact 5 and Fact 8 since $234 \in E$ and $24 \in D_3$.

SUBCASE 4.1.1.1: Assume $23 \in D_4$, then the following placement is implied: $33 \in D_4$ or D_3 using Fact 5 and Fact 6 since $334 \in E$ and $233 \notin E$. These two subcases lead to the decomposition quotients:

(P 4.13) $D_0 = \{11, 12, 13, 22\}$
C_4 $D_4 = \{14, 23, 33\}$
C_3 $D_3 = \{24\}$
C_2 $D_2 = \{34\}$
C_1 $D_1 = \{44\}$.

(P 4.14) $D_0 = \{11, 12, 13, 22\}$
C_4 $D_4 = \{14, 23\}$
C_3 $D_3 = \{24, 33\}$
C_2 $D_2 = \{34\}$
C_1 $D_1 = \{44\}$.

SUBCASE 4.1.1.2: Assume $23 \in D_3$, then the following placement is implied: $33 \in D_2$ using Fact 5 and Fact 6. Thus, the decomposition quotient is:

(P 4.15) $D_0 = \{11, 12, 13, 22\}$
C_4 $D_4 = \{14\}$
C_3 $D_3 = \{23, 24\}$
C_2 $D_2 = \{33, 34\}$
C_1 $D_1 = \{44\}$.

SUBCASE 4.1.2: Assume $24 \in D_2$, then the following placement is implied: $23 \in D_4$, D_3,, or D_5 using Fact 5 and Fact 8 since $234 \in E$ and $24 \in D_2$.

SUBCASE 4.1.2.1: Assume $23 \in D_4$, then the following placements are implied: $22 \in D_4$ using Fact 5 and Fact 8, and $33 \in D_4$ using Fact 4. Thus, the decomposition quotient is:

(P 4.16)
$$\begin{array}{ll} & D_0 = \{11, 12, 13\} \\ C_4 & D_4 = \{14, 22, 23\} \\ C_3 & D_3 = \{33\} \\ C_2 & D_2 = \{24, 34\} \\ C_1 & D_1 = \{44\}. \end{array}$$

SUBCASE 4.1.2.2: Assume $23 \in D_3$, then the following placements are implied: $22 \in D_4$ using Fact 5 and Fact 6, and $33 \in D_4$ using Fact 5 and Fact 8. Thus, the decomposition quotient is:

(P 4.17)
$$\begin{array}{ll} & D_0 = \{11, 12, 13\} \\ C_4 & D_4 = \{14, 22\} \\ C_3 & D_3 = \{23\} \\ C_2 & D_2 = \{24, 33, 34\} \\ C_1 & D_1 = \{44\}. \end{array}$$

SUBCASE 4.1.2.3: Assume $23 \in D_2$, then the following placements are implied: $33 \in D_2$ using Fact 7 and Fact 6, and $22 \in D_3$ using Fact 4. Thus, the decomposition quotient is:

(P 4.18)
$$\begin{array}{ll} & D_0 = \{11, 12, 13\} \\ C_4 & D_4 = \{14\} \\ C_3 & D_3 = \{22\} \\ C_2 & D_2 = \{23, 24, 33, 34\} \\ C_1 & D_1 = \{44\}. \end{array}$$

SUBCASE 4.2: Assume $14 \in D_3$, then the following placements are implied: $34 \in D_1$ using Fact 10, and $24 \in D_3$ or D_2 using Fact 7 and Fact 6 since $14 \in D_3$ and $124 \notin E$.

SUBCASE 4.2.1: Assume $24 \in D_3$, then the following placements are implied: $22 \in D_0$ using Fact 6, $33 \in D_2$ using Fact 4, $23 \in D_3$ using Fact 5 and Fact 8, and $13 \in D_4$ using Fact 4. Thus, the decomposition quotient is:

(P 4.19)
$$\begin{array}{ll} & D_0 = \{11, 12, 22\} \\ C_4 & D_4 = \{13\} \\ C_3 & D_3 = \{14, 23, 24\} \\ C_2 & D_2 = \{33\} \\ C_1 & D_1 = \{34, 44\}. \end{array}$$

SUBCASE 4.2.2: Assume $24 \in D_2$, then the following placement is implied: $23 \in D_4$, D_3, or D_2 using Fact 5 and Fact 8 since $234 \in E$ and $24 \in D_2$.

SUBCASE 4.2.2.1: Assume $23 \in D_4$, then the following placements are implied: 13, $22 \in D_4$ using Fact 5 and Fact 8, and $33 \in D_4$ or D_3 using Fact 5 and Fact 6 since $334 \in E$ and $233 \notin E$. These two subcases lead to the decomposition quotients:

(P 4.20) $D_0 = \{11, 12\}$
C_4 $D_4 = \{13, 22, 23, 33\}$
C_3 $D_3 = \{14\}$
C_2 $D_2 = \{24\}$
C_1 $D_1 = \{34, 44\}$

(P 4.21) $D_0 = \{11, 12\}$
C_4 $D_4 = \{13, 22, 23\}$
C_3 $D_3 = \{14, 33\}$
C_2 $D_2 = \{24\}$
C_1 $D_1 = \{34, 44\}$

SUBCASE 4.2.2.2: Assume $23 \in D_3$, then the following placements are implied: $22 \in D_4$ using Fact 5 and Fact 6, and $13 \in D_4$ or D_3 using Fact 5 and Fact 8 since $134 \in E$ and $23 \in D_3$.

SUBCASE 4.2.2.2.1: Assume $13 \in D_4$. Then the following placement is implied: $33 \in D_2$ using Fact 5 and Fact 6. Thus, the decomposition quotient is:

(P 4.22) $D_0 = \{11, 12\}$
C_4 $D_4 = \{13, 22\}$
C_3 $D_3 = \{14, 23\}$
C_2 $D_2 = \{24, 33\}$
C_1 $D_1 = \{34, 44\}$.

SUBCASE 4.2.2.2.2: Assume $13 \in D_3$, Then the following placement is implied: $33 \in D_1$ using Fact 10. Thus, the decomposition quotient is:

(P 4.23) $D_0 = \{11, 12\}$
C_4 $D_4 = \{22\}$
C_3 $D_3 = \{13, 14, 23\}$
C_2 $D_2 = \{24\}$
C_1 $D_1 = \{33, 34, 44\}$.

SUBCASE 4.2.2.3: Assume $23 \in D_2$, then the following placements are implied: $33 \in D_2$ using Fact 7 and Fact 6, and $13 \in D_4$ using Fact 4 and $22 \in D_3$ or D_2 using Fact 5 and Fact 8 since $223 \in E$ and $23 \in D_2$. These two subcases lead to the decomposition quotients:

(P 4.24) $D_0 = \{11, 12\}$
C_4 $D_4 = \{13\}$
C_3 $D_3 = \{14, 22\}$
C_2 $D_2 = \{23, 24, 33\}$
C_1 $D_1 = \{34, 44\}$

(P 4.25) $D_0 = \{11, 12\}$
C_4 $D_4 = \{13\}$
C_3 $D_3 = \{14\}$
C_2 $D_2 = \{22, 23, 33\}$
C_1 $D_1 = \{34, 44\}$

Figure 3.6 illustrates the decision tree used for finding the 3-regular hypergraph decomposition quotients with four classes and such that $11, 12 \in D_0$. Only vertex-pairs options that must be explored are shown.

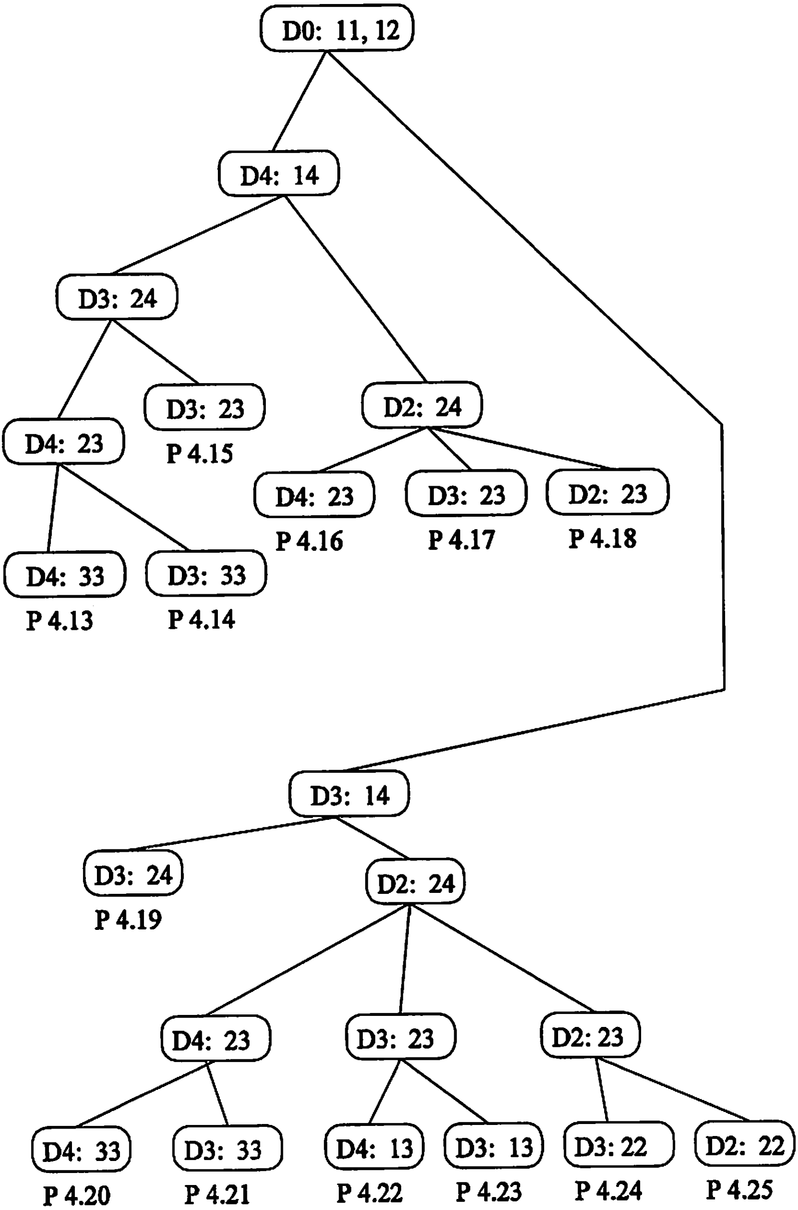

Figure 3.6: Decision tree for the decomposable quotients with three classes and 11, 12$\in D_0$.

As is demonstrated by the weighting functions of Table 3.4, the six 4-class decomposition quotients with $11 \in D_0$ and $12 \in D_4$ are edge-threshold.

Table 3.4: Weighting functions for the 4-class decomposition quotients.

Decomposition	**P4.1**	**P4.2**	**P4.3**	**P4.4**	**P4.5**
Threshold	9	9	6	8	6
Weight(C_1)	1	1	1	1	1
Weight(C_2)	2	3	2	3	2
Weight(C_3)	4	4	3	4	4
Weight (C_4)	8	8	5	7	5

Decomposition	**P4.6**	**P4.7**	**P4.8**	**P4.9**	**P4.10**
Threshold	6	8	6	8	8
Weight(C_1)	1	1	1	1	1
Weight(C_2)	2	2	2	3	2
Weight(C_3)	3	4	3	4	3
Weight (C_4)	4	6	4	5	6

Decomposition	**P4.11**	**P4.12**	**P4.13**	**P4.14**	**P4.15**
Threshold	7	9	9	11	12
Weight(C_1)	1	1	1	2	1
Weight(C_2)	2	3	2	3	3
Weight(C_3)	3	4	3	4	5
Weight (C_4)	5	6	5	5	6

Decomposition	**P4.16**	**P4.17**	**P4.18**	**P4.19**	**P4.20**
Threshold	11	10	12	11	9
Weight(C_1)	1	1	1	1	1
Weight(C_2)	3	3	4	2	2
Weight(C_3)	4	4	5	5	3
Weight (C_4)	6	5	6	6	6

Decomposition	**P4.21**	**P4.22**	**P4.23**	**P4.24**	**P4.25**
Threshold	8	7	8	9	11
Weight(C_1)	1	1	1	1	1
Weight(C_2)	2	2	2	3	4
Weight(C_3)	3	3	4	4	5
Weight (C_4)	5	4	5	5	6

4. Quotients from 3-Regular Hypergraphs with Five Classes.

In this section we examine decomposition quotients with five classes.

Theorem 5: There exist 142 distinct 3-regular hypergraph decomposition quotients with five classes.

Because of the length of the proof of Theorem 5 and its similarity to the proofs of Theorems 1-4 we do not include it in this paper. However, we do present a listing of the quotients themselves in Table 4.1. Each column in the table represents a quotient. A cell entry of k in row ij, column P5.x in the table indicates that in quotient P5.x the vertex-pair ij belongs to D_k. Also included in all but two columns are weighting functions that demonstrate that the decomposition quotient in that column is edge-threshold. As shown below the two quotients without weighting functions are not edge-threshold.

Theorem 6: There exist two distinct 3-regular hypergraph decomposition quotients with five classes that are not edge-threshold.

Proof: Assume that the 3-regular hypergraph decomposition quotient P5.57 (Table 4.1) is edge-threshold. Then there exists $t \in Z^+$ and a weighting function w from the set of vertices to the non-negative integers such that $\{x,y,z\} \in E$ iff $w(x)+w(y)+w(z) > t$. Thus since 135, 234 $\in E$ and 133, 144, 225 $\notin E$ we know:

$$\begin{array}{llcl}
w(1)+w(3)+w(5) > t & w(1)+w(3)+w(3) & & w(1)+w(5) > w(1)+w(3) \\
w(2)+w(3)+w(4) > t & w(1)+w(4)+w(4) & \Rightarrow & w(2)+w(3) > w(1)+w(4) \\
w(2)+w(3)+w(4) > t & w(2)+w(2)+w(5) & & w(3)+w(4) > w(2)+w(2)
\end{array}$$

Hence $w(5)-w(3) > w(3)-w(1) > w(4)-w(2) > w(5)-w(3)$ which is a contradiction. Therefore P5.57 cannot be threshold.

Similarly, assume that the 3-regular hypergraph decomposition quotient P5.131 (Table 4.1) is edge-threshold. Then there exists $t \in Z^+$ and a weighting function w from the set of vertices to the non-negative integers such that $\{x,y,z\} \in E$ iff $w(x)+w(y)+w(z)$ t. Thus since 333, 144, 225 $\in E$ and 234, 135 $\notin E$ we know:

$$\begin{array}{llcl}
w(3)+w(3)+w(3) > t & w(2)+w(3)+w(4) & & w(3)+w(3) > w(2)+w(4) \\
w(1)+w(4)+w(4) > t & w(2)+w(3)+w(4) & \Rightarrow & w(1)+w(4) > w(2)+w(3) \\
w(2)+w(2)+w(5) > t & w(1)+w(3)+w(5) & & w(2)+w(2) > w(1)+w(3)
\end{array}$$

Hence $w(3)-w(2) > w(4)-w(3) > w(2)-w(1) > w(3)-w(2)$ which is a contradiction. Therefore P5.131 cannot be threshold.

(Note in the P5.131 case, the assumption that 333∈E implies that [3] contains at least three points. Recall that this same assumption was made in constructing the quotients.)

(P5.57) $D_0 = \{11, 12, 22\}$
C_5 $D_5 = \{13, 14\}$
C_4 $D_4 = \{23, 33\}$
C_3 $D_3 = \{15, 24, 25\}$
C_2 $D_2 = \{34, 44\}$
C_1 $D_1 = \{35, 45, 55\}$

(P 5.131) $D_0 = \{11, 12, 13\}$
C_5 $D_5 = \{22, 23\}$
C_4 $D_4 = \{14, 15, 24\}$
C_3 $D_3 = \{33, 34\}$
C_2 $D_2 = \{25, 35\}$
C_1 $D_1 = \{44, 45, 55\}$

Table 4.1: Weighting functions for the 5-class decomposition quotients.

Decomposition Quotients P5.1-P5.16

11	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
12	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5
13	4	4	4	3	3	3	5	5	5	5	5	5	5	5	5	5
14	2	2	2	2	2	2	5	5	5	5	5	5	4	4	4	4
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
22	3	4	4	4	2	3	5	5	5	5	4	4	3	4	5	5
23	2	3	4	3	2	2	5	5	4	4	3	4	2	3	3	4
24	1	1	1	1	1	1	4	4	3	3	2	2	2	2	3	3
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
33	2	2	3	1	1	1	4	3	4	3	2	3	2	2	2	3
34	1	1	1	1	1	1	3	3	2	2	2	2	2	2	2	2
35	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
44	1	1	1	1	1	1	2	2	2	2	2	2	1	1	1	1
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	9	7	8	9	8	6	9	11	15	11	13	17	12	10	12	8
w1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
w2	3	2	2	2	3	2	2	2	4	3	4	5	4	3	3	2
w3	4	3	3	5	4	3	3	4	5	4	5	6	5	4	5	3
w4	6	5	6	7	6	4	4	5	7	5	6	8	6	5	6	4
w5	8	6	7	8	7	5	8	10	14	10	12	16	11	9	11	7

Decomposition Quotients P5.17-P5.32

11	5	5	5	5	5	5	5	5	5	0	0	0	0	0	0	0
12	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
13	5	5	4	4	4	4	3	4	4	3	4	4	4	4	4	4
14	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3
15	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
22	5	4	4	4	5	4	4	3	2	4	5	5	5	4	4	4
23	4	4	4	4	3	3	3	2	2	3	3	4	4	4	4	3
24	3	2	2	2	3	2	2	2	2	2	3	3	3	2	2	2
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
33	4	3	3	4	2	2	1	2	2	1	2	3	4	4	3	2
34	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
35	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
44	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	9	11	8	9	9	7	8	9	11	8	9	10	12	9	8	7
w1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
w2	2	3	2	2	2	2	2	3	4	2	2	2	2	2	2	2
w3	3	4	3	3	4	3	4	4	5	4	4	4	4	3	3	3
w4	5	6	5	6	5	4	5	5	6	5	5	6	8	6	5	4
w5	8	10	7	8	8	6	7	8	10	6	7	8	10	7	6	5

Decomposition Quotients P5.33-P5.48

11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
13	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5
14	3	3	5	5	5	4	5	4	4	4	4	5	5	5	5	4
15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
22	3	2	5	5	5	5	5	5	5	5	5	5	5	4	4	3
23	2	2	5	5	5	5	5	5	4	4	3	4	4	3	4	2
24	2	2	5	5	4	4	4	4	3	3	3	3	3	2	2	2
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
33	2	2	4	5	4	4	3	3	3	4	2	3	4	2	3	2
34	1	1	3	4	3	3	3	3	2	2	2	2	2	2	2	2
35	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
44	1	1	3	3	2	1	2	1	1	1	1	2	2	2	2	1
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
T	9	11	12	10	9	10	11	14	8	9	12	11	15	13	17	12
W1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
W2	3	4	2	2	2	2	2	2	2	2	3	3	4	4	5	4
W3	4	5	4	3	3	3	4	5	3	3	5	4	5	5	6	5
W4	5	6	5	4	4	5	5	7	4	5	6	5	7	6	8	6
W5	6	7	10	8	7	8	9	12	6	7	9	8	11	9	12	8

Decomposition Quotients P5.49-P5.64

11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	5	5	4	4	4	5	5	5	5	5	5	5	5	5	5	5
14	4	4	3	3	3	5	5	5	5	5	5	5	5	5	5	5
15	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3
22	4	4	5	5	5	0	0	0	0	0	5	5	5	5	5	5
23	3	4	4	4	3	5	5	5	4	4	5	5	5	5	5	5
24	2	2	3	3	3	4	4	4	3	3	5	4	5	4	4	5
25	1	1	2	2	2	3	3	3	3	3	2	2	2	2	2	2
33	2	3	4	3	2	5	4	3	4	3	4	4	5	5	3	5
34	2	2	1	1	1	4	3	3	2	2	3	3	4	4	3	5
35	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
44	1	1	1	1	1	2	2	2	2	2	3	2	3	2	2	4
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
T	10	11	12	10	9	15	13	11		15	12	9	10	11	11	11
w1	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1
w2	3	3	2	2	2	2	2	2		3	2	2	2	2	2	2
w3	4	4	4	4	4	4	4	4		6	4	3	3	3	4	3
w4	5	6	8	6	5	7	6	5		7	5	4	4	5	5	4
w5	7	8	9	7	6	11	9	7		9	9	6	7	8	8	8

Decomposition Quotients P5.65-P5.80

11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
14	5	5	5	5	5	4	4	4	4	4	4	4	4	4	4	4
15	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
22	4	4	5	5	4	5	5	5	0	0	5	5	0	5	4	4
23	4	4	4	4	3	5	5	5	4	4	4	4	3	3	4	4
24	2	2	3	3	2	4	4	4	3	3	3	3	3	3	2	2
25	2	2	2	2	2	2	2	2	3	3	2	2	3	2	2	2
33	4	3	4	3	2	5	4	3	4	3	4	3	2	2	4	3
34	2	2	2	2	2	4	3	3	2	2	2	2	2	2	2	2
35	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
44	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
T	21	17	15	11	13	12	10	14	12	12	9	8	14	12	12	11
w1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
w2	6	5	4	3	4	2	2	2	2	2	2	2	3	3	3	3
w3	7	6	5	4	5	3	3	5	4	5	3	3	6	5	4	4
w4	10	8	7	5	6	6	5	7	7	6	5	4	7	6	7	6
w5	14	11	10	7	8	9	7	11	8	7	6	5	8	7	8	7

Decomposition Quotients P5.81-P5.96

11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	5	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0
14	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5
15	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	4
22	4	3	2	0	0	0	0	0	0	5	5	0	5	0	5	0
23	3	2	2	0	0	0	5	5	5	5	5	4	4	4	4	3
24	2	2	2	4	4	4	4	4	4	4	4	3	3	3	3	3
25	2	2	2	4	4	4	3	3	3	2	2	3	2	3	2	3
33	2	2	2	5	4	3	5	4	3	3	4	4	4	3	3	2
34	2	2	2	4	3	3	4	3	3	3	3	2	2	2	2	2
35	1	1	1	3	3	3	2	2	2	2	2	2	2	2	2	2
44	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	10	12	14	11	13	17	11	9	11	17	12	21	15	13	11	17
w1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
w2	3	4	5	2	2	2	2	2	2	4	3	5	4	3	3	4
w3	4	5	6	3	4	6	3	3	4	6	4	7	5	5	4	7
w4	5	6	7	5	6	8	5	4	5	7	5	10	7	6	5	8
w5	6	7	8	6	7	9	7	5	6	10	7	11	8	7	6	9

Decomposition Quotients P5.97-P5.112

11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	5	5	5	5	5	5	5	5	5	5	5	5	5	4	4	4
15	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
22	5	3	4	4	0	0	0	0	0	5	5	5	5	0	0	0
23	3	2	3	4	5	5	5	5	5	5	5	5	4	5	5	5
24	3	2	2	2	5	5	5	5	5	5	5	5	3	4	4	4
25	2	2	2	2	3	3	3	3	3	2	2	2	2	3	3	3
33	2	2	2	3	5	5	5	4	3	5	4	3	3	5	4	3
34	2	2	2	2	5	5	4	3	3	4	3	3	2	4	3	3
35	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
44	2	2	2	2	5	4	3	3	3	3	3	3	2	1	1	1
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	15	15	13	17	12	11	10	12	14	13	18	23	11	12	10	14
w1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
w2	4	5	4	5	2	2	2	2	2	3	4	5	3	2	2	2
w3	6	6	5	6	3	3	3	4	5	4	6	8	4	3	3	5
w4	7	7	6	8	4	4	4	5	6	5	7	9	5	6	5	7
w5	8	8	7	9	8	7	6	7	8	8	11	14	6	8	6	8

Decomposition Quotients P5.113-P5.128

11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	4	4	4	4	4	0	0	0	0	0	0	0	0	0	0	0
15	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	5
22	5	5	5	5	5	0	0	5	5	4	4	0	0	0	0	0
23	4	4	3	5	5	4	4	4	4	3	4	0	0	0	0	5
24	3	3	3	4	4	3	3	3	3	2	2	5	5	4	4	4
25	2	2	2	2	2	3	3	2	2	2	2	4	4	4	4	3
33	4	3	2	4	3	4	3	4	3	2	3	4	3	4	3	4
34	2	2	2	3	3	2	2	2	2	2	2	3	3	3	3	3
35	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	2
44	1	1	1	1	1	2	2	2	2	2	2	3	3	2	2	2
45	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	12	16	20	14		24	16	18	14	16	20	12	14	16	20	12
w1	1	1	1	1		1	1	1	1	1	1	1	1	1	1	1
w2	3	4	5	3		6	4	5	4	5	6	2	2	3	3	3
w3	4	6	8	4		8	6	6	5	6	7	4	5	5	7	4
w4	6	8	10	7		11	7	8	6	7	9	5	6	7	9	5
w5	7	9	11	9		12	8	9	7	8	10	6	7	8	10	6

Decomposition Quotients P5.129-P5.142

11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	5	5	5	5	5	5	5	5	5	5	5	5	5	5
22	0	0	5	5	5	0	0	0	0	0	5	0	0	0
23	5	5	5	5	5	5	5	0	0	0	5	5	0	0
24	4	5	4	4	5	5	4	5	4	5	5	5	5	5
25	3	3	2	2	2	3	3	4	4	4	2	3	4	4
33	3	4	4	3	4	5	5	5	5	0	5	5	5	5
34	3	3	3	3	3	4	4	4	4	4	4	5	5	5
35	2	2	2	2	2	2	2	3	3	4	2	2	3	3
44	2	3	2	2	3	3	2	3	2	3	3	2	5	4
45	2	2	2	2	2	2	2	2	2	2	2	2	2	2
55	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	14	15	15	20	21	13	17	10	14	12	16	14	12	11
w1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
w2	3	3	4	5	5	3	4	2	3	2	4	4	2	2
w3	5	5	5	7	7	4	5	3	4	3	5	5	3	3
w4	6	6	6	8	8	5	7	4	6	5	6	7	4	4
w5	7	8	8	11	12	7	9	5	7	6	9	9	7	6

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