Degree sequence conditions for super-edge-connected graphs and digraphs

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Abstract

A graph or a digraph G is called super-edge-connected or super-λ, if every minimum edge cut consists of edges adjacent to or from a vertex of minimum degree. Clearly, if G is super-λ, then $\lambda(G) = \delta(G)$, where $\delta(G)$ is the minimum degree and $\lambda(G)$ is the edge-connectivity of G.

In this paper degree sequence conditions for graphs and digraphs as well as for bipartite graphs and digraphs to be super-λ are presented.

Keywords: *super-edge-connected graphs, edge-connectivity, degree sequence*

1. Introduction

We consider finite graphs and digraphs without loops and multiple edges. For a vertex $v \in V(D)$ of a digraph D, the *degree* of v, denoted by $d(v)$, is defined as the minimum value of its out-degree $d^+(v)$ and its in-degree $d^-(v)$. The *degree sequence* of D is defined as the nonincreasing sequence of the degrees of the vertices of D. For two sets $X, Y \subset V(D)$ let (X, Y) be the set of arcs from X to Y. A graph or a digraph G is called *super-edge-connected* or *super-λ*, if every minimum edge cut is trivial, that means, that every minimum edge cut consists of edges adjacent to or from a vertex of minimum degree. Clearly, if G is super-λ, then $\lambda(G) = \delta(G)$, where $\delta(G)$ is the minimum degree and $\lambda(G)$ is the edge-connectivity of G. For other graph theory terminology we follow Chatrand and Lesniak [3].

Sufficient conditions for graphs and digraphs to be super-λ were given by several authors, for example: Kelmans [8], Lesniak [9], Boesch and Tindell [1], Fàbrega and Fiol [4], [5], Fiol [6], and Soneoka [10]. In this paper, in particular, degree sequence conditions for graphs and digraphs as well as for bipartite graphs and digraphs to be super-λ are given. Different examples will show that these conditions are best possible.

2. Degree sequence conditions for arbitrary graphs and digraphs

We start with a simple but useful lemma.

Lemma 2.1. Let D be a digraph of edge-connectivity λ. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \max\{2, \delta(D)\}$.

Proof. Clearly, by the hypothesis, $|X|, |Y| \geq 2$. Now let $\delta = \delta(D) \geq 3$ and suppose, without loss of generality, that $2 \leq |X| \leq \delta - 1$. Then we obtain the contradiction

$$|X|\delta \leq \sum_{x \in X} d^+(x) \leq |X|(|X|-1) + \lambda \leq (\delta - 1)(|X|-1) + \delta. \quad \square$$

Corollary 2.2 (Fiol [6]) If D is a digraph of order n with $\delta(D) \geq (n+1)/2$, then D is super-λ.

Corollary 2.3 (Kelmans [8]) If G is a graph of order n with $\delta(G) \geq (n+1)/2$, then G is super-λ.

Proof. Given the graph G, define the digraph D on the vertex set $V(G)$ by replacing each edge of G by two arcs in opposite directions and apply Corollary 2.2 to D. $\square$

Theorem 2.4 Let D be a digraph of order n with edge-connectivity λ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. If $\delta \geq \lfloor n/2 \rfloor + 1$ or if $\delta \leq \lfloor n/2 \rfloor$ and

$$\sum_{i=1}^{k}(d_i + d_{n+i-\delta}) \geq k(n-2) + 2\delta + 1,$$

for some k with $1 \leq k \leq \delta$, then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. Then, according to Lemma 2.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \max\{2, \delta\}$. This leads to $\delta \leq \lfloor n/2 \rfloor$.
If $S \subseteq X$ and $T \subseteq Y$ are two k-sets with $1 \leq k \leq \delta$, then

$$\sum_{v \in S} d^+(v) \leq k(|X| - 1) + \lambda$$

and

$$\sum_{v \in T} d^-(v) \leq k(|Y| - 1) + \lambda$$

and thus, in total

$$\sum_{v \in S \cup T} d(v) \leq k(n-2) + 2\lambda. \quad (1)$$

Now choose S and T to contain the k vertices in X and Y of highest degree, respectively. Then $S \cup T$ contains the k vertices of highest degree but not the $\delta - k$ of lowest degree in D. Hence,

$$\sum_{v \in S \cup T} d(v) \geq \sum_{i=1}^{k} (d_i + d_{n+i-\delta}),$$

which yields together with (1) a contradiction to the hypothesis. □

Analogously to the proof of Corollary 2.3, Theorem 2.4 leads to an undirected version, which is related to a degree condition of Bollobás [2] for graphs with equal edge-connectivity and minimum degree.

Corollary 2.5 Let G be a graph of order n with edge-connectivity λ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. If $\delta \geq \lfloor n/2 \rfloor + 1$ or if $\delta \leq \lfloor n/2 \rfloor$ and

$$\sum_{i=1}^{k} (d_i + d_{n+i-\delta}) \geq k(n-2) + 2\delta + 1,$$

for some k with $1 \leq k \leq \delta$, then G is super-λ.

Example 2.6 The following example shows that Corollary 2.5 is best possible in the sense that $\sum_{i=1}^{k}(d_i + d_{n+i-\delta}) \geq k(n-2) + 2\delta$ for some $k \leq \delta$ does not guarantee that the graph is super-λ.

Let H_1 and H_2 be two copies of the complete graph K_p with $p \geq 5$, $V(H_1) = \{x_1, x_2, \ldots, x_p\}$, and $V(H_2) = \{y_1, y_2, \ldots, y_p\}$. We define the graph G_p as the union of H_1 and H_2 together with the new edges

$x_1y_1, x_1y_2, \ldots, x_1y_{p-3}$, x_2y_1, and x_2y_2. Then $n(G_p) = n = 2p$, $\delta(G_p) = \delta = p-1 = n/2-1$, $\lambda(G_p) = \delta(G_p)$, and

$$\sum_{i=1}^{\delta}(d_i + d_{n+i-\delta}) = 2\delta^2 + 2\delta = \delta(n-2) + 2\delta,$$

however, G_p is not super-λ.

If we replace each edge of G_p by two arcs in opposite direction, then we obtain a digraph, which shows that also Theorem 2.4 is best possible.

As a generalization of a result of Goldsmith and White [7] for graphs, Xu [11] has given in 1994 the following sufficient condition for equality of edge-connectivity and minimum degree of a digraph.

Theorem 2.7 (Xu [11]) Let D be a digraph of order n. If there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with $d(v_i) + d(w_i) \geq n$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, then $\lambda = \delta$.

Examples show that the conditions of Xu do not lead to super-λ digraphs, in general. However, we can prove the following related result.

Corollary 2.8 Let D be a digraph of order n with $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with $d(v_i) + d(w_i) \geq n$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$. If there exists as least one pair (v_j, w_j) such that $d(v_j) + d(w_j) \geq n + 1$ and $\min\{d(v_j), d(w_j)\} \leq d_{n+1-\delta}$, and if in the case that n is odd, the $\lfloor n/2 \rfloor$ pairs contain the δ vertices of lowest degree, then D is super-λ.

Proof. If $\delta \geq \lfloor n/2 \rfloor + 1$, then we are done by Theorem 2.4.

If $\delta \leq \lfloor n/2 \rfloor$, then from the $\lfloor n/2 \rfloor$ pairs of vertices choose δ pairs $(v_1', w_1'), (v_2', w_2'), \ldots, (v_\delta', w_\delta')$ containing the δ vertices of lowest degree of v_i and w_i as well as the pair (v_j, w_j) with $d(v_j) + d(w_j) \geq n + 1$. Since by the hypotheses such a choice is possible, in both cases n is odd and n is even, we conclude that

$$\begin{aligned} \sum_{i=1}^{\delta}(d_i + d_{n+i-\delta}) &\geq \sum_{i=1}^{\delta}(d(v_i') + d(w_i')) \\ &\geq \delta n + 1 = \delta(n-2) + 2\delta + 1. \end{aligned}$$

With $k = \delta$ in Theorem 2.4, this inequality implies that D is super-λ. □

The following examples will show that Corollary 2.8 as well as its undirected version are best possible.

Example 2.9 Let n be even such that $n = 2\delta + 2p$ with $\delta, p \geq 2$. Let H_1 be the complete graph K_δ with vertex set $\{x_1, x_2, \ldots, x_\delta\}$ and H_2 the complete graph $K_{n-\delta}$ with vertex set

$$\{y_1, y_2, \ldots, y_\delta\} \cup \{u_1, u_2, \ldots, u_p\} \cup \{v_1, v_2, \ldots, v_p\}.$$

We define the graph H as the union of H_1 and H_2 together with the edges $x_1y_1, x_2y_2, \ldots, x_\delta y_\delta$. Then $n(H) = n$ and $\delta(H) = \delta = \lambda(H)$. Furthermore, $d(x_i) + d(y_i) = n$ for $i = 1, 2, \ldots, \delta$ and $d(u_i) + d(v_i) \geq n + 1$ for $i = 1, 2, \ldots, p$. Clearly, the graph H is not super-λ.

Hence, we see that in the case that the order n is even, the condition that there exists one pair (v_j, w_j) with $d(v_j) + d(w_j) \geq n + 1$ such that $\min\{d(v_j), d(w_j)\} \leq d_{n+1-\delta}$ is necessary.

Now let n be odd such that $n = 2\delta - 1 + 2p$ with $\delta, p \geq 2$. Let H_1 be the complete graph K_δ with vertex set $\{x_1, x_2, \ldots, x_\delta\}$ and H_2 the complete graph $K_{n-\delta}$ with vertex set

$$\{y_1, y_2, \ldots, y_{\delta-1}\} \cup \{u_1, u_2, \ldots, u_p\} \cup \{v_1, v_2, \ldots, v_p\}.$$

We define the graph G as the union of H_1 and H_2 together with the edges $x_1y_1, x_2y_2, \ldots, x_{\delta-1}y_{\delta-1}$ and $x_\delta y_{\delta-1}$. Then $n(G) = n$ and $\delta(G) = \delta = \lambda(G)$. Furthermore, $d(x_i) + d(y_i) = n$ for $i = 1, 2, \ldots, \delta - 2$, $d(x_{\delta-1}) + d(y_{\delta-1}) = n + 1$, and $d(u_i) + d(v_i) \geq n + 1$ for $i = 1, 2, \ldots, p$. Clearly, the graph G is not super-λ.

Hence, we see that in the case that the order n is odd, the condition that the $\lfloor n/2 \rfloor$ pairs contain the vertices of degree $d_n, d_{n-1}, \ldots, d_{n+1-\delta}$, is necessary. In our example, the $\lfloor n/2 \rfloor$ pairs do not contain the vertex x_δ, and thus these pairs only contain the $\delta - 1$ vertices of lowest degree.

Next we present degree sequence conditions that consider only the lower end of the degree sequence.

Theorem 2.10 Let D be a digraph of order n and edge-connectivity λ with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 2$. If $\delta \geq \lfloor n/2 \rfloor + 1$ or if $\delta \leq \lfloor n/2 \rfloor$ and

$$\sum_{i=1}^{2k} d_{n+1-i} \geq \max\{k(n-1) + 1, (k-1)n + 2\delta + 1\}$$

for some k with $1 \leq k \leq \delta$, then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. Then, according to Lemma 2.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \delta$. This leads to $\delta \leq \lfloor n/2 \rfloor$.

Now let $S \subseteq X$ and $T \subseteq Y$ be two k-sets with $1 \leq k \leq \delta$. If we choose S such that the number of arcs of (X,Y) incident with S is minimal, then we conclude

$$\sum_{v \in S} d^+(v) \leq k(|X|-1) + \delta - \min\{\delta, |X|-k\}. \tag{2}$$

If we choose T such that the number of arcs of (X,Y) incident with T is minimal, then we conclude

$$\sum_{v \in T} d^-(v) \leq k(|Y|-1) + \delta - \min\{\delta, |Y|-k\}. \tag{3}$$

Case 1. Let $\delta \leq |X| - k$ and $\delta \leq |Y| - k$. The inequalities (2) and (3) imply the following contradiction to the hypothesis:

$$\begin{aligned}\sum_{i=1}^{2k} d_{n+1-i} &\leq \sum_{v \in S \cup T} d(v) \leq k(|X|-1) + k(|Y|-1) \\ &= k(n-2) < k(n-1) + 1\end{aligned}$$

Case 2. Let $\delta \leq |X| - k$ and $\delta \geq |Y| - k$. In view of Lemma 2.1, we have $-|Y| \leq -\delta$. Hence (2) and (3) lead to the following contradiction to the hypothesis:

$$\begin{aligned}\sum_{i=1}^{2k} d_{n+1-i} &\leq \sum_{v \in S \cup T} d(v) \leq k(|X|-1) + k(|Y|-1) + \delta - |Y| + k \\ &\leq kn - k + \delta - \delta = k(n-1) < k(n-1) + 1\end{aligned}$$

Case 3. The case $\delta \geq |X| - k$ and $\delta \leq |Y| - k$ can be proved analogously to Case 2.

Case 4. Let $\delta \geq |X| - k$ and $\delta \geq |Y| - k$. Then (2) and (3) yield to the following contradiction to the hypothesis:

$$\begin{aligned}\sum_{i=1}^{2k} d_{n+1-i} &\leq \sum_{v \in S \cup T} d(v) \\ &\leq k(|X|-1) + \delta - |X| + k + k(|Y|-1) + \delta - |Y| + k \\ &\leq kn + 2\delta - n = (k-1)n + 2\delta \\ &< (k-1)n + 2\delta + 1 \ \square\end{aligned}$$

Corollary 2.11 Let G be a graph of order n and edge-connectivity λ with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 2$. If $\delta \geq \lfloor n/2 \rfloor + 1$ or if $\delta \leq \lfloor n/2 \rfloor$ and

$$\sum_{i=1}^{2k} d_{n+1-i} \geq \max\{k(n-1)+1, (k-1)n+2\delta+1\}$$

for some k with $1 \leq k \leq \delta$, then G is super-λ.

Now we give examples which show that both bounds in Theorem 2.10 as well as in Corollary 2.11 are best possible and necessary.

Example 2.12 Let H_1 and H_2 be two copies of the complete graph K_p with $p \geq 2$, $V(H_1) = \{x_1, x_2, \ldots, x_p\}$, and $V(H_2) = \{y_1, y_2, \ldots, y_p\}$. We define the graph H as the union of H_1 and H_2 together with the new edges $x_1y_1, x_2y_2, \ldots, x_py_p$. Then $n(H) = n = 2p$, $\delta(H) = \delta = p$, $\lambda(H) = \delta(H)$,

$$\sum_{i=1}^{2\delta} d_{n+1-\delta} = 2\delta^2$$

and H is not super-λ. Furthermore, $\delta(n-1)+1 = 2\delta^2 - \delta + 1$ and $(\delta-1)n + 2\delta + 1 = 2\delta^2 + 1$. Consequently, we see that for $k = \delta$ the second bound in Corollary 2.11 and Theorem 2.10 is important and best possible.

Example 2.13 Let H_1 be the complete graph K_p with $p \geq 2$, $V(H_1) = \{x_1, x_2, \ldots, x_p\}$ and let H_2 be the complete graph K_{2p+1} with $V(H_2) = \{y_1, y_2, \ldots, y_{2p+1}\}$. We define the graph H as the union of H_1 and H_2 together with the new edges $x_1y_1, x_2y_2, \ldots, x_py_p$. Then $n(H) = n = 3p+1$, $\delta(H) = \delta = p$, $\lambda(H) = \delta(H)$,

$$\sum_{i=1}^{2\delta} d_{n+1-\delta} = 3\delta^2$$

and H is not super-λ. Furthermore, $\delta(n-1)+1 = 3\delta^2 + 1$ and $(\delta-1)n + 2\delta + 1 = 3\delta^2$. Consequently, we see that for $k = \delta$ the first bound in Corollary 2.11 and Theorem 2.10 is important and best possible.

Finally, we present examples which show that Theorem 2.10 and Corollary 2.11 are independent of Theorem 2.4 and Corollary 2.5.

Example 2.13 Let H_1 and H_2 be two copies of the complete graph K_p with $p \geq 3$, $V(H_1) = \{x_1, x_2, \ldots, x_p\}$, and $V(H_2) = \{y_1, y_2, \ldots, y_p\}$.

We define the graph H as the union of H_1 and H_2 together with the new edges $x_1y_1, x_2y_2, \ldots, x_{p-1}y_{p-1}$ and $x_{p-1}y_p$. Then $n(H) = n = 2p$, $\delta(H) = \delta = p - 1$ and

$$\sum_{i=1}^{2\delta} d_{n+1-\delta} = \delta n - 1 \geq \max\{\delta(n-1)+1, (\delta-1)n + 2\delta + 1\}.$$

Thus, with respect to Corollary 2.11, the graph H is super-λ. Furthermore,

$$\sum_{i=1}^{k}(d_i + d_{n+i-\delta}) = kn < k(n-2) + 2\delta + 1,$$

for $1 \leq k \leq \delta$, and hence Corollary 2.5 does not show that H is super-λ.

Example 2.14 Let H_1 be the complete graph K_p with vertex set $V(H_1) = \{x_1, x_2, \ldots, x_p\}$ and $p \geq 3$, and let H_2 be the complete graph K_{p+1} with $V(H_2) = \{y_1, y_2, \ldots, y_{p+1}\}$. We define the graph H as the union of H_1 and H_2 together with the new edges $x_3y_3, x_4y_4, \ldots, x_{p-1}y_{p-1}$ and x_1y_1, x_1y_2, x_py_p, and x_py_{p+1}. Then $n(H) = n = 2p + 1$, $\delta(H) = \delta = p - 1$ and

$$\sum_{i=1}^{2\delta} d_{n+1-\delta} = \delta n \geq \max\{\delta(n-1)+1, (\delta-1)n + 2\delta + 1\}.$$

Thus, by Corollary 2.11, the graph H is super-λ. Furthermore,

$$\sum_{i=1}^{k}(d_i + d_{n+i-\delta}) \leq kn < k(n-2) + 2\delta + 1,$$

for $1 \leq k \leq \delta$, and hence we cannot apply Corollary 2.5 to show that H is super-λ.

Example 2.15 Let H_1 and H_2 be two copies of the complete graph K_p with $p \geq 4$, $V(H_1) = \{x_1, x_2, \ldots, x_p\}$, and $V(H_2) = \{y_1, y_2, \ldots, y_p\}$. We define the graph H as the union of H_1 and H_2 together with the new edges $x_1y_1, x_1y_2, \ldots, x_1y_p$ and x_2y_1. Then $n(H) = n = 2p$, $\delta(H) = \delta = p - 1$ and

$$\sum_{i=1}^{\delta}(d_i + d_{n+i-\delta}) = \delta n + 2 \geq \delta(n-2) + 2\delta + 1.$$

Thus, by Corollary 2.5, the graph H is super-λ. Furthermore, for $1 \leq k \leq \delta - 1$, we have

$$\sum_{i=1}^{2k} d_{n+1-\delta} \leq k(n-1) < k(n-1) + 1,$$

and for $k = \delta$ we see that

$$\sum_{i=1}^{2\delta} d_{n+1-\delta} \leq \delta(n-1) + 1 < (\delta - 1)n + 2\delta + 1.$$

The last two inequalities show that Corollary 2.11 does not yield that H is super-λ.

3. Degree sequence conditions for bipartite graphs and digraphs

In the sequel let G be a graph or a digraph with bipartition $V(G) = (V', V'')$. We adopt the convention that for every subset X of $V(G)$, we denote the set $X \cap V'$ by X' and $X \cap V''$ by X''.

Lemma 3.1. Let D be a bipartite digraph of edge-connectivity λ. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X'|, |X''|, |Y'|, |Y''| \geq \max\{1, \delta(D) - 1\}$. Furthermore, if $\delta = \delta(D) \geq 3$, then $|X|, |Y| \geq 2\delta - 1$.

Proof. In view of Lemma 2.1, $|X|, |Y| \geq \max\{2, \delta\}$. Hence, there exists a vertex $u \in X$ such that all vertices of $N^+(u)$, with exception of one vertex, are contained in X. If $\delta \geq 2$ and, without loss of generality, $u \in X''$, then it follows $|X'| \geq \delta - 1$. Now there exists a vertex $v \in X'$ such that all vertices of $N^+(v)$, with exception of one vertex, are contained in X'', and hence $|X''| \geq \delta - 1$.

In the case $\delta = 1$, there exists a vertex $u \in X$ with $N^+(u) \subset X$. If, without loss of generality, $u \in X''$, then it follows $|X'| \geq 1$. In the case $|X'| = 1$ we are done. Otherwise, there exists a vertex $v \in X'$ with $N^+(v) \subset X''$, and hence $|X''| \geq 1$.

Similarly one can show that $|X''|, |Y'|, |Y''| \geq \max\{1, \delta - 1\}$.

Now let $\delta \geq 3$ and suppose, without loss of generality, that $|X'| = |X''| = \delta - 1$. Then we obtain the contradiction

$$(2\delta - 2)\delta = |X|\delta \leq \sum_{x \in X} d^+(x) \leq (\delta - 1)(2\delta - 2) + \delta. \quad \square$$

Corollary 3.2 (Fiol [6]) If D is a bipartite digraph of order n with $\delta(D) \geq \lfloor (n+2)/4 \rfloor + 1$, then D is super-λ.

The following degree sequence condition is an analogue to Theorem 2.4 for bipartite digraphs.

Theorem 3.3 Let D be a bipartite digraph of order n with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 3$ and edge-connectivity λ. If $\delta \geq \lfloor (n+2)/4 \rfloor + 1$ or if $\delta \leq \lfloor (n+2)/4 \rfloor$ and

$$\sum_{i=1}^{k} (d_i + d_{n+i-2\delta+1}) \geq (k+1)(n-2\delta) + 2\delta + 1,$$

for some k with $1 \leq k \leq 2\delta - 1$, then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. Then, according to Lemma 3.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2\delta - 1$ and $|X'|, |X''|, |Y'|, |Y''| \geq \delta - 1$. This leads to $\delta \leq \lfloor (n+2)/4 \rfloor$.

Now let $S \subseteq X$ and $T \subseteq Y$ be two k-sets with $1 \leq k \leq 2\delta - 1$ and define $S' = S \cap X'$, $S'' = S \cap X''$, $T' = T \cap Y'$, and $T'' = T \cap Y''$. By Lemma 3.1, we can assume, without loss of generality, that $|X'|, |Y'| \geq \delta$. This implies

$$\begin{aligned}
\sum_{v \in S} d^+(v) &= \sum_{v \in S'} d^+(v) + \sum_{v \in S''} d^+(v) \\
&\leq |S'||X''| + |S''||X'| + \lambda \\
&\leq |S'|(|X| - \delta) + |S''|(|X| - \delta + 1) + \lambda \\
&\leq k(|X| - \delta) + |S''| + \lambda \\
&\leq k(|X| - \delta) + (|X| - \delta) + \lambda \\
&= (k+1)(|X| - \delta) + \lambda.
\end{aligned}$$

Similarly, we have

$$\sum_{v \in T} d^-(v) \leq (k+1)(|Y| - \delta) + \lambda,$$

and thus, in total

$$\sum_{v \in S \cup T} d(v) \leq (k+1)(n - 2\delta) + 2\lambda. \tag{4}$$

Now choose S and T to contain the k vertices in X and Y of highest degree, respectively. Then $S \cup T$ contains the k vertices of highest degree but not the $2\delta - 1 - k$ of lowest degree in D. Hence,

$$\sum_{v \in S \cup T} d(v) \geq \sum_{i=1}^{k} (d_i + d_{n+i-2\delta+1}),$$

which yields together with (4) a contradiction to the hypothesis. □

As above, we obtain the following corollary for undirected bipartite graphs.

Corollary 3.4 Let G be a bipartite graph of order n with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 3$ and edge-connectivity λ. If $\delta \geq \lfloor (n+2)/4 \rfloor + 1$ or if $\delta \leq \lfloor (n+2)/4 \rfloor$ and

$$\sum_{i=1}^{k}(d_i + d_{n+i-2\delta+1}) \geq (k+1)(n-2\delta) + 2\delta + 1,$$

for some k with $1 \leq k \leq 2\delta - 1$, then G is super-λ.

The next examples will show that Corollary 3.4 a s well as Theorem 3.3 are best possible.

Example 3.5 Let H_1 and H_2 be two copies of the complete bipartite graph $K_{p,p+1}$ with $p \geq 3$, $V(H_1) = \{x_1, x_2, \ldots, x_p\} \cup \{y_1, y_2, \ldots, y_{p-1}\}$, and $V(H_2) = \{u_1, u_2, \ldots, u_p\} \cup \{v_1, v_2, \ldots, v_{p-1}\}$. We define the bipartite graph H as the union of H_1 and H_2 together with the new edges $x_1u_1, x_2u_2, \ldots, x_pu_p$. Then $n(H) = n = 4p-2$, $\delta(H) = \delta = p = (n+2)/4$, $\lambda(H) = \delta$, and

$$\sum_{i=1}^{2\delta-1}(d_i + d_{n+i-2\delta+1}) = pn = \delta(4\delta - 2) = 2\delta(n-2\delta) + 2\delta,$$

however, H is not super-λ.

Again, there is an analogue to Theorem 2.10 for bipartite digraphs.

Theorem 3.6 Let D be a bipartite digraph of order n with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 3$ and edge-connectivity λ. If $\delta \geq \lfloor (n+2)/4 \rfloor + 1$ or if $\delta \leq \lfloor (n+2)/4 \rfloor$ and

$$\begin{aligned}\sum_{i=1}^{2k} d_{n+1-i} \geq & \max\{(k+1)(n-2\delta)+1, (k+1)(n-2\delta)+k-\delta+2, \\ & k(n+2-2\delta)+1\}\end{aligned}$$

for some k with $1 \leq k \leq 2\delta - 1$, then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. Then, according to Lemma 3.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2\delta - 1$ and $|X'|, |X''|, |Y'|, |Y''| \geq \delta - 1$. This leads to $\delta \leq \lfloor (n+2)/4 \rfloor$.

Now let $S \subseteq X$ and $T \subseteq Y$ be two k-sets with $1 \leq k \leq 2\delta - 1$ and define $S' = S \cap X'$, $S'' = S \cap X''$, $T' = T \cap Y'$, and $T'' = T \cap Y''$. By Lemma 3.1, we can assume, without loss of generality, that $|X'|, |Y'| \geq \delta$. If we choose S such that the number of arcs of (X, Y) incident with S is minimal, then we conclude

$$\begin{aligned}
\sum_{v \in S} d^+(v) &= \sum_{v \in S'} d^+(v) + \sum_{v \in S''} d^+(v) \\
&\leq |S'||X''| + |S''||X'| + \delta - \min\{\delta, |X| - k\} \\
&\leq |S'|(|X| - \delta) + |S''|(|X| - \delta + 1) + \delta - \min\{\delta, |X| - k\} \\
&\leq k(|X| - \delta) + |S''| + \delta - \min\{\delta, |X| - k\} \\
&\leq k(|X| - \delta) + (|X| - \delta) + \delta - \min\{\delta, |X| - k\}
\end{aligned}$$

and this implies

$$\sum_{v \in S} d^+(v) \leq (k+1)(|X| - \delta) + \delta - \min\{\delta, |X| - k\}. \tag{5}$$

If we choose T such that the number of arcs of (X, Y) incident with T is minimal, then we conclude similarly

$$\sum_{v \in T} d^-(v) \leq (k+1)(|Y| - \delta) + \delta - \min\{\delta, |Y| - k\}. \tag{6}$$

Case 1. Let $\delta \leq |X| - k$ and $\delta \leq |Y| - k$. The inequalities (5) and (6) imply the following contradiction to the hypothesis:

$$\begin{aligned}
\sum_{i=1}^{2k} d_{n+1-i} &\leq \sum_{v \in S \cup T} d(v) \leq (k+1)(|X| - \delta) + (k+1)(|Y| - \delta) \\
&= (k+1)(n - 2\delta)
\end{aligned}$$

Case 2. Let $\delta \leq |X| - k$ and $\delta \geq |Y| - k$. In view of Lemma 3.1, we have $-|Y| \leq -2\delta + 1$. Hence (5) and (6) lead to the following contradiction to the hypothesis:

$$\begin{aligned}
\sum_{i=1}^{2k} d_{n+1-i} &\leq \sum_{v \in S \cup T} d(v) \\
&\leq (k+1)(|X| - \delta) + (k+1)(|Y| - \delta) + \delta - |Y| + k \\
&\leq (k+1)(n - 2\delta) + k + 1 - \delta
\end{aligned}$$

Case 3. The case $\delta \geq |X| - k$ and $\delta \leq |Y| - k$ can be proved analogously to Case 2.

Case 4. Let $\delta \geq |X| - k$ and $\delta \geq |Y| - k$. Then (5) and (6) yield to the following contradiction to the hypothesis:

$$\begin{aligned}\sum_{i=1}^{2k} d_{n+1-i} &\leq \sum_{v \in S \cup T} d(v) \\ &\leq (k+1)(|X| - \delta) + \delta - |X| + k \\ &+ (k+1)(|Y| - \delta) + \delta - |Y| + k \\ &= (k+1)(n - 2\delta) - n + 2\delta + 2k = k(n + 2 - 2\delta) \quad \Box\end{aligned}$$

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