On the Cordiality of the t-Uniform Homeomorphs - II (Complete Graphs)

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Abstract: Let G be a simple graph with vertex set V and edge set E. A vertex labeling $f : V \longrightarrow \{0,1\}$ induces an edge labelling $\overline{f} : E \longrightarrow \{0,1\}$ defined by $\overline{f}(uv) = | f(u) - f(v) |$. Let $v_f(0), v_f(1)$ denote the number of vertices v with $f(v) = 0$ and $f(v) = 1$ respectively. Let $e_f(0), e_f(1)$ be similarly defined. A graph is said to be **cordial** if there exists a vertex labeling f such that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

A t-uniform homeomorph $P_t(G)$ of G is the graph obtained by replacing all edges of G by vertex disjoint paths of length t. In this paper we show that (1) $P_t(K_{2n})$ is cordial for all $t \geq 2$. (2) $P_t(K_{2n+1})$ is cordial iff (a) $t \equiv 0 (\bmod 4)$ OR (b) t is odd and n is not $\equiv 2(\bmod 4)$ OR (c) $t \equiv 2(\bmod 4)$ and n is even.

Introduction

Throughout this paper, all graphs are finite, simple and undirected. A **t-uniform homeomorph** $P_t(G)$ of a graph G is the graph obtained by replacing all edges of G by vertex disjoint paths of length t, i.e. H is obtained from G by introducing $t-1$ new vertices on each edge of G.

Let G be a graph with vertex set V and edge set E. A binary labeling $f : V \longrightarrow \{0,1\}$ induces an edge labeling $\overline{f} : E \longrightarrow \{0,1\}$ defined by $\overline{f}(uv) = | f(u) - f(v) |$. By $v_f(0)$ and $v_f(1)$ we mean the number of vertices with $f(v) = 0$ and $f(v) = 1$ respectively. Similarly by $e_f(0)$ and $e_f(1)$ we mean the number of edges labeled 0 and 1 respectively. A graph G is said to be **cordial**, if there exists a binary vertex labeling f of G such that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$ and f is called a **cordial labeling** of G. Cahit [1] introduced the concept of cordial graphs as a weaker version of both graceful and harmonious graphs. In the same paper he proved the following:

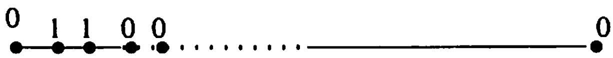
Theorem A: The complete graph K_n is cordial iff $n \leq 3$.

Theorem B: If G is an Eulerian graph with e number of edges, where $e \equiv 2(\bmod 4)$, then G has no cordial labeling.

In [1], we investigated the cordiality of $P_t(G)$, where G was cordial. It is a natural question whether $P_t(G)$ is cordial when G itself is not cordial. In this paper we prove that $P_t(K_{2n})$ is cordial for all n, t and $P_t(K_{2n+1})$ is cordial whenever the number of edges is not congruent to $2(\bmod 4)$.

For a binary labeling g of a graph G we introduce $t-1 = k$ vertices on every edge and label some of them as follows: write $k = 4q + r, r =$

0, 1, 2, 4. Every path given by the original edge uv of G can be written as $\{u, v_1, v_2, \cdots, v_{4q}, \cdots, v_k, v\}$ We define $f(u) = g(u), f(v) = g(v)$.



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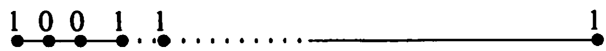

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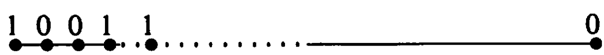

Type (c)

For the new vertices, we give three types of labelings shown above.
Type (a): If $f(u) = f(v) = 0$, define $f(v_i) = 1$ for $i \equiv 1, 2(\bmod 4)$ and $f(v_i) = 0$ for $i \equiv 0, 3(\bmod 4)$.
Type (b): If $f(u) = f(v) = 1$, define $f(v_i) = 0$ for $i \equiv 1, 2(\bmod 4)$ and $f(v_i) = 1$ for $i \equiv 0, 3(\bmod 4)$.
Type (c): If $f(u) = 1, f(v) = 0$, define $f(v_i) = 0$ for $i \equiv 1, 2(\bmod 4)$ and $f(v_i) = 1$ for $i \equiv 0, 3(\bmod 4)$.

We call this the **basic partial labeling** of the homeomorph $P_t(G)$.

Cordial Labeling of $P_t(K_{2n})$

Let g be a binary labeling of vertices of K_{2n} in which exactly n vertices have received label 0 and the remaining n vertices have received the label 1. Then $v_g(0) = v_g(1) = n$ and $e_g(0) = n^2 - n$ while $e_g(1) = n^2$. On each edge of K_{2n} introduce $t-1$ new vertices so as to obtain t- uniform homeomorph $P_t(K_{2n})$ of K_{2n}.

Theorem 1: $P_t(K_{2n})$ is cordial for all $t \geq 2$.
Proof: Let $k = t - 1 = 4q + r, r = 0, 1, 2, 3$. We have introduced k new vertices on each of the edges of K_{2n}. Let f be the basic partial labeling of $P_t(K_{2n})$ obtained from the labeling g given above.

We note that, of all the vertices and the edges labeled so far, exactly half have received the label 0 and half have received label 1. Now, the vertices yet to be labeled are $v_i, i > 4q$ on each path of type $\{u, v_1, v_2, \cdots, v_{k-1}, v_k, v\}$, where u and v are the original vertices of the complete graph K_{2n}.

Case (1): $r = 1$, i.e. $k = 4q + 1$. This means that exactly one vertex remains to be labeled on each of these paths.

There are exactly $(n^2-n)/2$ paths of type (a) and exactly $(n^2-n)/2$ paths of type (b). In both the types, $f(v_{4q}) = f(v)$. For these n^2-n paths, define $f(v_{4q+1}) = 0$. Similarly, there are exactly n^2 paths of type (c), for which $f(u) = 1 = f(v_{4q}), f(v) = 0$. For $n^2-n+\lfloor n/2 \rfloor$ of these paths, define $f(v_{4q+1}) = 1$ and for the remaining $\lceil n/2 \rceil$ paths, define $g(v_{4q+1}) = 0$.

It is easy to see that $e_f(0) = e_f(1) = 4nq + 2n^2 - n$. If n is even, $v_f(0) = v_f(1)$ and if n is odd, $v_f(0) = v_f(1) + 1$. Hence f is a cordial labeling of $P_{k+1}(K_{2n})$.

Case (2): $r = 2$, i.e. $k = 4q+2$. First we assume that n is not $\equiv 2(\bmod 4)$.

With the basic partial labeling f, we need only to label the vertices of type v_{4q+1}, v_{4q+2} on each path given by the edges of original copy of K_{2n}. For n^2-n paths of type (a) and (b) and n^2-n paths of type (c), define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$. At this stage, the map f has labeled $6(n^2-n)$ more edges and exactly half of them have received label 0 and the remaining half have received label 1.

Now out of the remaining n paths of type (c), we form two groups (A) and (B) of size $3s$ and s respectively, where $n = 4s + \alpha$. In group (A) define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$. In group (B), define $f(v_{4q+1}) = 0, f(v_{4q+2}) = 1$.

If $\alpha = 0$, then $e_f(0) = e_f(1)$ and f is a cordial labeling.

If $\alpha = 1$, there is an extra path of type (b). For this path, define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$. Then $e_f(0) = e_f(1) + 1$ and f is a cordial labeling.

If $\alpha = 3$, there are three additional paths of type (b) . For two of these paths, define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$, while for the third path, define $f(v_{4q+1}) = 0, f(v_{4q+2}) = 1$. Then $e_f(1) = e_f(0)+1$ and f is a cordial labeling.

Case (3): $r = 3$, i.e. $k = 4q + 3$. Let n be not $\equiv 1(\bmod 2)$.

With the basic partial labeling f, we need only to label the vertices of type $v_{4q+1}, v_{4q+2}, v_{4q+3}$ on each path.

For $(n^2-n)/2$ paths of type (a), define $f(v_{4q+1}) = 1 = f(v_{4q+2})$ and $f(v_{4q+3}) = 0$. For the $(n^2-n)/2$ paths of type (b), define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1$. On each of these paths, two more edges have received the label 0 and two edges have received the label 1.

Now there are n^2 paths of type (c) with $f(v_{4q}) = 1, f(v) = 0$. Since we are taking n to be even, n^2 is also even. For $n^2/2$ of these paths, let $f(v_{4q+1}) = 1 = f(v_{4q+2})$ and $f(v_{4q+3}) = 0$. Hence, on each of these paths, three edges get label 0 and one edge gets label 1. For the remaining $n^2/2$ paths, let $f(v_{4q+1}) = 0 = f(v_{4q+2})$ and $f(v_{4q+3}) = 1$. On each of these paths, one edge gets the label 0 and three edges get label 1, making f cordial.

Case (4): $r = 4$, i.e. $k = 4q + 4$. Let n be not $\equiv 2(\bmod 4)$.

With the basic partial labeling f, we need only to label the vertices of type $v_{4q+1}, v_{4q+2}, v_{4q+3}$ and v_{4q+4} on each path given by the edges of original copy of K_{2n}. For $(n^2 - n)/2$ paths of type (a), define $f(v_{4q+1}) = 1 = f(v_{4q+2})$ and $f(v_{4q+3}) = 0 = f(v_{4q+4})$. For the $(n^2-n)/2$ paths of type (b), define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1 = f(v_{4q+4})$. On each of these paths, three edges receive the label 0 and two receive the label 1.

There are now n^2 paths of type (c). For $n^2 - n$ of these paths, let $f(v_{4q+1}) = 0 = f(v_{4q+2})$ and $f(v_{4q+3}) = 1 = f(v_{4q+4})$. Hence, on each of these paths, two edges get label 0 and three edges get label 1. So the edge labels have balanced out so far.

For the remaining n paths of type (c), form two groups (A) and (B) of sizes $3s$ and s respectively, where $n = 4s + \alpha$. For each path in group (A), let $f(v_{4q+1}) = 0 = f(v_{4q+2})$ and $f(v_{4q+3}) = 1 = f(v_{4q+4})$. For each path in group (B), let $f(v_{4q+1}) = 1 = f(v_{4q+2}), f(v_{4q+3}) = 0 = f(v_{4q+4})$.

If $\alpha = 0$, the map f is cordial. If $\alpha = 1$, one path of type (b) still remains. For this path, define $f(v_{4q+1}) = 0 = f(v_{4q+2})$ and $f(v_{4q+3}) = 1 = f(v_{4q+4})$. Then one can see that $e_f(1) = e_f(0) + 1$ and f is a cordial labeling.

If $\alpha = 3$, three paths of type (b) remain. For two of these paths, define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1 = f(v_{4q+4})$ and for the remaining path, define $f(v_{4q+1}) = 1 = f(v_{4q+2}), f(v_{4q+3}) = 0 = f(v_{4q+4})$. It is clear that $e_f(0) = e_f(1) + 1$ and f is a cordial labeling.

There now remain three cases to be disposed of: (5) $k \equiv 2(\bmod 4)$, $n \equiv 2(\bmod 4)$, (6) $k \equiv 0(\bmod 4), n \equiv 2(\bmod 4)$, (7) $k \equiv 3(\bmod 4), n \equiv 1(\bmod 2)$. For these cases, we now proceed as follows: Let g be a binary labeling of K_{2n} which gives label 0 to $n + 1$ vertices and label 1 to the remaining $n - 1$ vertices of K_{2n}. Then $v_g(0) = n + 1, v_g(1) = n - 1, e_g(0) = n^2 - (n - 1)$ and $e_g(1) = n^2 - 1$. Let f be the basic partial labeling of $P_{k+1}(K_{2n})$ given by g. There are $(n^2+n)/2$ paths of type (a), $(n-1)(n-2)/2$ paths of type (b) and $n^2 - 1$ paths of type (c).

Case (5): Let $k = 4q + 2$ and $n = 4s + 2$. For $n^2 - (n - 1)$ paths of type (a) and (b) together, in which $f(v_{4q}) = f(v)$, define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$. On each of these paths, there is, now, one additional edge with label 0 and two edges with label 1.

There are now $n^2 - 1$ paths of type (c) with $f(v_{4q}) = 1, f(v) = 0$. On one of these paths, define $f(v_{4q+1}) = 1 = f(v_{4q+2})$.

If $n = 2$, there will be two paths remaining. For both of them, define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$. If $n \geq 3$, on $n^2 - n$ of these paths, define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$. On each of these paths, there is now one additional edge with label 1 and two additional edges with label 0.

There now remain $n - 2 = 4s$ paths of type (c) with $f(v_{4q}) = 1, f(v) = 0$. Form two groups $(A), (B)$ of these paths of sizes $3s, s$ respectively. For every path in the group (A), define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0$ and for each path in the group (B), define $f(v_{4q+1}) = 0, f(v_{4q+2}) = 1$. It is easy to see that $e_f(1) = e_f(0) + 1$. Hence f is a cordial labeling of $P_{k+1}(K_{2n})$.

Case (6): $k = 4q + 4$ and $n \equiv 2(\bmod 4)$.

We again start with the basic partial labeling f. There now remain on each path, the vertices of type $v_{4q+1}, \cdots, v_{4q+4}$ to be labeled. For $(n^2 + n)/2$ paths of type (a), define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 1, f(v_{4q+3}) = f(v_{4q+4}) = 0$. On $(n - 1)(n - 2)/2$ paths of type (b), define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1 = f(v_{4q+4})$.

There now remain $n^2 - 1$ paths of type (c). On one of these paths, define $f(v_{4q+1}) = 0, f(v_{4q+2}) = 1 = f(v_{4q+3}) = f(v_{4q+4})$. If $n = 2$, on the remaining two paths, define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1 = f(v_{4q+4})$.

If $n > 2$, for $n^2 - n$ of these paths, define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1 = f(v_{4q+4})$. Out of the remaining $n - 2$ paths, form two groups $(A), (B)$ of sizes $3s, s$ respectively, where $n = 4s + 2$. For each path in the group (A), define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1 = f(v_{4q+4})$. For each path in the group (B), define $f(v_{4q+1}) = 1 = f(v_{4q+2}), f(v_{4q+3}) = 0 = f(v_{4q+4})$. One can easily see that now $e_f(0) = e_f(1)$ and hence f is a cordial labeling.

Case (7): $k = 4q + 3$ and $n \equiv 1(\bmod 2)$. We start by the basic partial labeling. On each path, there now remain three vertices to be labeled. On each of the $(n^2 - n + 2)/2$ paths of type (a), define $f(v_{4q+1}) = 1 = f(v_{4q+2}), f(v_{4q+3}) = 0$. For the remaining $n - 1$ paths of type (a), with $f(v_{4q}) = 0 = f(v)$. For these paths define $f(v_{4q+1}) = 1, f(v_{4q+2}) = 0 = f(v_{4q+3})$.

On each of the $(n - 1)(n - 2)/2$ paths of type (b), define $f(v_{4q+1}) = 0 = f(v_{4q+2}), f(v_{4q+3}) = 1$.

Out of $n^2 - 1$ paths of type (c) for which $f(v_{4q}) = 1, f(v) = 0$, take $(n^2 - 1)/2$ paths and define $f(v_{4q+1}) = 1 = f(v_{4q+2}), f(v4q + 3) = 0$ and for the remaining $(n^2 - 1)/2$ paths, define $f(v_{4q+1}) = f(v_{4q+3}) = 0, f(v_{4q+2}) = 1$. One can see that $v_f(0) = v_f(1) + 1$ and $e_f(0) = e_f(1)$. Hence f is a cordial labeling. □

Cordial Labeling of $P_t(K_{2n+1})$

We now discuss the cordiality of the t-uniform homeomorph $P_t(K_{2n+1})$ of the complete graph K_{2n+1}. Let $V(K_{2n+1})$ and $E(K_{2n+1})$ denote respectively, the vertex and edge set of K_{2n+1}. Then, $|V(P_t(K_{2n+1}))| = 2n+1+(t-1)n(2n+1)$ and $e = |E(P_t(K_{2n+1}))| = tn(2n+1)$.

Theorem 2: The t-uniform homeomorph, $P_t(K_{2n+1}), t > 1$, of of the complete graph K_{2n+1} is cordial iff (i) $t \equiv 0(\mod 4)$ OR (ii) t is odd and n is not $\equiv 2(\mod 4)$ OR (iii) n is even and $t \equiv 2(\mod 4)$.
Proof: Clearly, $P_t(K_{2n+1})$ is Eulerian. Moreover, $e \equiv 2(\mod 4)$, whenever $n \equiv 2(\mod 4)$ and t is odd or $t \equiv 2(\mod 4)$ and n is odd. Hence by Theorem B, $P_t(K_{2n+1})$ is not cordial in these cases.

We now proceed to show that in all the remaining cases $P_t(K_{2n+1})$ is cordial. If e is not $\equiv 2(\mod 4)$, then either (i) $t \equiv 0(\mod 4)$ OR (ii) t is odd and n is not $\equiv 2(\mod 4)$ OR (iii) n is even and $t \equiv 2(\mod 4)$.

Let g be a binary labeling of K_{2n+1} which assigns label 0 to $n+1$ vertices and label 1 to remaining n vertices. Hence, $e_g(0) = (n^2+n)/2 + (n^2-n)/2 = n^2$ and $e_g(1) = n^2+n$. Let $k = t-1$. Write $k = 4q+r, r = 1,2,3,4$. Let f be the basic partial labeling given by g.

Of all the edges that have been labeled thus far by f, exactly half have received label 0 and half have received label 1. More over, thus far $\left\lceil \frac{(2n+1)(4qn+1)}{2} \right\rceil$ vertices have received label 0 and $\left\lfloor \frac{(2n+1)(4qn+1)}{2} \right\rfloor$ vertices have received label 1. We need now label only the vertices $v_i, i > 4q$, on each path of $P_t(K_{2n+1})$. There are $n(n+1)/2$ paths of type (a), $n(n-1)/2$ paths of type (b) and n^2+n paths of type (c).

Case (1): $r = 1$, i.e. $t \equiv 2(\mod 4)$ and n is even.

In this case only one vertex remains to be labeled on each path. For each path of type (b), let $f(v_k) = 0$. For $n^2/2$ paths of type (a), define $f(v_k) = 0$ while for the remaining $n/2$ paths of type (a), define $f(v_k) = 1$.

For n^2 paths of type (c), define $f(v_k) = 1$ and for the remaining n paths of type (c), define $f(v_k) = 0$.

Then, $v_f(0) = (3n)/2 + (n^2+1) + 2qn(2n+1) = v_f(1) + 1$ and $e_f(0) = (3n^2)/2 + n/2 + 2qn(2n+1) = e_f(1)$. Hence f is a cordial labeling.

Case (2): $r = 2$, i.e. $t \equiv 3(\mod 4)$ and n is not $\equiv 2(\mod 4)$.

For all the paths of type (a) and (b) and for n^2 paths of type (c), define $f(v_{k-1}) = 1, f(v_k) = 0$.

For the remaining n paths of type (c), let $n = 4s + \alpha, \alpha = 0, 1, 3$.

For $3s$ of these paths, let $f(v_{k-1}) = 1, f(v_k) = 0$. On the remaining s paths, let $f(v_{k-1}) = 0, f(v_k) = 1$.

If $\alpha = 0$, one can see that $v_f(0) = v_f(1) + 1$ and $e_f(0) = e_f(1)$.

If $\alpha = 1$, on the remaining path define $f(v_{k-1}) = 1, f(v_k) = 0$.

If $\alpha = 3$, on the two of the remaining paths, let $f(v_{k-1}) = 1, f(v_k) = 0$ and on the third path, let $f(v_{k-1}) = 0, f(v_k) = 1$. It can be easily verified that in this case $v_f(0) = v_f(1) + 1$ and $e_f(0) = e_f(1) + 1$. Hence f is a cordial labeling.

Case (3): $r = 3$, i.e. $t \equiv 0(\bmod 4)$ and n is even.

On the $n(n-1)/2$ paths of type (b), $n/2$ paths of type (a) and $n(n+1)/2$ paths of type (c), define $f(v_{k-2}) = 1, f(v_{k-1}) = 0 = f(v_k)$.

There remain $n^2/2$ paths of type (a). On each of them, define $f(v_{k-2}) = 1 = f(v_{k-1}), f(v_k) = 0$. For the remaining $n(n+1)/2$ paths of type (c), define $f(v_{k-2}) = 1 = f(v_k), f(v_{k-1}) = 0$. One can easily see that this is a cordial labeling.

Case (4): $r = 4$, i.e. $t \equiv 1(\bmod 4)$ and n is not $\equiv 2(\bmod 4)$.

For all the paths of type (b) and n^2 paths of type (c), let $f(v_{k-3}) = 0 = f(v_{k-2}), f(v_{k-1}) = 1 = f(v_k)$. For all the paths of type (a), define $f(v_{k-3}) = 1 = f(v_{k-2}), f(v_{k-1}) = 0 = f(v_k)$.

For the remaining n paths of type (c), let $n = 4s + \alpha, \alpha = 0, 1, 3$. For $3s$ of these paths, define $f(v_k - 3) = 0 = f(v_{k-2}), f(v_{k-1}) = 1 = f(v_k)$. On s paths, define $f(v_{k-3}) = 1 = f(v_{k-2}), f(v_{k-1}) = 0 = f(v_k)$,

If $\alpha = 1$, for the remaining path, define $f(v_{k-3}) = 0 = f(v_{k-2})$, $f(v_{k-1}) = 1 = f(v_k)$. If $\alpha = 3$, for two of the remaining three paths, let $f(v_{k-3}) = 0 = f(v_{k-2}), f(v_{k-1}) = 1 = f(v_k)$ and for the last path, let $f(v_{k-3}) = 1 = f(v_{k-2}), f(v_{k-1}) = 0 = f(v_k)$. Then f is easily seen to be cordial.

Case (5): $r = 3$, i.e. $t \equiv 0(\bmod 4)$ and n is odd.

In this case, we have to change the basic binary labeling of K_{2n+1}. Let g be a binary labeling of K_{2n+1} in which $n+2$ vertices receive label 0 and $n-1$ vertices receive label 1. Then $v_g(0) = n+2, v_g(1) = n-1, e_g(0) = (n+2)(n+1)/2 + (n-1)(n-2)/2 = n^2 + 2, e_g(1) = n^2 + n - 2$.

Let f be the basic partial labeling given by g. Of the vertices labeled so far by f, exactly $n + 2 + 2qn(2n+1)$ vertices have received the label 0 and $n - 1 + 2qn(2n+1)$ vertices have received the label 1. We need now label only the vertices of type v_{k-2}, v_{k-1}, v_k on each path.

There are $(n+2)(n+1)/2$ paths of type (a), $(n-1)(n-2)/2$ paths of type (b) and $n^2 + n - 2$ paths of type (c).

For exactly half the paths of type (c), define $f(v_{k-2}) = 1 =$

$f(v_{k-1}), f(v_k) = 0$. On the remaining half, let $f(v_{k-2}) = 0 = f(v_{k-1})$, $f(v_k) = 1$.

In all, there are $n^2 + 2$ paths of type (a) and (b) together. On $(n^2 - 1)/2 + 3$ of these, let $f(v_{k-2}) = 1 = f(v_{k-1}), f(v_k) = 0$. On the $(n^2 - 1)/2$ remaining paths, let $f(v_{k-2}) = 1, f(v_{k-1}) = 0 = f(v_k)$.

One can easily see that $v_f(0) = 2nq(2n + 1) + 3n^2 + (5n + 1)/2 = v_f(1)$ and $e_f(0) = 2nq(2n + 1) + 4n^2 + 2n = e_f(1)$. Hence f is a cordial labeling of $P_t(K_{2n+1})$. □

Remark: In [1] as well as in this paper on cordiality of t-uniform homeomorphs we have taken a fixed t. This is not necessary. We can take a graph G with edges $e_1, \cdots, e_q$ along with some positive integers $t_1, \cdots, t_q$. If we replace the edge e_j by a path of length $t_j, 1 \leq j \leq q$ and if $t_\alpha \equiv t_\beta \bmod 4$ for all $1 \leq \alpha, \beta \leq q$, then all the results in teses two papers are valid. The same proofs are valid.

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Some applications of combinatorial designs to extremal graph theory

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ABSTRACT. In this paper, we give a few applications of combinatorial design theory to a few problems in extremal graph theory. Using known results in combinatorial design theory, we have unified, simplified, and extended results on a few problems.

1 Introduction and Definitions

In this short paper, we give a few applications of combinatorial design theory to extremal graph theory. Before we proceed, we need some basic terminology.

Let G be an additively written group of order n. A k-subset D of G is a (n, k, λ)-*difference packing* if every nonzero element of G has at most λ representations as a difference $d - d'$ with elements from D. The difference packing is *abelian*, *cyclic* etc., if the group G has the respective property.

A (m, n, k, λ) *relative difference set (RDS)* R in an abelian group G relative to a normal subgroup N is a k-subset of G with the following property: the list of differences $a - b$ with distinct elements $a, b \in R$ contains each element in $G \setminus N$ exactly λ times. Moreover, no (non-identity) element in N has such a representation. Therefore N is called the *forbidden subgroup*. The meaning of the parameters m and n is as follows: the order of the group G is mn; the order of N is n. Note that each coset of N contains at most one element from R.

The following series are known to exist. For references, see [15].

1. $(p^2 + p + 1, 1, p + 1, 1)$-RDS exists whenever p is a prime power.

2. (p^a, p^b, p^a, p^{a-b})-RDS exists whenever p is a prime and $a \geq b \geq 0$.

3. $(q+1, q-1, q, 1)$-RDS exists for all prime powers q.

4. For any prime power q and any divisor d of $q-1$, $(q+1, \frac{q-1}{d}, q, d)$-RDS exists.

It is easy to see that a (m, n, k, λ)-RDS is a (mn, k, λ) difference packing. Series 1 is the well-known difference set for desarguesian finite projective plane of order p. Series 3 was obtained by Bose [3] in 1942 by proving that the affine plane of order q is 1-rotational. Series 4 can be obtained from Series 3, since the relative difference set $(q+1, q-1, q, 1)$ exists over Z_{q^2-1}. There are many more known series of relative difference sets. We refer the reader to [15] for a survey.

A *transversal design of order* n or *group size* n, *block size* r, and *index* λ, denoted $\mathrm{TD}_\lambda(r, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where:

(1) V is a set of rn elements;

(2) $\mathcal{G}$ is a partition of V into r classes (called *groups*), each of size n;

(3) $\mathcal{B}$ is a collection of r-subsets of V (called *blocks*);

(4) every unordered pair of elements from V is either contained in exactly one group and in no blocks, or is contained in exactly λ blocks, but not both.

It is easy to see that there are λn^2 blocks in a $\mathrm{TD}_\lambda(r, n)$.

A *Steiner system* $S(2, l, v)$ is a pair $(V, \mathcal{B})$, where V is a v-set, whose elements are called *points*, and $\mathcal{B}$ is a family of l-subsets of V called *blocks*, such that any two distinct points belong to exactly one block. An $S(2, l, v)$ is said to be *resolvable* if the blocks can be partitioned into classes so that every point appears once in each class.

2 Turán type problems

Given a family of r-uniform hypergraphs (or r-graphs) $\mathcal{F}$, we say that an r-graph $\mathcal{G}$ is $\mathcal{F}$-free if $\mathcal{G}$ contains no subhypergraph isomorphic to any element in $\mathcal{F}$. Let $\mathrm{ex}(n, F)$ be the maximum number of edges in an n vertex $\mathcal{F}$-free r-graph. If $\mathcal{F} = \{K_k^{(r)}\}$, the complete r-graph on k vertices, then $\mathrm{ex}(n, \mathcal{F})$ is the Turán number $t_r(n, k)$. The determination of $\lim_{n\to\infty} \frac{t_r(n,k)}{\binom{n}{r}}$ is perhaps the most fundamental open problem in extremal hypergraph theory. We consider the related question of determining $\mathrm{ex}(n, \mathcal{F})$ when $\mathcal{F} \neq \{K_k^{(r)}\}$. In this section, we give a unified treatment of some known results using difference packings.

First we consider the case when $r = 2$ and $F = K_{2,t+1}$.

Theorem 2.1. *Suppose there exists a (n, k, t) difference packing, D, over G. Let α be the number of $g \in G$ such that $2g \in D$. Then there exists a graph with $\frac{nk-\alpha}{2}$ edges on n vertices which does not contain any $K_{2,t+1}$.*

Proof: Construct a $|G| \times |G|$ adjacency matrix, A of the graph G, by $A(i, j) = 1$ if and only if $i + j \in D$ where $i, j \in G$ and $i \neq j$. We show that this graph does not contain any $K_{2,t+1}$. Suppose there exists a $K_{2,t+1}$ on $\{x, y, a_1, a_2, \ldots, a_{t+1}\}$. Then there exists $d_1, d_2, \ldots, d_{2t+2} \in D$, such that $x + a_i = d_i$ for $i = 1, 2, \ldots, t + 1$, and $y + a_i = d_{t+1+i}$ for $i = 1, 2, \ldots, t + 1$. Clearly, $d_1 \neq d_2 \neq \cdots \neq d_{t+1}$ and $d_{t+2} \neq d_{t+3} \neq \cdots \neq d_{2t+2}$. Also, we must have $x = d_{t+1+i} - d_i$ for $i = 1, 2, \ldots, t + 1$. Hence, D is not a (v, k, t) difference packing, a contradiction. The number of edges in the graph can be counted in an obvious manner. □

Corollary 2.2. (Erdös, Rényi and Sós [6] and Brown [4]) $ex(K_n, C_4) = \frac{1+o(1)}{2} n^{1.5}$.

Proof: The lower bound can be obtained by applying Theorem 2.1. If p is a prime power, then there exists a $(p^2+p+1, 1, p+1, 1)$-RDS over Z_{p^2+p+1}. Clearly, $\alpha = p + 1$. Simple calculation yields the lower bound. The upper bound is proved by Kövári, Sós and Turán [9]. □

Corollary 2.3. (Füredi [7]) $ex(K_n, K_{2,t+1}) = \frac{1}{2} t^{0.5} n^{1.5} + O(n^{\frac{4}{3}})$.

Proof: If $q \equiv 1 \pmod t$, a $(q + 1, \frac{q-1}{t}, q, t)$-RDS exists, the lower bound can be obtained by applying Theorem 2.1. The upper bound is proved by Kövári, Sós and Turán [9]. □

In the remainder of this section, we consider the Turán numbers for r-partite r-graphs. Let r and t be fixed positive integers, $r \geq 2$. Let $\mathcal{H}$ be the complete r-partite r-graph $K^{(r)}(1, 1, \ldots, 1, 2, t + 1)$ consisting of $r - 2$ sets of size 1, one set of size 2, and one set of size $t + 1$.

Theorem 2.4. *Suppose there exists an $(n, k, t + 1)$ difference packing, D, over G. Then there exists an r-uniform hypergraph on n points avoiding $\mathcal{H}$, with the number of hyperedges at least $\frac{k}{n}\binom{n}{r}$.*

Proof: The hypergraph has vertex set G. Any hyperedge of the form $\{x_1, x_2, \ldots, x_r\}$ is in the hypergraph if and only if $x_1 + x_2 \cdots + x_r \in D$ and $x_1 \neq x_2 \neq \cdots \neq x_r$. It is easy to see that the hypergraph does not contain any $K^{(r)}(1, 1, \ldots, 1, 2, t + 1)$. Suppose there exists a r-uniform r-paritite hypergraph $K^{(r)}(1, 1, \ldots, 12, t + 1)$. Consider the 2-graph induced by the $r - 2$ parts of size 1 in the $K^{(r)}(1, 1, \ldots, 1, 2, t + 1)$. Then, the resulting 2-graph would contain a $K_{2,t+1}$, which is impossible by Theorem 2.1. There are in total $\binom{n}{r}$ hyperedges in the hypergraph. So, we can translate the difference packing from D to $D + i$, which has the maximum number of hypergraphs. Hence, we obtain the required lower bound on the number of hyperedges. □

Theorem 2.5. (Muyabi) [13] *If $q \equiv 1 \pmod t$ is a prime power, then there exists a r-uniform hypergraph on $n = \frac{q^1-1}{t}$ points with at least $\frac{q+1}{n}\binom{n}{r}$ hyperedges, such that the hypergraph avoids any r-uniform r-partite graph $K^{(r)}(1,1,\ldots,1,2,t+1)$.*

Proof: There exists a D, $(\frac{q^2-1}{t}, q, t)$-difference packing, by the existence of RDS. The results follow by Theorem 2.4 □

3 Multicolor Ramsey theory

The multicolor Ramsey number $r_k(G)$ is the smallest integer n for which any k-coloring of the edges of the complete graph K_n must produce a monochromatic 4-cycle. It was proved by Chung and Graham [5] that $r_k(C_k) \geq k^2 - k + 2$ whenever $k-1$ is a prime power. It has recently been shown by Lazebnik and Woldar [10] that $r_k(C_k) \geq k^2 + 2$ when k is an odd prime power. In this section, we give a uniform treatment to both constructions. As a by-product, we give an extension of Lazebnik and Woldar's construction.

Theorem 3.1. *Let G be an abelian group of order n. Suppose $G = D_1 \cup D_2 \cup \cdots \cup D_k$ such that each D_i is a (n, k_i, t) difference packing over G where $|G| = n$. Then, $r_k(K_{2,t+1}) \geq n+1$.*

Proof: Since D_i is a $(n, k, t+1)$-difference packing, we can construct a graph on vertex set C with edges $\{x, y\}$ if and only if $x + y \in D_i$. It was shown in Theorem 2.1 that the graph obtained in this manner does not contain any $K_{2,t+1}$. Since D_i partitions G, if we obtain k graphs using $D_1, D_2, \ldots, D_k$ in turn, we can partition the edge of K_n. Hence, $r_k(K_{2,t+1}) \geq n+1$. □

The generalization of the above lemma to hypergraphs is immediate, and thus omitted.

The following is immediate from the partition of Z_{k^2-k+1} into difference packings.

Corollary 3.2. [5] *$r_k(C_4) \geq k^2 - k + 2$ when $k-1$ is a prime power.*

Corollary 3.3. *$r_k(C_4) \geq k^2 + 1$ for all prime powers k.*

Proof: Let D be a $(k, k, k, 1)$-RDS over G. Let N be the normal subgroup in the RDS. It is clear that $\{\{D+i\} : i \in N\}$ partitions all elements in G. The result follows from Theorem 3.1. □

In a certain situations, we can further increase the bound by 1.

Theorem 3.4. *$r_k(C_4) \geq q^2 + 2$ when q is a prime power.*

Proof: It suffices to exhibit a q-coloring of the edges of K_{q^2+1} in which there is no monochromatic C_4. Fix a vertex v of K_{q^2+1} and denote the subgraph induced by its set of neighbors by G. As G is isomorphic K_{q^2}, we can q-color its edges in Corollary 3.3. Thus, it remains only to color the edges $\{u, v\}$ for $u \in V(G)$, and we do this by simply assigning color i to the edges of the form $\{v, x\}$ where x is in the ith coset of the normal subgroup in the $(q, q, q, 1)$-RDS. Suppose, by way of contradiction, that a monochromatic 4-cycle exists, say of color j. Clearly v must be one of its vertices, so denote the consecutive vertices of this cycle by v, w, x, y. Then w and y are in the same coset of the normal subgroup. Also, $w+x = d_1+j$ and $y+x = d_2+j$, where $d_1, d_2 \in D$, $d_1 \neq d_2$ and $j \in N$, where N is the normal subgroup. Then, we have $w-y = d_1-d_2$. Since w and y are in the same coset, $w-y$ must be in the normal subgroup. Hence, D is not a relative difference set. □

The proof of the above theorem when p is odd was first proved by Lazebnik and Woldar. We have extended the result to the case in which p is an even prime power.

4 Splittable colorings of graphs and hypergraphs

Let $\mathcal{K}_n^k$ denote the complete k-uniform hypergraph on n vertices; i.e. we have a ground set of n elements and we take all the k-sets to be edges. Often, we will denote the ground set $\{1, 2, \ldots, n\}$ by $[n]$. An r-coloring of $\mathcal{E}(\mathcal{K}_n^k)$ is a coloring of the k-sets with r colors.

Given a coloring of the vertices and edges of $\mathcal{K}_n^k$, a *totally monochromatic m-clique*, for $k \leq m \leq n$, is a $\mathcal{K}_m^k$ whose edges (k-sets) all get the same color.

An r-coloring of $\mathcal{E}(\mathcal{K}_n^k)$ is *(r, m)-splittable* if there is a coloring of the ground set with r colors so that no totally monochromatic m-clique is produced.

Let $f_r^k(m)$ be the minimum n for which there is an r-colorring of $\mathcal{E}(\mathcal{K}_n^k)$ which is not (r, m)-splittable. This means that we can find an r-colorring of the k-sets of $\mathcal{K}_n^k$ with the property that every r-vertex coloring produces a totally monochromatic m-clique. An r-colorring of $\mathcal{E}(\mathcal{K}_n^k)$ where every set of size $\lceil \frac{n}{r} \rceil$ has a monochromatic m-clique in every color is such a non-(r, m)-splittable coloring and will be called an *(r, m)-balanced* coloring. $g_n^k(m)$ is then defined as the minimum n for which there is an r-colorring $\mathcal{E}(\mathcal{K}_n^k)$ is defined as the minimum n for which there is an r-colorring of $\mathcal{E}(\mathcal{K}_n^k)$ which is (r, m)-balanced.

Füredi and Ramamurthi [8] have defined and obtained various bounds on $f_r^k(m)$ and $g_r^k(m)$. The upper bound construction in [8] is different when $k = 2$ and when $k \geq 3$. The construction when $k \geq 3$ is fairly complicated. The purpose of this note is to give a unified construction for all $k \geq 2$.

Furthermore, the upper bound that is given in the construction is slightly better when $k \geq 3$.

Using an algebraic construction, Füredi and Ramamurthi [8] proved the following theorem.

Theorem 4.1. *$f_r^k(m) \leq g_r^k(m) \leq \frac{q^2-1}{t}$ where $q \equiv 1 \pmod{t}$ is a prime power such that $q \geq r(m+1) - 1$ and $t < k$.*

Theorem 4.2. *If there exists a $TD_{k-1}(r, v)$ and $v > \frac{r(m-1)}{k-1}$, then $f_r^k(m) \leq g_r^k(m) \leq v^2(k-1)$.*

Proof: Let $n = v^2(k-1)$. We will show that $\mathcal{K}_n^k$ has an (r, m)-balanced coloring. The vertices of $\mathcal{K}_n^k$ is the block in the transversal designs. For any point in group j, x in the transversal designs, let $A_1, A_2, \ldots, A_{(k-1)v}$ be all blocks such that $x \in A_i$ for $i = 1, 2, \ldots, (k-1)v$. We color all k subsets of $A_1, A_2, \ldots, A_{(k-1)v}$ in color i. We color all remaining hyperedges arbitrarily. First, we need to show that any k subsets receive at most one color. Suppose $B_1, B_2, \ldots, B_k$ are k blocks in the transversal design which receive two colors. Then there must be a pair of point y and z such that y, z are all in of the k blocks $B_1, B_2, \ldots, B_k$. Hence, it contradicts the fact that any pair of points are on at most $k-1$ blocks.

Let S be a set of $|S|$ points in $\mathcal{K}_n^k$. By interchanging the roles of points and blocks, these $|S|$ points become $|S|$ blocks in the transversal designs. Consider color class i. Since every block intersects group i, if $|S| \geq \lceil \frac{n}{r} \rceil > (m-1)v$, some point in group i must be on at least m blocks. Then these m blocks intersecting in a point in group i define a totally monochromatic clique of size m of color class i. □

By taking multiple copies of an affine plane of order q, $TD_{k-1}(q, q)$ exists for all prime powers q. Hence, we have the following corollary.

Corollary 4.3. *$f_r^k(m) \leq g_r^k(m) \leq (k-1)q^2$ where q is a prime of a power such that $q > \frac{r(m-1)}{k-1}$.*

$TD_{k-1}(r, v)$ is known to exist when v is sufficiently large (see [2]). Hence, we have the following.

Corollary 4.4. *For fixed r and a sufficiently large m, $f_r^k(m) \leq g_r^k(m) \leq (\lfloor \frac{r(m-1)}{k-1} \rfloor + 1)^2(k-1)$.*

5 Certain coloring of $K_{n,n}$ avoiding monochromatic C_4

In this section, we study a problem considered by Axenovich, Füredi, and Mubayi [1] on the generalized Ramsey theory.

Given graphs G and H, a coloring of $E(G)$ is called (H, q)-coloring if the edges of every copy of $H \subset G$ together receive at least q colors. Let $r(G, H, q)$ denote the minimum number of colors in an (H, q)-coloring of G.

Axenovich, Füredi, and Mubayi proved the following.

Theorem 5.1. *If n is odd, then $r(K_{n,n}, C_4, 3) \leq n$. If n is even, then $r(K_{n,n}, C_4, 3) \leq n+1$.*

They commented that improving these upper bound is very difficult. Eichhorn improved it by one when $n = 4, 12, 20, 36, 60$ by exhibiting $(C_4, 3)$-colorings of $K_{n,n}$ with n colors. We apply a known Latin square result to improve the result.

Theorem 5.2. *If $n \geq 3$, $r(K_{n,n}, C_4, 3) \leq n$.*

Proof: Eichhorn has proved the theorem for instances in which $n = 4$. Suppose $n \geq 3$, $n \neq 4$. There exists a Latin square of order n which does not contain a subsquare of order 2 [12]. The (i, j)th entry in the Latin square represents the color of $x_i y_j$, where the partite sets of G are $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. □

Axenovich, Füredi and Mubayi also proved the following lower bound on $r(K_{n.n}, C_4, 3)$.

Theorem 5.3. $r(K_{n,n}, C_4, 3) > \lfloor \frac{2n}{3} \rfloor$.

In proving the lower bound, they did not make use of the fact that the C_4 can contain an alternating C_4, a 2-colored C_4 whose edges alternate between its two colors when viewed cyclically.

Hence, Axenovich, Füredi, and Mubayi defined the following: a *weak* $(C_4, 3)$-coloring of $K_{n,n}$ is a coloring of the edges of $K_{n,n}$ in which every copy of C_4 has at least three colors or is alternately 2-colored. Let $r'(K_{n,n}, C_4, 3)$ denote the minimum number of colors in a weak $(C_4, 3)$-coloring of $K_{n,n}$. Using a sophisticated theorem of Pippenger and Spencer [14] and the probabilistic method, they proved the following.

Theorem 5.4. *As $n \to \infty$, $r'(K_{n,n}, C_4, 3) \leq \frac{3n}{4}(1 + o(1))$.*

A simple use of resolvable designs can lead to the following construction.

Theorem 5.5. *If there exists a weak C_4 coloring on $K_{n,n}$ with r color classes and a resolvable $S(2, n, v)$, then there exists a weak C_4 coloring on $K_{v,v}$ with $\frac{r(v-1)}{n-1} + 1$ color classes.*

Proof: Let C be the coloring on $K_{n,n}$ with r colors. By taking edges from other color classes, we can obtain a coloring on $K_{n,n}$ with $r + 1$ colors such that one of the color class is a 1-factor on $K_{n,n}$. Taking a resolvable $S(2, n, v)$ on V, we will color the edges of $V \times \{0, 1\}$. Our method is to give r colors to each parallel class in the Steiner system. For every parallel class with blocks $B_1, B_2, \ldots, B_{\frac{v}{n}}$, we color the edges on $B_i \times \{0, 1\}$ by the color class of $K_{n,n}$. Repeat the same procedure for every parallel class. Since B_i in each parallel class is disjoint, each color class is clearly a star. Finally, we add a new parallel class corresponding to the 1-factor. In total, we have

r color classes from each of the $\frac{v-1}{n-1}$ parallel classes and 1 color class for a 1-factor. Hence, we obtain the required number of color classes. It is not too difficult to check that the required conditions in [1] are satisfied. For details, we refer the reader to [1]. □

The above construction basically shows that if one can find a good coloring with few colors when n is small, one can then obtain a good upper bound asymptotically. Due to the complexity of the proof, we refer the reader to [11]. In fact, using other techniques from combinatorial design theory, it is indeed possible the prove the following.

Theorem 5.6. [11] *There exists a constant C such that $r'(K_{n,n}, C_4, 3) \leq \frac{2n}{3} + C$ for all $n \geq 1$.*

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