Powers of Asteroidal Triple-free Graphs with Applications*

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Abstract. An asteroidal triple is an independent set of three vertices in a graph such that every two of them are joined by a path avoiding the closed neighborhood of the third. Graphs without asteroidal triples are called AT-free graphs. In this paper, we show that every AT-free graph admits a vertex ordering that we call a 2-cocomparability ordering. The new suggested ordering generalizes the cocomparability ordering achievable for cocomparability graphs. According to the property of this ordering, we show that every proper power G^k ($k \geq 2$) of an AT-free graph G is a cocomparability graph. Moreover, we demonstrate that our results can be exploited for algorithmic purposes on AT-free graphs.

Keywords: Asteroidal triple, AT-free graphs, Cocomparability graphs, Powers of graphs.

1. Introduction

An *asteroidal triple* (AT for short) of a graph is an independent set of three vertices such that every two of them are joined by a path avoiding the closed neighborhood of the third. A graph is *asteroidal triple-free* (AT-free for short) if it does not contain an AT. Lekkerkerker and Boland [24] first introduced the concept of asteroidal triples to characterize interval graphs. A graph is an *interval graph* if and only if it is AT-free and chordal (a graph that every cycle of length at least four has a chord).

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Corneil, Olariu and Stewart [10] recently obtained a collection of structural properties for AT-free graphs. However, up to now a nice characterization of AT-free graphs such as a geometric intersection model, an elimination scheme of vertices or edges is not known. The recognition of AT-free graphs can be solved in $O(n^3)$ time [10] (a better time complexity for graphs which are sparse or have a sparse complement refers to [22]). For other algorithmic problems like maximum independent set, steiner set and various domination-type problems on AT-free graphs, polynomial time algorithms can be found in [3, 6, 9, 11, 22]. Möhring [26] showed that if a graph G is AT-free, then every minimal triangulation of G must be an interval graph. This implies that the treewidth and the pathwidth of every AT-free graph are equal. Conversely, Parra and Scheffler [27] (and independently by Corneil et al. [10]) showed that a minimal triangulation of a graph G is an interval graph only if the given graph G is AT-free. The algorithmic problems like maximum clique [6], clique partition [6], treewidth [19], minimum fill-in [19], and vertex ranking [20] on AT-free graphs have known to be NP-complete.

The class of AT-free graphs contains various well-known classes of graphs including interval, permutation, trapezoid and cocomparability graphs. For an overview of these classes of graphs, please refer to [5, 17]. Note that AT-free graphs are not perfect in the sense of Berge [4] since the complements of odd chordless cycles with at least five vertices are AT-free. In fact, Cheah [8] showed that the class of cocomparability graphs is strictly contained in the class of perfect AT-free graphs (i.e., graphs that are both perfect and AT-free). Moreover, many results related to the powers of AT-free graphs and cocomparability graphs have been studied. Raychaudhuri [28] showed that if G^k is AT-free, then so is G^{k+1}. This implies that every proper power G^k ($k \geq 2$) of an AT-free graph G is AT-free. Damaschke [13] (and independently by Flotow [15]) established a similar property showing that the class of cocomparability graphs is closed under powers. Recently, Ho et al. [18] obtained some forbidden structures of the powers of AT-free graphs. Using these forbidden structures, they showed that every proper power of an AT-free graph is a perfect graph. A question given from their paper to ask is whether every proper power of AT-free graph is a cocomparability graph.

It is well-known that cocomparability graphs are characterized by the existence of the cocomparability ordering. Most useful characterizations and efficient algorithms on cocomparability graphs are developed from this vertex ordering (see e.g. [2, 13, 23]). It would be interesting to see whether AT-free graphs also possess some vertex ordering which can be exploited for algorithmic purposes. In this paper, we show that every AT-free graph admits a vertex ordering which is a generalization of the cocomparability ordering, and such an ordering can be produced in $O(M(n))$ time, where $M(n)$ denotes the time complexity of multiplying two $n \times n$ matrices of integers (currently is known to be $O(n^{2.376})$ time [12]). According to the property of this ordering, we extend the partial results of the previous researchers [13, 15, 18, 28] about the powers of AT-free graphs and cocomparability graphs. In particular, we answer the question of Ho et al. [18] in the affirmative, i.e., all proper powers of AT-free graphs are cocomparability graphs. Moreover, we demonstrate that our result can be used for solving the distance k-domination, k-stability, and the graph k-clustering problems for $k \geq 2$ on AT-free graphs in a more efficient way.

2. Terminologies and notations

All graphs considered in this paper are undirected, simple (i.e., without loops and multiple edges) and connected. Let $G = (V, E)$ be a graph with vertex set V of size n and edge set E of size m. The *distance* of two vertices $u, v \in V$, denoted by $d_G(u, v)$, is the number of edges of a shortest path from u to v in G. The *diameter*, denoted by *diam*(G), is the maximum of all the distances $d_G(u, v)$ for $u, v \in V$. The *k-th power* G^k of a graph G is the graph with the same vertex set as G such that two vertices are adjacent in G^k if and only if their distance in G is at most k. For convenience, we write $G^k = (V, E^k)$ as the k-th power of G, where $E^k = \{(u, v) : u, v \in V$ and $d_G(u, v) \leq k\}$. In particular, we call G^2 the *square* of G, and G^k for $k \geq 2$ a *proper power* of G.

The *open neighborhood* $N_G(u)$ of a vertex u in a graph $G = (V, E)$ is the set $\{v \in V : (u, v) \in E\}$; and the *closed neighborhood* $N_G[u]$ is $N_G(u) \cup \{u\}$. A vertex $u \in V$ is *pendant* if $|N_G(u)| = 1$. For a nonempty subset $S \subset V$, we denote by $G[S]$ the subgraph of G induced by S. Also, we let $N_G(S)$ stand for the set of vertices in G that have a neighbor in S, and use $G - S$ to denote the subgraph of G induced by the set $V \backslash S$ (i.e., $\{v \in V : v \notin S\}$). A graph $G' = (V', E')$ is said to arise from G by an *S-contraction* if G' is obtained from $G - S$ by adding a new vertex s to $G - S$ such that $V' = (V \backslash S) \cup \{s\}$ and $E' = \{(u, v) \in E : u, v \in V \backslash S\} \cup \{(w, s) : w \in (V \backslash S) \cap N_G(S)\}$.

A path joining two vertices u and v is termed a *u-v path*. Two paths P and P' are connected by a common endvertex is denoted by $P \oplus P'$. A vertex u *misses* a path P if there are no vertices of P adjacent to u; otherwise, we say that u *intercepts* P. Let x be an arbitrary vertex of a graph G. For each vertex $u \in V$, $D_G(u, x)$ denotes the set of vertices that intercept all u-x paths in G; i.e., a vertex $v \notin D_G(u, x)$ if and only if there exists a u-x path in G such that v misses the path. Note that $D_G(x, x) = N_G[x]$ and

$D_G(u,x) = N_G[u] \cup N_G[x]$ for $(u,x) \in E$. A *dominating set* of a graph G is a set $D \subseteq V$ such that every vertex not in D is adjacent to some vertex of D. A pair of vertices $x, y \in V$ is a *dominating pair* if the vertices in each x-y path constitute a dominating set of G; i.e., $D_G(x,y) = V$.

Let π be a vertex ordering of a graph $G = (V, E)$. For any two vertices $u, v \in V$, we write $u <_\pi v$ if u comes before v in π. Note that, if π is clear from the context, we simply write $u < v$. When no ambiguity arises, the subscript G in the notations $d_G(u,v)$, $N_G(u)$, $N_G[u]$ and $D_G(u,x)$ can be omitted. For graph-theoretic terminologies and notations not mentioned here we refer to [17].

3. A generalization of cocomparability graphs

A graph $G = (V, E)$ is a *comparability graph* if it has a *transitive ordering*, i.e., there exists a linear order $<$ on the vertex set of G such that for every choice of vertices u, v, w with

$$u < v < w,\ (u,v) \in E,\ \text{and}\ (v,w) \in E\ \text{implies}\ (u,w) \in E.$$

There is a linear time algorithm to produce a transitive ordering when the input is a comparability graph [25], however there is no known algorithm for testing whether a given ordering is transitive faster than $O(M(n))$. A *cocomparability graph* $G = (V, E)$ is the complement of a comparability graph, or equivalently, it has a *cocomparability ordering*, i.e., there exists a linear order $<$ on the set of its vertices such that for every choice of vertices u, v, w with

$$u < v < w\ \text{and}\ (u,w) \in E\ \text{implies}\ (u,v) \in E\ \text{or}\ (v,w) \in E.$$

The following definition generalizes the concept of cocomparability ordering and cocomparability graphs.

Definition 1. Let $G = (V, E)$ be a graph and k a positive integer. A *k-cocomparability ordering* (k-CCPO) of G is an ordering of its vertices such that for every choice of vertices u, v, w with

$$u < v < w\ \text{and}\ d(u,w) \le k\ \text{implies}\ d(u,v) \le k\ \text{or}\ d(v,w) \le k.$$

A graph G is called a *k-cocomparability* graph if it admits a k-CCPO.

Because the distance between any two vertices in a graph is no more than its diameter, every graph G must be a k-cocomparability graph for some $k \le diam(G)$. Since we can construct a 2-cocomparability graph from an arbitrary graph by adding a new vertex adjacent to all vertices, the class of k-cocomparability graphs, $k \ge 2$, is not closed under taking induced subgraphs. From the above definition together with the notion of powers of graphs, the following lemma can easily be derived from the fact that if π is a k-CCPO of G then it is a cocomparability ordering of G^k, and conversely.

Lemma 1. *A graph G is a k-cocomparability graph if and only if G^k is a cocomparability graph.*

Lemma 2. *Let $G = (V, E)$ be a k-cocomparability graph, $k \ge 1$, and π a k-CCPO of G. For every three vertices $u < v < w$, if $d(u, w) \le k + 1$, then one of the following is true:* (i) $d(u, v) \le k$; (ii) $d(v, w) \le k$; *or* (iii) $d(u, v) = d(v, w) = k + 1$.

Proof. Since π is a k-CCPO, if $d(u, w) \le k$, then $d(u, v) \le k$ or $d(v, w) \le k$. Thus no further proof is necessary for this case. In the following, we consider $d(u, w) = k + 1$ and assume $d(u, v) > k$ and $d(v, w) > k$. Let P be a shortest path joining u and w in G, and let u' be the vertex adjacent to u in P; i.e., $(u, u') \in E$ and $d(u', w) = k$. Clearly, $u' \ne v$ and either $v < u'$ or $u' < v$. For the former case, since $u < v < u'$, $(u, u') \in E$ and $d(u, v) > k$, it implies $d(v, u') \le k$. For the latter case, since $u' < v < w$, $d(u', w) = k$ and $d(v, w) > k$, it implies $d(v, u') \le k$. Thus $k < d(v, u) \le d(v, u') + d(u', u) \le k + 1$. This shows that $d(u, v) = k + 1$. By a similar argument, let w' be the vertex adjacent to w in P, we can show that $d(v, w) = d(v, w') + d(w', w) = k + 1$. □

An immediate consequence obtained from Lemma 2 is that all the classes of k-cocomparability graphs, $k \ge 1$, constitute a hierarchy by set inclusion. Thus we have the following result.

Lemma 3. *If G is a k-cocomparability graph for any positive integer k, then it is also a $(k+1)$-cocomparability graph. In particular, every k-CCPO is a $(k + 1)$-CCPO in G.*

By Lemma 1 and Lemma 3, we conclude the following property.

Theorem 1. *A graph G is a k-cocomparability graph if and only if every power G^s for $s \ge k$ is a cocomparability graph.*

The *cocomparability number* of a graph G, denoted by $ccp(G)$, is the smallest integer k such that G admits a k-CCPO. Note that $ccp(G) = 1$ if and only if $\overline{G}$ (the complement of G) is a comparability graph. In what follows, we will show how to find the cocomparability number of an arbitrary graph G if $ccp(G) \geq 2$.

Lemma 1 shows that $ccp(G) = k$ if and only if G^k is a cocomparability graph but G^{k-1} is not. So we can recognize $\overline{G^k}$ and $\overline{G^{k-1}}$ by using the algorithm of [30] to determine $ccp(G)$. For constructing the powers of G, the square G^2 can be produced from the matrix multiplication on the adjacent matrix of G. So, if $ccp(G) = 2$, we can determine this case in $O(M(n))$ time. In general, to construct G^k and G^{k-1}, we first establish the *distance matrix* of G; i.e., an $n \times n$ matrix $D = (d_{u,v})$ such that each element $d_{u,v} = d_G(u, v)$ for $u, v \in V$. Seidel [29] showed that the distance matrix of a general graph can be computed in $O(M(n) \log n)$ time. Thus, a naive algorithm for finding the specific integer k over the range from 1 to $diam(G)$ can be implemented by using the binary search technique. For each step in the search, we use the distance matrix D to build the corresponding G^k and G^{k-1}, and then carry out the recognition for their complements. The process of each step takes $O(M(n))$ time. Consequently, the search totally requires $O(M(n) \log d)$ time where $d = diam(G)$. Since $d < n$, we conclude the following.

Theorem 2. *The problem of finding $ccp(G)$ of a graph G can be solved in $O(M(n) \log n)$ time.*

4. AT-free graphs are 2-cocomparability graphs

In this section, we will show that every AT-free graph admits a 2-CCPO. For an AT-free graph $G = (V, E)$, a vertex $u \in V$ is called *pokable* if the graph obtained from G by adding a pendant vertex adjacent to u is AT-free. A *pokable dominating pair* is a dominating pair such that both the vertices are pokable. An existential proof given in [10] shows that every connected AT-free graph contains a pokable dominating pair. Besides, an $O(n + m)$ time algorithm for finding pokable dominating pairs of a connected AT-free graph can also be found in [11].

We now assume that $G = (V, E)$ is a connected AT-free graph with at least two vertices and let $\{x, y\}$ be a pokable dominating pair of G. Define a binary relation R on G such that for every two vertices $u, v \in V$: $u \, R \, v \iff D_G(u, x) = D_G(v, x)$. Clearly, R is an equivalence relation and the vertices of G can be partitioned into equivalence classes. Let $C_1, C_2, \ldots, C_k (k \geq 1)$

be the corresponding equivalence classes of G. A class C_i is termed *non-trivial* if $|C_i| \geq 2$. A non-trivial class is *valid* if it induces a connected subgraph of G. Let $C(u)$ be the equivalence class containing the vertex u. In particular, we call $C(y)$ the *dominating equivalence class* with respect to x. Note that $C(y)$ is always valid and a vertex $u \in C(y)$ if and only if $D_G(u,x) = V$; i.e., $\{x,u\}$ is also a dominating pair of G. The following interesting property appeared in [10].

Lemma 4. (Corneil et al. [10]) *Let C be any valid equivalence class in an AT-free graph G. If a graph G' is obtained from G by a C-contraction, then G' remains AT-free. In particular, G' is connected whenever G is.*

In the above lemma, if C is distinct from $C(x)$ and $C(y)$, then $\{x,y\}$ remains a pokable dominating pair in G'. On the other hand, if y' is the vertex of G' obtained by contracting $C(y)$ in G, then $\{x,y'\}$ forms a pokable dominating pair in G'. According to this property, a natural consequence is that every connected AT-free graph G can be decomposed by the following way. Let $G_0, G_1, \ldots, G_h$ be a sequence of graphs defined as follows:

(i) $G_0 = G$ contains a pokable dominating pair $\{x,y\}$;
(ii) For every $i = 0, \ldots, h-1$, let R_i be the equivalence relation defined on G_i by setting $u \; R_i \; v \iff D_{G_i}(u,x) = D_{G_i}(v,x)$, and let G_{i+1} be the graph obtained from G_i by a C-contraction, where C is an arbitrary valid equivalence class in G_i;
(iii) G_h consists of a single vertex.

In [10], such a sequence $G_0, G_1, \ldots, G_h$ is called *involutive*, and it has been proved that every connected AT-free graph has an involutive sequence. For an example to illustrate the concept of the decomposition property on AT-free graphs please refer to [10].

In the remainder of this section, we consider that each contraction of an AT-free graph is restricted to a dominating equivalence class, and such an involutive sequence is called a *dominating involutive sequence*. Formally, in a dominating involutive sequence $G_0, G_1, \ldots, G_h$, the graph G_{i+1} is obtained from G_i by a $C(y_i)$-contraction, where $y_0 = y$ and y_i $(i \geq 1)$ is the vertex of G_i obtained by contracting $C(y_{i-1})$ in G_{i-1}, and G_h consists of a single vertex y_h. For convenience, we refer to y_i $(i > 0)$ as a *super vertex* of G_i except y_0, and let V_i be the set containing all non-super vertices of G_i. For $i = 0, \ldots, h-1$, let $U_i = V_i \setminus V_{i+1}$; i.e., $U_0 = C(y_0)$ and $U_i = C(y_i) \setminus \{y_i\}$ for $i \geq 1$. Clearly, $\bigcup_{i=0}^{h-1} U_i = V$.

With respect to a dominating involutive sequence of an AT-free graph G, we construct a vertex ordering Π according to the following rule: for any two vertices $u \in U_i$ and $v \in U_j$, u comes before v in Π whenever $i < j$. Otherwise, if u and v are contained in the same set U_i, ties are broken arbitrarily. The following lemmas are helpful for proving that Π is a 2-CCPO of G.

Lemma 5. *Let $u \in V_i$ where $1 \leq i \leq h-1$. Then $D_{G_i}(u,x) \backslash \{y_i\} \subseteq D_{G_{i-1}}(u,x)$.*

Proof. For the case $u = x$, $D_{G_i}(x,x) \backslash \{y_i\} = N_{G_i}[x] \backslash \{y_i\} \subseteq N_{G_i}[x] \subseteq N_{G_{i-1}}[x] = D_{G_{i-1}}(x,x)$. We now consider that u and x are two distinct vertices in G_i. Let $w \in D_{G_i}(u,x) \backslash \{y_i\}$; i.e., w is a non-super vertex and it intercepts all x-u paths in G_i. Suppose that $w \notin D_{G_{i-1}}(u,x)$. Then, there is a path P joining x and u in G_{i-1} such that w misses the path. Since G_i is obtained from G_{i-1} by a $C(y_{i-1})$-contraction, P must pass through a vertex $z \in C(y_{i-1})$ such that w misses the two subpaths of P where one from x to z and the other from z to u. Also, since $z \in C(y_{i-1})$, $\{x,z\}$ is a dominating pair in G_{i-1}. Thus w cannot miss any x-z path in G_{i-1}, a contradiction. □

Lemma 6. *Let $H = G_i - \{y_i\}$ where $1 \leq i \leq h-1$. Then H is connected.*

Proof. By Lemma 4, if G is connected, then so is G_i. Suppose that H is disconnected; i.e., y_i is a cut vertex in G_i. Let $H(x)$ be the connected component of H containing x. Also, let $u \in V_i$ be any vertex in another component. Then every x-u path must pass through y_i. Since $\{x, y_i\}$ is a dominating pair in G_i, every vertex not in $H(x)$ is adjacent to y_i. This implies that $\{x,u\}$ is also a dominating pair of G_i. Thus $D_{G_i}(u,x) = V_i \cup \{y_i\}$. Since y_i is obtained from the $C(y_{i-1})$-contraction, u must be adjacent to at least one vertex of $C(y_{i-1})$ in G_{i-1}. Let $W = C(y_{i-1}) \cap N_{G_{i-1}}(u)$. Since $u \notin C(y_{i-1})$, there is a vertex z in G_{i-1} such that $z \notin D_{G_{i-1}}(u,x)$. By Lemma 5, $z \notin D_{G_i}(u,x) \backslash \{y_i\} = V_i$ (in fact, $z \in C(y_{i-1}) \backslash W$). Let P be a path joining x and u in G_{i-1} such that z misses the path, and let $w \in W$ be the vertex adjacent to u in P. Clearly, z misses the subpath of P from x to w in G_{i-1}; i.e., $z \notin D_{G_{i-1}}(w,x)$. This contradicts the fact that $D_{G_{i-1}}(w,x)$ contains all vertices of G_{i-1} since $w \in W \subset C(y_{i-1})$. □

Lemma 7. *Let $u,v \in V_i$ where $1 \leq i \leq h-1$. Then $d_{G_i}(u,v) = d_{G_{i-1}}(u,v)$.*

Proof. Due to the concept of contracting operation, clearly, $d_{G_i}(u,v) \leq d_{G_{i-1}}(u,v)$. If P is any shortest path joining u and v in G_i that does not pass through y_i, then P is also a shortest u-v path in G_{i-1}. Thus $d_{G_i}(u,v) = d_{G_{i-1}}(u,v)$ in this case. In particular, the equality holds whenever u and v

are adjacent in G_i. Let $H = G_i - \{y_i\} = G_{i-1} - C(y_{i-1})$. By Lemma 6, H is connected. Let $P(u,v)$ be a shortest u-v path in H. If $|P(u,v)| = d_{G_i}(u,v)$, then $P(u,v)$ is a shortest u-v path in G_i which does not pass through y_i. Thus no further proof is necessary for this case. So we only need to consider that $|P(u,v)| > d_{G_i}(u,v) \geq 2$ in the following proof.

Suppose to the contrary. Assume $d_{G_i}(u,v) < d_{G_{i-1}}(u,v)$; i.e., there is a shortest u-v path P^* in G_{i-1} which passes through at least two vertices of $C(y_{i-1})$. We denote $V(P^*)$ as the set of vertices contained in P^*. Consider the following two cases.

Case 1: either $u = x$ or $v = x$. Without loss of generality assume $u = x$. Let s be a vertex at a shortest distance to x in $V(P^*) \cap C(y_{i-1})$, and $P^*(x,s)$ the subpath of P^* from x to s. Since P^* is a chordless x-v path containing at least two vertices of $C(y_{i-1})$, v misses $P^*(x,s)$ in G_{i-1}. Also, since $s \in C(y_{i-1})$, $P^*(x,s)$ is a dominating path in G_{i-1}. However, this is impossible because v misses $P^*(x,s)$.

Case 2: u,v,x are three distinct vertices. Let s (resp. t) be a vertex at a shortest distance to u (resp. to v) in $V(P^*) \cap C(y_{i-1})$. Also, let $P^*(u,s)$ (resp. $P^*(v,t)$) be the subpath of P^* from u to s (resp. from v to t). Since P^* is chordless, v misses $P^*(u,s)$ and u misses $P^*(v,t)$, respectively, in G_{i-1}. By the fact $s,t \in C(y_{i-1})$, every x-s path or x-t path is a dominating path of G_{i-1}. Recall that $H = G_{i-1} - C(y_{i-1})$ is connected and $|P(u,v)| \geq 3$. Let $P(x,u)$ be a shortest x-u path in H. Clearly, $P(u,v)$ and $P(x,u)$ are contained in G_{i-1}. Since $P(x,u) \oplus P^*(u,s)$ is a dominating path in G_{i-1} and v misses $P^*(u,s)$, either $P(x,u)$ contains v or there is a vertex v' contained in $P(x,u)$ such that v and v' are adjacent. For the former case, let $P(x,v)$ denote the subpath of $P(x,u)$ from x to v. Since $|P(u,v)| \geq 3$ and v is contained in the shortest path $P(x,u)$, u misses $P(x,v)$ in G_{i-1}. As a result, u also misses $P(x,v) \oplus P^*(v,t)$ in G_{i-1}. However, this is impossible because $P(x,v) \oplus P^*(v,t)$ is a dominating path in G_{i-1}. For the latter case, let $P(x,v')$ and $P(v',u)$ denote the subpaths of $P(x,u)$ where one from x to v' and the other from v' to u. Since $|P(u,v)| \geq 3$ and the vertex v' contained in the shortest path $P(x,u)$ is adjacent to v, $|P(v',u)| \geq 2$ and u misses $P(x,v')$ in G_{i-1}. Consequently, u misses $P(x,v') \oplus (v',v) \oplus P^*(v,t)$ in G_{i-1}. This leads to a contradiction that $P(x,v') \oplus (v',v) \oplus P^*(v,t)$ is a dominating path in G_{i-1}. □

Lemma 8. *Let $u \in U_i$, $v \in U_j$ and $w \in U_k$ where $0 \leq i \leq j \leq k \leq h-1$. If $d_{G_i}(u,w) \leq 2$, then $d_{G_i}(u,v) \leq 2$ or $d_{G_i}(v,w) \leq 2$.*

Proof. Clearly, u,v,w are contained in G_i, and v,w are contained in G_j. Suppose that $d_{G_i}(u,w) \leq 2$. Let $P(w,u)$ be a shortest w-u path in G_i

(i.e., $|P(w,u)| \leq 2$) and let $P(x,w)$ be a shortest x-w path in G_k. By Lemma 7, $|P(x,w)| = d_{G_k}(x,w) = d_{G_j}(x,w) = d_{G_i}(x,w)$. Thus $P(x,w)$ is also a shortest x-w path in G_j and G_i, respectively. Since $u \in U_i$, $P(x,w) \oplus P(w,u)$ is a dominating path in G_i. So v intercepts $P(x,w)$ or $P(w,u)$ in G_i. Since $|P(w,u)| \leq 2$, if v intercepts $P(w,u)$, it is clear that $d_{G_i}(u,v) \leq 2$ or $d_{G_i}(v,w) \leq 2$. The following proof only consider the case that v intercepts $P(x,w)$ and does not intercept $P(w,u)$ (in particular $(v,w) \notin E$).

Suppose that v intercepts $P(x,w)$ at v' ($\neq w$); i.e., v' is contained in $P(x,w)$ and $v' = v$ or $(v,v') \in E$. Let $P(x,v) = P(x,v') \oplus (v',v)$ where $P(x,v')$ is the subpath of $P(x,w)$ from x to v'. Note that if v and v' are the same vertex, then $P(x,v) = P(x,v')$ in this case. Clearly, $P(x,v)$ is contained in G_j. Since $v \in U_j$, $P(x,v)$ is a dominating path in G_j. Thus w intercepts $P(x,v)$ in G_j. Since $P(x,w)$ is a shortest path, if w intercepts $P(x,v')$, then $(v',w) \in E$. By Lemma 7, $d_{G_i}(v,w) = d_{G_j}(v,w) \leq d_{G_j}(v,v') + d_{G_j}(v',w) \leq 2$. On the other hand, if w intercepts (v',v), then it is obvious that $d_{G_i}(v,w) = d_{G_j}(v,w) = 2$. □

Lemma 7 shows that if G is an AT-free graph with an involutive sequence $G_0, G_1, \ldots, G_h$, then the distance of any two non-super vertices of G_i for $i = 0, \ldots, h-1$ is the same as their distance in G. Hence, we can use $d_G(s,t)$ instead of $d_{G_i}(s,t)$ in Lemma 8, where $s,t \in \{u,v,w\}$ are two distinct vertices. That is, for any three vertices $u < v < w$ in Π, if $d_G(u,w) \leq 2$ then $d_G(u,v) \leq 2$ or $d_G(v,w) \leq 2$. This shows that Π is a 2-CCPO of G and thus we have the following theorem.

Theorem 3. *AT-free graphs are 2-cocomparability graphs.*

Figure 1 shows a 2-cocomparability graph with a 2-CCPO $a < b < c < d < e < f < g$, which contains an asteroidal triple $\{c,d,e\}$. Thus, the class of AT-free graphs is properly contained in the class of 2-cocomparability graphs. Furthermore, by Theorem 1 and Theorem 3, we obtain the following consequence which extends the partial results of [13, 15, 18, 28] about the powers of AT-free graphs and cocomparability graphs.

Theorem 4. *Every proper power of an AT-free graph is a cocomparability graph.*

5. Applications

We close this paper with the following discussions. It is well-known that the distance k-domination and k-stability problems are NP-complete on gen-

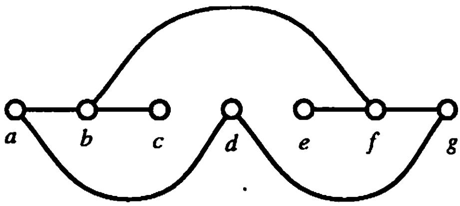

Fig. 1. A 2-cocomparability graph that is not AT-free.

eral graphs [16] (for further results please refer to [7]). Solving the distance k-domination and k-stability problems on a graph G can be reduced to solving the domination and the stability problems on G^k. The best known algorithms for solving the domination and stability problems on AT-free graphs can be run in time $O(n^6)$ [22] and $O(n^4)$ [6], respectively. Based on the result of Theorem 4, for $k \geq 2$, the distance k-domination and k-stability problems on AT-free graphs can be solved by applying the best known algorithms for the domination and the stability problems on cocomparability graphs instead of AT-free graphs to yield more efficient algorithms. Thus, the distance k-domination problem on AT-free graphs can be solved in $O(nm^2)$ time using the algorithm of [2]. The stability problem on cocomparability graphs can be implemented by finding the maximum clique on its complement. Because there is a linear time algorithm for finding a maximum clique of a comparability graph with a given transitive ordering [17,25], the distance k-stability problem on AT-free graphs can be solved in $O(M(n) \log n)$ time for $k > 2$ and $O(M(n))$ time for $k = 2$.

Another application is the graph k-clustering (partition) problem which is defined as follows: Let k be a fixed positive integer. Given a graph $G = (V, E)$ and positive integer l, determine whether there is a partition of V into disjoint subsets $C_1, C_2, \ldots, C_l$ such that $diam(G[C_i]) \leq k$ for every $i \in \{1, 2, \ldots, l\}$. For a special case when fixed $k = 1$, the problem is also called the *clique partition* and has known to be NP-complete (see [GT15] of [16]). It should be noted that clique partition remains NP-complete for AT-free graphs [6].

Deogun, Kratsch, and Steiner [14] showed that the k-clustering problem is NP-complete for any fixed $l \geq 3$. Furthermore, there is an $\varepsilon > 0$ such that there is no polynomial time approximation algorithm for minimizing the size of a partition into a k-clustering that has worst-case performance ratio n^ε, unless P=NP. In addition, they gave a polynomial time approximation algorithm of constant worst-case ratio at most 3 for solving the k-clustering problem with the minimum partition size on a super class of AT-free graphs, called the diametral path graphs. Abbas and Stewart [1] recently showed that the k-clustering problem remains NP-complete on bipartite graphs (for any fixed $k \geq 2$) and chordal graph (even when the input is restricted on those graphs that are both split graphs and undirected path graphs for $k = 2$). In contrast, they also provided linear time algorithms for solving the k-clustering problem on interval graphs and bipartite permutation graphs.

From the definition of k-clustering problem, an easy observation is that for $k \geq 2$, solving the k-clustering problem on a graph G can be reduced to solving clique partition on the power G^k. Recall that every comparability graph G has a transitive ordering on vertices, which makes it possible to design a linear time algorithm for solving clique partition problem on the complement of G [25]. From the reduction together with Theorem 4, we conclude that the k-clustering problem on AT-free graphs can be solved in $O(M(n) \log n)$ time for $k > 2$ and $O(M(n))$ time for $k = 2$.

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