Mutually Eccentric Vertices in Graphs

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Abstract

The distance $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining u and v. The eccentricity $e(v)$ of vertex v is the distance to a vertex farthest from v. In a graph G, an eccentric vertex of v is a vertex farthest from v, that is, a vertex u for which $d(u,v) = e(v)$. Given a set X of vertices in G, the vertices of X are mutually eccentric provided that for any pair of vertices u and v in X, u is an eccentric vertex of v and v is an eccentric vertex of u. In this paper, we discuss problems concerning sets of mutually eccentric vertices in graphs.

1. Introduction

For a connected graph G, the ***distance*** $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The ***eccentricity*** $e(v)$ of vertex v is the distance to a vertex farthest from v. The ***radius*** $r(G)$ of G is the minimum eccentricity and the ***diameter*** $d(G)$ is the maximum eccentricity.

A finite nonempty set X of positive integers is an ***eccentric set*** if there exists a graph G all of whose eccentricities are elements of X and X has no additional elements. Behzad and Simpson [1] characterized eccentric sets. For a graph G, the sequence of the vertex eccentricities in ascending order is the ***eccentric sequence*** of G. The eccentric sequence of a graph was studied by Lesniak [12], who showed, among other thing, that a sequence S of positive integers is the eccentric sequence of some graph if and only if some subsequence S' of S is eccentric.

If u is a farthest vertex from v, then u is an ***eccentric vertex of v***, and we say that u is an ***eccentric vertex*** if it is an eccentric vertex of at least one vertex of G. In a graph G with vertex set $V(G)$, vertices in a set $X \subseteq V(G)$ are ***mutually eccentric*** if for all pairs $u, v \in X$, u is eccentric for v and v is eccentric for u. In this paper we study mutually eccentric sets of vertices with particular emphasis on relations to centrality concepts of graphs.

2. Eccentric Vertices and Centrality

There are over two dozen (see [3] or [4]) centrality concepts for graphs. Many are related to facility location problems (see [2] or [3]). Which centrality concept one uses depends on the application. The ***center*** $C(G)$ of graph G is the set of vertices of minimum eccentricity, and such vertices are called central vertices. The center is related to the location of an emergency facility such as a hospital, police station, or fire station, where response time to a farthest possible point should be minimized. The ***periphery*** $P(G)$ is the set of vertices of maximum eccentricity, and those vertices are called peripheral. Thus $C(G) = \{v : e(v) = r(G)\}$ and $P(G) = \{v : e(v) = d(G)\}$. A graph is ***self-centered*** if every vertex is in the center, that is, $C(G) = V(G)$.

Another way of describing an eccentric vertex is to say that u is an eccentric vertex of v if $d(u,v) = e(v)$. If each vertex of G has exactly one eccentric vertex, then G is a ***unique eccentric point graph (u.e.p. graph)***. A simple class of u.e.p. graphs are the paths P_{2n} on an even number of vertices.

Theorem 1 (Parthasarathy and Nandakumar [14]). A u.e.p. graph G is self-centered if and only if each vertex in G is an eccentric vertex. ∎

Recall that a graph G is ***diameter-maximal*** if for all $e \in E(\overline{G})$, $d(G+e) < d(G)$, that is, adding any edge to G will decrease it diameter. These graphs were characterized by Ore [13]. Parthasarathy and Nandakumar also looked at u.e.p. graphs with restricted diameter.

Theorem 2 (Parthasarathy and Nandakumar [14]). If G is a u.e.p. graph with diameter 3, then either G is diameter-maximal or G is self-centered. ∎

Some interesting and sometimes surprising results develop when studying eccentric vertices. Consider, for example, graph L, in Figure 1. In this graph, u and v are mutually eccentric, but neither has a peripheral vertex as an eccentric vertex. Buckley and Lewinter [5] studied graphs for which none of the peripheral vertices is an eccentric vertex of any central vertex (L is not such a graph) and showed, among other things, that for such a graph $d(G) \geq 6$.

A ***uniform central graph (ucg graph)*** is a graph for which every central vertex has the same set of eccentric vertices. Choi and Manickam [7] defined this class of graphs, provided a characterization of *ucg* graphs, and gave constructions for such graphs. A simple example of a *ucg* graph is graph H in Figure 1. Gu [9] showed for any graph G it is always possible to embed G as an induced subgraph in a *ucg* graph H so that $C(H) = V(G)$. In fact, she showed that the minimum number of vertices that need to be added to G in such an embedding is 2, 4, or 6 (which value depends on the structure of G) and characterized graphs in

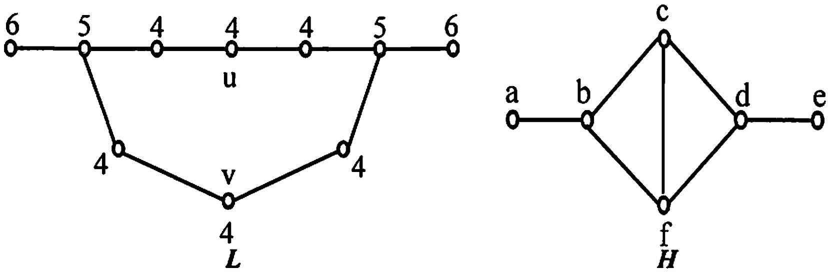

Figure 1

each category. We note that in addition to graph H of Figure 1 being a *ucg* graph, it also has another interesting property, namely, that the eccentric set of the central vertices is the periphery. Such graphs, called S-graphs, were studied by Buckley and Lewinter [5].

3. Mutually Eccentric Vertices

Buckley and Lewinter [5,6] defined a graph G to be an F-graph if $|C(G)| \geq 2$ and for all $u, v \in C(G)$, $d(u,v) = r(G)$. The motivation behind F-graphs is the desire to have central vertices separated as far as possible from one another so that separate emergency facilities could be located at different central vertices. By doing so, it would be less likely to have an ambulance in a traffic accident with a police car when each is rushing to the site of a separate emergency. Another way of describing an F-graph is as a graph G with $|C(G)| \geq 2$ where the central vertices are mutually eccentric.

Theorem 3 (Buckley and Lewinter [6]). If $r(G) = 1$ and $|C(G)| \geq 2$, then G is an F-graph. For all $a, b \in N$, with $a \geq 2$, there exists an F-graph G with $r(G) = a$ and $d(G) = b$ if and only if $a + 1 \leq b \leq 2a$. ∎

We have found that if one prescribes the radius and the number of central vertices, we can construct an F-graph with those specifications.

Theorem 4. For all positive integers r and k with $k \geq 2$, there exists an F-graph with radius r having k central vertices.

Proof. For $r = 1$ use the complete graph K_k. For r even, join two vertices u and v by k paths of length r. Then attach a pendent path of length $r/2$ at u and also one at v (See Figure 2a). For r odd ($r > 1$), begin with two disjoint copies of K_{k+1}. Label their vertices v_i and w_i, $1 \leq i \leq k+1$. For $1 \leq i \leq k$, join v_i and

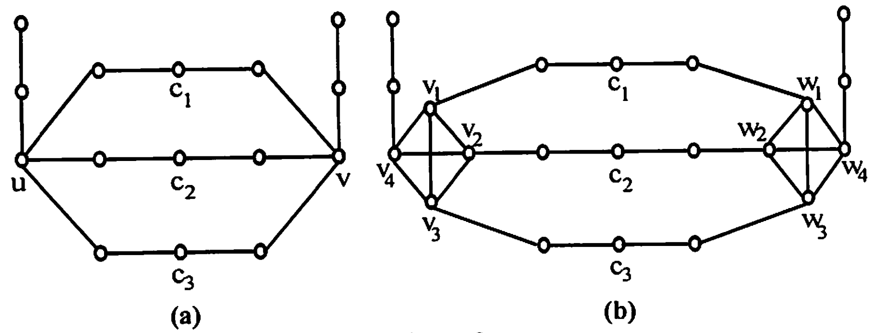

Figure 2

w_i by a path of length $r-1$. Then attach a pendent path of length $(r-1)/2$ at v_i and also one at w_i. (See Figure 2b). ∎

We can determine whether a given graph G of order n is an F-graph in the following way.

Algorithm 5. *F*-graph.

1. Find the distance matrix $D(G)$ by doing a breadth-first search from each vertex.
2. Identify central vertices by determining the rows of $D(G)$ whose maximum value is minimum over all of $D(G)$. [The minimum of the maximum values is $r(G)$ and is also the eccentricity of the vertices corresponding to those rows.]
3. Verify that in the principal submatrix $D_C(G)$ of $D(G)$ determined by the central vertices, all off-diagonal entries of $D_C(G)$ equal $r(G)$. ∎

The complexity of Algorithm 5 is $O(n^3)$. This is determined by step 1. Steps 2 and 3 each require no more than n^2 checks.

Algorithm 5 can easily be modified to solve the following problem, also in $O(n^3)$ time: Given a set X of vertices in graph G, are the vertices in X mutually eccentric. To do so, we perform step 1, then verify that the rows of $D(G)$ determined by X have the same maximum value. If not, then the vertices of X are not mutually eccentric. If so, then let m be the common maximum and proceed to verify that all off-diagonal entries in the principal submatrix $D_X(G)$ equal m. Consider the following decision problems.

MUTUALLY ECCENTRIC SET (MES)
INSTANCE: A connected graph G and a positive integer k.
QUESTION: Does G contain a set of mutually eccentric vertices of size at least k.

CLIQUE NUMBER (CLIQUE)
INSTANCE: A connected graph G of order n and a positive integer $k \leq n$.
QUESTION: Is the size of the largest complete subgraph (the largest clique) at least k.

CLIQUE is a well-known NP-complete problem. We shall use that fact in proving the following result.

Theorem 6. MES is NP-complete.
Proof. We must do two things. First, we must show that MES $\in NP$. Then we must construct a reduction from a known NP-complete problem to MES. The first part is straightforward, since it is easy to verify in polynomial time a 'yes' instance to MES. That is, given a graph G and a set $X \subseteq V(G)$ where $|X| \geq k$, it is easy to verify in polynomial time (as indicated immediately following Algorithm 5) whether X is a mutually eccentric set of vertices.

We show that MES is NP-hard by showing that CLIQUE $\propto$ MES. The reduction algorithm has as input an instance of the clique problem, namely a connected graph G of order n and an integer $k \leq n$. We construct, in polynomial time, a connected graph H of order $n+1$ that has a set of mutually eccentric vertices of size k if and only if G has a clique of size k. Let $H = \{x\} + \overline{G}$, that is, the join of an isolated vertex with the complement of G. This construction is done in $O(n^2)$ time. It remains for us to show that this is a reduction.

We must show that G has a clique of size k if and only if H has a set of k mutually eccentric vertices. Suppose that C is a clique of size k in G. Then $V(C)$ is a set of k mutually eccentric vertices in H. To see this, note that for all $u, v \in V(C)$, $uv \notin E_H(C)$. Also, for all $w \in V(H)$, $w \neq x$, $wx \in E(H)$. Thus, for all $u, v \in V(C)$, $e_H(u) = e_H(v) = d_H(u,v)$. So all vertices of C are mutually eccentric in H. For the converse, suppose that Y is a set of k mutually eccentric vertices in H. Since G is connected, x is the unique central vertex of $H = \{x\} + \overline{G}$. Furthermore, $e_H(x) = 1$, and $e_H(v) = 2$ for all $v \in H$. Thus, the set Y of k mutually eccentric vertices in H must be mutually at distance 2 from one another. Thus, the vertices of Y are mutually nonadjacent in H, so they are mutually adjacent in G. Hence, G has a clique of size k if and only if H has a set of k mutually eccentric vertices. ∎

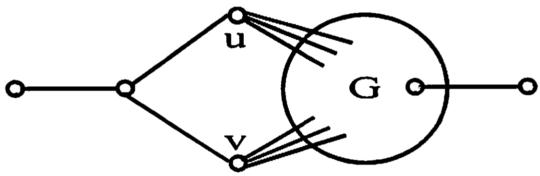

Figure 3

It was shown in [6] that it is possible to embed any graph G as an induced subgraph in an F-graph H, and at most 5 additional vertices are required in such an embedding. The construction is shown in Figure 3 where the center consists of the mutually eccentric vertices u and v.

Kyš [11] developed several constructions that enable one to obtain new F-graphs from old ones. He also obtained the following characterization of F-graphs.

Theorem 7 (Kyš [11]). Let G be a graph with $r(G) \geq 2$ and $C(G) = \{x_1, x_2, ..., x_k\}$, $k \geq 2$. Then G is an F-graph if and only if the neighborhoods $N_G(x_i)$ satisfy the following in $H = G - C(G)$:

(1) $d_H(N_G(x_i), N_G(x_j)) = r(G) - 2$, $i \neq j$,

(2) For all $u \in V(H)$, $d_H(u, N_G(x_i)) \leq r(G) - 1$, and

(3) For all $u \in V(H)$, there exists $v \in V(H)$ such that $d_G(u,v) > r(G)$ and $d_H(u, N_G(x_i)) + d_H(v, N_G(x_j)) \geq r(G) - 1$ for all i. ∎

4. Eccentric Partition Graphs

A graph G is an ***eccentric partition graph (e.p.g.)*** if $V(G)$ can be partitioned into $V_1, V_2, ..., V_t$ such that $|V_i| \geq 2$ for each i, and vertices within each V_i are mutually eccentric. For example, the graph M in Figure 4 is an *e.p.g.* with partition $V_1 = \{a, g\}$, $V_2 = \{b, d\}$, $V_3 = \{c, f\}$, and $V_4 = \{e, h\}$.

It is easy to verify that the eccentric partition for graph M in Figure 4 is unique. However, for an arbitrary *e.p.g.* graph G, the eccentric partition of $V(G)$

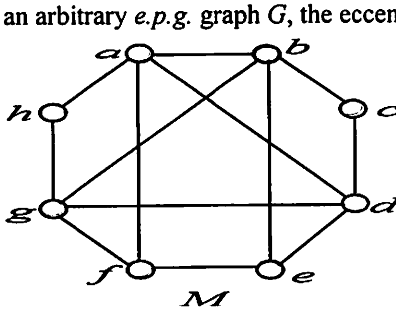

Figure 4

is not necessarily unique. For example, consider the bipartite graph G of order 9 and size 12 with vertex set $V(G) = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, x\}$ and edge set $E(G) = \{a_i b_j, c_i d_j, xb_i, xd_i : i = 1,2; j = 1,2\}$. There are seven ways to partition the partite set $B = \{b_1, b_2, d_1, d_2\}$ into mutually eccentric sets of size at least two, and there are two ways to partition $A = \{a_1, a_2, c_1, c_2, x\}$ into mutually eccentric sets of size at least two depending on whether x is grouped with the a's or the c's. Thus there are fourteen eccentric partitions of graph G.

The ***eccentric closure*** $ec(v)$ of vertex v is the set found recursively in the following way: (1) $v \in ec(v)$, and (2) If u is an eccentric vertex of any vertex in $ec(v)$, then $u \in ec(v)$. Graph G is ***eccentrically closed*** if for each $v \in V(G)$, the vertices within $ec(v)$ are mutually eccentric. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, let $EC(G) = \{ec(v_i) : 1 \le i \le n\}$. As usual with sets, if $ec(v_i) = ec(v_j)$, that set is only listed once within $EC(G)$.

Theorem 8. If a nontrivial graph G is eccentrically closed, then $EC(G)$ is an eccentric partition of $V(G)$.

Proof. If G is eccentrically closed, then the vertices within each eccentric closure are mutually eccentric. To show that $EC(G)$ is a partition of $V(G)$, we must show that $ec(v_i) \cap ec(v_j) \ne \varnothing$ implies that $ec(v_i) = ec(v_j)$. If $ec(v_i) \ne ec(v_j)$, then without loss of generality, there exists $u \in ec(v_i)$ such that $u \notin ec(v_j)$. Suppose that $w \in ec(v_i) \cap ec(v_j)$. Note that if $w = v_i$ or $w = v_j$, then by the way closures are formed, $ec(v_i) = ec(v_j)$. So assume that $w = v_i$ and $w = v_j$. Since u and w are both eccentric vertices of v_i and G is eccentrically closed, u and w are mutually eccentric. But then $w \in ec(v_j)$ forces $u \in ec(v_j)$ by condition (2) of eccentric closure formation. This contradiction shows that $ec(v_i) = ec(v_j)$ whenever $ec(v_i) \cap ec(v_j) \ne \varnothing$. Thus $EC(G)$ is a partition of $V(G)$. ∎

The converse of Theorem 8 does not hold. For example, graph M of Figure 4 was shown to be an *e.p.g.*, however, M is not eccentrically closed; the vertices within $ec(a) = \{a, c, e, f, g, h\}$ are not mutually eccentric.

Theorem 9. If G is eccentrically closed, then G is self-centered.

Proof. Let G be an eccentrically closed graph, and suppose that G is not self-centered. Then there exists some vertex $v \in C(G)$ that is adjacent to a vertex $u \notin C(G)$. Since the eccentricities of adjacent vertices differ by at most one, $e(v) = r$ and $e(u) = r + 1$. Let u' be an eccentric vertex for u. Then

$d(u,u') = r+1$. Also, $d(v,u') = r$ because if it were less, there would be a path of length less than $r+1$ from u to u' via v. But now $d(v,u') = r$ and $e(v) = r$ implies that u' is an eccentric vertex for v. So $u' \in ec(v)$, but u' and v are not mutually eccentric since $e(v) = r$ and $e(u') \geq r+1$. Thus G is not eccentrically closed, a contradiction. Therefore, if G is eccentrically closed, then G is self-centered. ▮

The converse to Theorem 9 is not true. For example, for $k \geq 2$ the odd cycles C_{2k+1} are self-centered but not eccentrically closed.

Harary and Norman [10] showed that the center of any connected graph lies in a single block. This implies that any self-centered graph contains no cutvertices. As a result, we have the following corollary to Theorem 9.

Corollary 10. If G is eccentrically closed, then G contains no cutvertices. ▮

Recall that G is an ***antipodal graph*** if for each vertex v there exists a unique vertex v' such that $d(v,v') = d(G)$. It is easy to verify that an antipodal graph is self-centered, a *u.e.p.* graph, an *e.p.g.* graph, eccentrically closed, and for each v, $|ec(v)| = 2$. This immediately provides an abundance of eccentrically closed graphs. For example n-cubes are antipodal graphs, so they are eccentrically closed. Except for the tetrahedron, the regular polyhedra are antipodal graphs. However, all five regular polyhedra are eccentrically closed. The following result is easy to prove.

Theorem 11. For $n_i \geq 2$, the complete multipartite graphs $K_{n_1,n_2,\ldots,n_k}$ are eccentrically closed. Furthermore, if $n_i \geq 3$ for some i, then $K_{n_1,n_2,\ldots,n_k}$ is not antipodal. ▮

Corollary 12. For all $n \geq 5$, there exists a non-antipodal, eccentrically closed graph on n vertices. ▮

We shall examine NP-completeness for eccentric partitions. First we consider the following decision problems.

CLIQUE PARTITION (CP)

INSTANCE: A graph $G = (V,E)$ and a positive integer $K \leq |V|$.

QUESTION: Can the vertices of G be partitioned into $k \leq K$ disjoint sets $V_1, V_2, \ldots, V_k$ such that for each i, $1 \leq i \leq k$, $\langle V_i \rangle$ is complete.

NONTRIVIAL CLIQUE PARTITION (NCP)
INSTANCE: A $G = (V, E)$ and a positive integer $K \le |V|/2$.
QUESTION: Can the vertices of *G* be partitioned into $k \le K$ disjoint sets $V_1, V_2, \ldots, V_k$, each of size at least two, such that for each *i*, $1 \le i \le k$, $\langle V_i \rangle$ is complete.

Comparing CP to NCP, one sees that NCP has the extra restriction that each V_i has at least two vertices. CLIQUE PARTITION is known to be NP-complete. It appears as problem GT15 in Garey and Johnson [8]. We shall use that fact in proving the following result, which will later be used to help establish the NP-completeness of a decision problem concerning eccentric partitions.

Lemma 13. NONTRIVIAL CLIQUE PARTITION (NCP) is NP-complete.
Proof. We must do two things. First, we must show that NCP $\in NP$. Then we must construct a reduction from a known *NP*-complete problem to NCP. The first part is straightforward, since it is easy to verify in polynomial time a 'yes' instance to NCP. That is, given a graph and a partition of its vertex set V into $k \le K$ disjoint sets $V_1, V_2, \ldots, V_k$, each of size at least two, it is easy to verify in polynomial time whether $\langle V_i \rangle$ is complete for each *i*. We can achieve this by verifying that the off-diagonal entries of the principal submatrix of the adjacency matrix determined by V_i contains all ones for each *i*. Each entry of the adjacency matrix is fetched and check at most once so the verification algorithm has order $O(n^2)$.

We show that NCP is NP-hard by showing that CP $\propto$ NCP. The reduction algorithm has as input an instance of the CP, namely, a graph *G* of order *n* and an integer $k \le n$. We construct, in polynomial time, a graph *H* of order $2n$ such that there is a partition of $V(H)$ into $k \le n$ disjoint sets of size at least two, each of which induces a complete graph, if and only if there exists a partition of $V(G)$ into $k \le n$ disjoint sets, each of which induces a complete graph. For a given graph *G*, let *H* be the join $G + G$. Thus *H* consist of two copies of *G* with each vertex in the first copy adjacent to every vertex in the second copy. Clearly, *H* can be constructed from *G* in polynomial time. It remains to show that this is a reduction.

Suppose that for graph *G* of order *n*, $V_1, V_2, \ldots, V_k$ is a partition of $V(G)$ into $k \le n$ disjoint sets, each of which induces a complete graph. Let the two copies of *G* in $H = G + G$ be denoted G_1 and G_2, and let the vertex sets in G_2 corresponding to each of the V_i partitioning $V(G_1)$ be denoted W_i. Then $V_1 \cup W_1, V_2 \cup W_2, \ldots, V_k \cup W_k$ is a partition of $V(H)$ into $k \le n$ sets of size at least two such that $\langle V_i \cup W_i \rangle$ is complete for each *i*. For the converse, suppose that for $k \le n$, $X_1, X_2, \ldots, X_k$ is a partition of $V(H)$ into disjoint sets of size at least two, each of which induces a complete graph. For each *i*, let $Y_1, Y_2, \ldots, Y_k$ be the restriction of X_i to G_1, that is, $Y_i = X_i \cap V(G_1)$. For any X_i that is wholly contained in G_2, its corresponding $Y_i = \varnothing$. Delete such Y_i from the list $Y_1, Y_2, \ldots, Y_k$ and relabel to get $Y_1, Y_2, \ldots, Y_j$. Then $Y_1, Y_2, \ldots, Y_j$ is a partition of $V(G_1) = V(G)$ into $j \le k \le n$ sets such that $\langle Y_i \rangle$ is complete for each *i*. Thus there is a partition of $V(G)$ into at most *k* sets, each of which induces a complete graph if and only if there is a partition of $V(H)$ into at most *k* sets of size at least two, each of which induces a complete graph. ∎

Now consider the following decision problem.

ECCENTRIC PARTITION (EP)
INSTANCE: A graph $G = (V, E)$ and a positive integer $K \le |V|/2$.
QUESTION: Can the vertices of *G* be partitioned into $k \le K$ disjoint sets $V_1, V_2, \ldots, V_k$, each of size at least two, such that for each *i*, $1 \le i \le k$, the vertices in V_i are mutually eccentric.

Theorem 14. ECCENTRIC PARTITION (EP) is NP-complete.
Proof. It is easy to verify in polynomial time a 'yes' instance to EP. That is, given a graph and a partition of its vertex set *V* into $k \le K$ disjoint sets $V_1, V_2, \ldots, V_k$, each of size at least two, we can verify in polynomial time whether the vertices of V_i are mutually eccentric for each *i*. We can achieve this by generating the distance matrix *D*. Then for each set V_i, we verify that the maximum entries in the rows of *D* corresponding to vertices of V_i are equal, that is, the vertices of V_i have the same eccentricity e_i. And finally, we verify that the off-diagonal entries of the principal submatrix of the distance matrix determined by V_i all equal e_i for each *i*. Generating the distance matrix is done in $O(n^3)$ steps. Then verifying that each vertex in V_i has the same eccentricity is done in $O(n^2)$ steps, and checking the entries in the principal submatrices of *D* determined by each V_i is also done in at $O(n^2)$ steps. So the verification algorithm has order $O(n^3)$.

We show that EP is NP-hard by showing that NCP $\propto$ EP. The reduction algorithm has as input an instance of the NCP, namely, a graph *G* of order *n* and an integer $k \le n/2$. We construct, in polynomial time, a graph *H* of order $n+2$ such that there is a partition of $V(H)$ into $k+1 \le n/2+1$ sets (of size at least two) of mutually eccentric vertices, if and only if there exists a partition of $V(G)$ into $k \leq n/2$ disjoint sets of size at least two, each of which induces a complete graph. For a given graph G of order n, let $H = K_2 + \overline{G}$. Graph H can be constructed from G in $O(n^2)$ time. We must show that this is a reduction.

For graph G of order n, suppose that $V_1, V_2, \ldots, V_k$ is a partition of $V(G)$ into $k \leq n/2$ sets of size at least two, each of which induces a complete graph. Then in $H = K_2 + \overline{G}$, the vertices coming from G all have eccentricity two. Furthermore, the vertices within a given V_i are mutually nonadjacent in H and since they also have eccentricity two, they are necessarily mutually eccentric. Let V^* be the two vertices from K_2 in H. Those two vertices have eccentricity one in H so they are mutually eccentric. Thus $V_1, V_2, \ldots, V_k, V^*$ is a partition of $V(H)$ into $k+1 \leq n/2+1$ sets (of size at least two) of mutually eccentric vertices. For the converse, suppose that $W_1, W_2, \ldots, W_{k+1}$ is a partition of $V(H)$ into $k+1 \leq n/2+1$ sets (of size at least two) of mutually eccentric vertices. Then since the vertices from the K_2 in H are the only two vertices that have eccentricity one in H, they must comprise one of the sets, say W_{k+1}. Then the sets $W_1, W_2, \ldots, W_k$ are mutually eccentric sets in H. Since all the vertices in those sets have eccentricity two in H, vertices within each W_i, $1 \leq i \leq k$, are mutually at distance two (therefore mutually nonadjacent) in H. Since those vertices came from $\overline{G}$ in $H = K_2 + \overline{G}$, vertices within each W_i, $1 \leq i \leq k$, must be mutually adjacent in G. Thus $W_1, W_2, \ldots, W_k$ is a set of $k \leq n/2$ sets of size at least two, each of which induces a complete graph. Thus there is a partition of $V(G)$ into at most k sets of size at least two, each of which induces a complete graph if and only if there is a partition of $V(H)$ into at most $k+1$ sets of mutually eccentric vertices, each of size at least two. ∎

5. Acknowledgements

A preliminary version of this paper was presented at the DIMACS Connect Institute at Rutgers University in July 2000. We would like to thank one of the participants there, Professor Sulamita Klein of Universidade Federal do Rio de Janeiro for her suggestion that we look at the NP-completeness of eccentric sets of vertices and eccentric partitions. We also thank Professor Steve Hedetniemi of Clemson University who informed us via e-mail that his colleague Professor Alice McRae of Appalachian State University had simultaneously obtained the NP-completeness result stated in Lemma 13.

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