Circumferences of 2-Connected Claw-Free Graphs [1]

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Abstract

A known result due to Matthews and Sumner is that every 2-connected claw-free graph on n vertices contains a cycle of length at least $min\{2\delta+4, n\}$, and is Hamiltonian if $n \leq 3\delta+2$. In this paper, we show that every 2-connected claw-free graph on n vertices which does not belong to one of three classes of exceptional graphs contains a cycle of length at least $min\{4\delta-2, n\}$, hereby generalizing several known results. Moreover, the bound $4\delta-2$ is almost best possible.

1. Introduction

In this paper we deal with finite simple graphs. Let G be a graph. We denote by $\delta(G)$ (or δ) and $k(G)$ the minimum degree and the connectivity of G, respectively. A graph is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. For a vertex v of G, the neighborhood $N(v)$ of v is the set of all vertices that are adjacent to v. For a subgraph H of a graph G and a subset S of $V(G)$, we denote by $G-H$ and $G[S]$ the induced subgraphs of G by $V(G)-V(H)$ and S, respectively. We denote by $N_H(S)$ the set of all vertices of H adjacent to some vertex of S, and let $N(S)=\bigcup_{x\in S}N(x)$ and $d_H(S)=|N_H(S)|$. For A and B in $V(G)$, let $E_G(A,B)=\{uv\in E(G): u\in A \text{ and } v\in B\}$ and $e_G(A,B)=|E_G(A,B)|ls$. For a cycle C with a fixed orientation, and two vertices x and y on C, we define the segment $S=C[x,y]=xCy$ to be the set of vertices on C from x to y (including x and y) and $C^-[y,x]=yC^-x$ to be a traversal of the $C[x,y]$ in the opposite sense according to the orientation of C. Let $C(x,y)=C[x,y]-\{x,y\}$, and x^+ and x^- denote the successor and the predecessor of x according to the orientation of C, respectively. Let $P=P[x_1,x_i]=x_1x_2...x_i$ be a (x_1,x_i)-path in G. Then uPv denotes the path $ux_1x_2...x_iv$ or $ux_ix_{i-1}...x_1v$, and $P[x_u,x_v]=x_ux_{u+1}...x_v$ denotes a subpath of P for $x_u,x_v\in V(P)$. Let $x^{++}=(x^+)^+$ and $x^{--}=(x^-)^-$. Other notation and terminology not defined here can be found in [1].

There have been many results in recent years dealing with Hamiltonian cycles and circumferences in claw-free graphs (see [2]-[13]). Matthews and Sumner [12] proved the following result.

Theorem 1 (Matthews and Sumner [12]). *Every 2-connected, claw-free graph G on n vertices contains a cycle of length at least $min\{2\delta+4, n\}$, and is Hamiltonian if $n\leq 3\delta+2$.*

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M. Li and Z. Li [10] showed the following theorem.

Theorem 2 (M. Li and Z. Li [10]). *Every 3-connected claw-free graph G on n vertices contains a cycle of length at least* $min\{4\delta - 5, n\}$.

Theorem 3 (M. Li [7]). *Every 2-connected k-regular claw-free graph G on n vertices contains a cycle of length at least* $min\{4k - 2, n\}$.

Let J_1 be the set of all graphs defined as follows: Any graph G in J_1 can be decomposed into three disjoint connected subgraphs G_1, G_2 and G_3 such that $E_G(G_i, G_j) = \{u_iu_j, v_iv_j\}$ for $i \neq j$ and $i, j = 1, 2, 3$ (where $u_i \neq v_i \in V(G_i)$ for $i = 1, 2, 3$). Note that Figure 1 is a representation of such a graph.

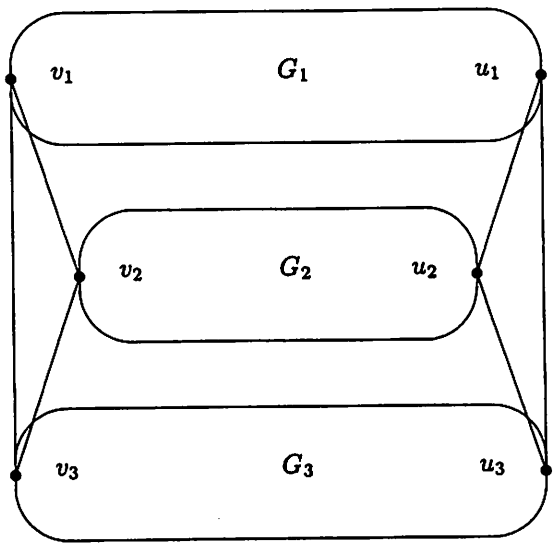

Fig. 1 J_1

M. Li [9] proved the following result which will be used in the proof of Theorem 5.

Theorem 4 (M. Li [9]). *Every 2-connected claw-free graph $G \notin J_1$ on n vertices contains a cycle of length at least* $min\{n, 3\delta + 2\}$. *Moreover, the bound $3\delta + 2$ is best possible.*

Let J_2 be the set of all graphs defined as follows: Any graph G in J_2 can be decomposed into four disjoint hamiltonian connected subgraphs G_1, G_2, G_3 and G_4 such that

$$E_G(G_i, G_{i+1}) = \{u_iv_i\} \text{ for } i = 1, 2 \text{ and } E_G(G_3, G_1) = \{u_3v_3\}$$

(where $v_i \neq u_{i+1}$ for $i = 1, 2$ and $v_3 \neq u_1$),

$$E_G(G_4, G_1 \cup G_2 \cup G_3) = \{w_i v_i, w_i u_i \text{ for } i = 1,2,3\}$$

(where $w_1 \neq w_2 \neq w_3 \neq w_1$). Note that Figure 2 is a representation of such a graph.

Let J_3 be the set of all graphs defined as follows: Any graph G in J_3 can be decomposed into four disjoint connected subgraphs G_1, G_2, G_3 and G_4 such that

$$E_G(G_1, G_2) = \{v_1v_2\},\ E_G(G_1, G_3) = \{v_1v_3\} \text{ and } E_G(G_2, G_3) = \{v_2v_3\},$$

and $N(V(G_i)) \cap V(G_4) = \{w_i\}$ for $i = 1,2,3$, where w_1, w_2, w_3 are three distinct vertices of G_4. Note that Figure 3 is a representation of such a graph. Let $J = J_1 \cup J_2 \cup J_3$.

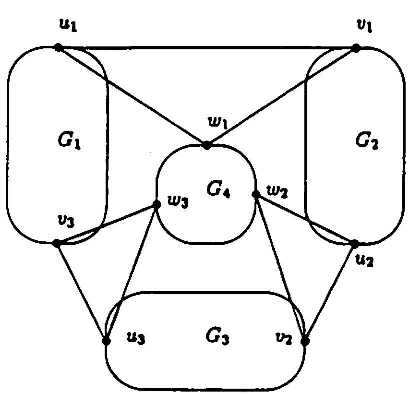

Fig.2 J_2

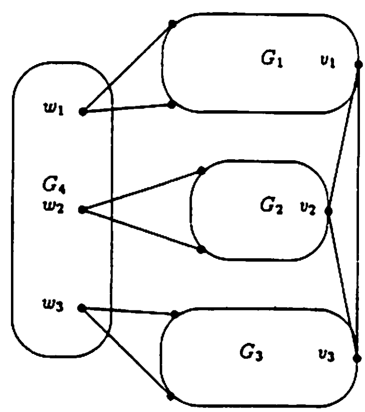

Fig. 3 J_3

In this paper, our purpose is to strengthen Theorems 1-4, and to show the following result whose proof will occur in Section 3.

Theorem 5. *Every 2-connected claw-free graph $G \notin J$ on n vertices contains a cycle of length at least $min\{4\delta - 2, n\}$.*

Note that the bound $4\delta - 2$ is almost best possible since there is a graph J_4 on $5\delta - 1$ vertices whose circumference is 4δ, where J_4 is defined as follows: J_4 can be decomposed into five disjoint complete subgraphs G_1, G_2, G_3, G_4 and G_5 such that

(a). $E_{J_4}(G_i, G_j) = \emptyset$ for $i, j = 1,2,3$ and $i \neq j$ and $E_{J_4}(G_4, G_5) = \emptyset$,

(b). G_i has $\delta - 1$ vertices for $i = 1,2,3$ and G_j has $\delta + 1$ vertices for $j = 4,5$,

(c). $N_{G_j}(V(G_i)) = \{w_{ij}\}$ (where w_{1j}, w_{2j}, w_{3j} are three distinct vertices of G_j) for $i = 1,2,3$ and $j = 4,5$.

(d). w_{ij} is adjacent to all vertices of G_i for $i = 1, 2, 3$ and $j = 4, 5$. Furthermore, if we add a new condition: $n \leq 5\delta - 5$ in Theorem 5, the following result was obtained by M. Li [6] and H. Li et al.[4], independently.

Theorem 6 (M. Li [6], H. Li et al. [4]). *Every 2-connected claw-free graph $G \notin J$ on at most $5\delta - 5$ vertices is hamiltonian.*

We guess that if other new classes of exceptional graphs were added into J, Theorems 5 and 6 might further be generalized (see Problem 7). Perhaps, similar techiques in this paper can be used in Problem 7.

Problem 7. *There exists other class J^0 of exceptional graphs such that every 2-connected claw-free graph G on n vertices contains a cycle of length at least $min\{5\delta - c, n\}$ for some constant c, unless G belongs to $J \cup J^0$.*

Let G be a graph such that for every $x \in V(G)$, $G[N(x)]$ is either a clique or a union of two disjoint cliques. We call such a graph a *closed claw-free graph*. Then by Lemma 1 in [13], we know that G is a line graph of a triangle-free graph H. Z.Ryjacek [13] introduced a closure concept in claw-free graphs and proved the following very important result which is used in the proof of our theorem.

Theorem 8 (Ryjacek [13]). *If G is a claw-free graph, then there is a closed claw-free graph $cl(G)$ (called the closure of G) such that*

(1). G is a spanning subgraph of $cl(G)$, and

(2). the length of a longest cycle in both G and $cl(G)$ is the same.

(3). $cl(G)$ is the line graph of some triangle-free graph.

2. Lemmas

In this section, we introduce seven lemmas which will be used in the proof of Theorem 5. We start with five lemmas from previous papers.

Lemma 1 [7]. *Let $C = (c_1c_2...c_nc_1)$ be a longest cycle in a 2-connected claw-free graph G, and let P be a path of order h in $G - C$ such that the end-vertices a and b of P are joined to vertices c_i and c_j $(i < j)$, respectively.*

(a). We have that $c_i^+c_i^-, c_j^+c_j^- \in E(G)$ and $c_i^+, c_i^- \notin N(c_j)$ and $c_j^+, c_j^- \notin N(c_i)$.

(b). If at least one of $c_ic_t, c_i^-c_t \in E(G)$ and at least one of $c_jc_g, c_j^+c_g \in E(G)$ with $i < t < j, i < g < j, g \neq t$ and $c_t, c_g \in C(c_i^+, c_j^-)$, then $|g - t - 1| \geq h$, $|C(c_t, c_j^-)| \geq h$ and $|C(c_i^+, c_g)| \geq h$.

(c). If $c_gc_t \in E(G)$ with $i < t < j < g$, then $j - t + |C(c_g, c_i)| \geq h$.

Lemma 2 [7]. *Let C be a longest cycle in a 2-connected claw-free graph G and P a path of order h in $G - C = R$ such that its end-vertices a and b are joined to vertices c_i and c_j on C, respectively. Suppose that uc_i, vc_j are edges in $E(G)$ such that $u, v \in C(c_i^+, c_j^-)$, $d_R(u) = d_R(v) = 0$ and $u \in C(c_i, v]$. Then*

(a). $uc_i^+, uc_i^-, vc_j^+, vc_j^- \in E(G)$, and $G[N_R(c_j)]$ and $G[N_R(c_i)]$ are complete.

(b). $u \neq v$.

(c). If $xu^-, yv^+ \in E(G)$ and $x, y \in C(c_j^+, c_i^-)$, then $x \neq y$, $|C(c_j^+, x)| \geq h$, $|C(y, c_i^-)| \geq h$ and $|C(x, y)| \geq h$ (or $|C(y, x)| \geq h$).

(d). If $xu^-, yv^+ \in E(G)$ and $x, y \in C(u, v)$, then $|C(x, v)| \geq h$, $|C(u, y)| \geq h$ and $|C(y, x)| \geq h$ (or $|C(x, y)| \geq h$).

Lemma 3 [9]. *Let C be a longest cycle in an m-connected ($m \geq 2$) claw-free graph G, and H a component of $G - C$ such that $|V(H)| \geq 3$. If H is hamiltonian connected, then there exists some vertex v in H such that*

$$|V(C)| \geq s(d(v) - s + 4) + (m - s)(|V(H)| - s + 3),$$

whenever $0 \leq s \leq |V(H)| + 3$.

Lemma 4 [5]. *Let C be a longest cycle in a 2-connected claw-free graph G and H a component of $G - C$. If H contains a cut-vertex, then there exist nonadjacent vertices u and v in H such that*

$$|V(C)| \geq 2(d(u) + d(v)) + 4 \text{ and } |V(G)| \geq 3(d(u) + d(v)) + 3.$$

Lemma 5 [11]. *Let G be a 2-connected graph on at least $2\delta \geq 2k$ vertices. Then there is a cycle of length at least $2k$ containing x and y for any two distinct vertices x and y.*

In the proof of our theorem, we need the following two lemmas.

Lemma 6. *Let H be a 2-connected closed claw-free graph on n vertices. Then for any vertex v and any two distinct vertices x and y in $V(H) - \{v\}$, H has an $x - y$ path $P = P[x, y]$ containing v and all neighbors of v and connecting x and y.*

Proof. Let v be an arbitrary vertex in H. If $H[N(v)]$ is connected, then $H[N(v) \cup \{v\}]$ is a complete subgraph of H, and if $H[N(v)]$ is not connected, then $H[N(v)]$ is the union of two cliques in H, since we have assumed that H is a closed claw-free graph. Let $S = N(v) \cup \{v\}$ and $x, y \in V(H) - \{v\}$. If $G[N(v)]$ is connected, then we easily prove that there is a path $P = P[x, y]$ containing all vertices of S and connecting x and y since H is 2-connected. Hence we are done. So $H[N(v)]$ is not connected. Let one clique of $H[N(v)]$ be A and the other be B. If one of $\{x, y\}$ belongs to A and the other belongs to B, then it is easy to see that there is a path $P = P[x, y]$ connecting x and y and containing all vertices of S. Hence we are done. If both x and y belong to the same clique of $N(v)$ such as A, then, by Lemma 5, there is a cycle C' containing x and z, where z belongs to B. It is easy to see that there is a subpath $Q = Q[a, b]$) on C' such that $a \in A$, $b \in B$ and $V(Q) - \{a, b\}$ and S are disjoint. Hence we easily see that there is a path $P = P[x, y]$ containing all vertices of S. Hence we are done. So we may assume that at least one of the $\{x, y\}$ does not belong to $A \cup B$, say $x \notin S$. If y belongs to $A \cup B$ (say $y \in A$), then choose a vertex z in B. Clearly, there is a path $P = P[y, z]$ connecting y and z and containing only all vertices of S. Since H is 2-connected, there is a path $Q = Q[z, x]$ such that $y \notin V(Q)$. Let w be the first vertex on Q from x such that $w \in V(P)$. If $w \in B$, then it is easy to see that there is a path $P_1 = P_1[y, w]$ containing all vertices of S. Hence $P_1[y, w] \cup Q[w, x]$ is a path we require. So we have $w \in A \cup \{v\}$. If $w \in A$, then consider the first vertex w_1 on Q from z such that $w_1 \in A$. Thus $G[S \cup V(Q[w_1, z])]$ is 2-connected and there is a path $R_1 = R_1[w, y]$ containing all vertices of S. Hence $R_1 \cup Q[w, x]$ is what we require. Trivially $w \neq v$. It follows that $x, y \notin A \cup B$.

Since H is 2-connected, there are two internally vertex-disjoint paths P_1 and P_2 connecting y and v, and a path $P_3 = P_3[x, v]$ such that $y \notin V(P_3)$. Since $N(v)$ is contained in $A \cup B$, there is a vertex v_1 on P_3 such that $P_3 = P_3[x, v_1]v$ and $v_1 \in A \cup B$, and there is a vertex u_i on P_i such that $P_i = P_i[y, u_i]v$ and $u_i \in A \cup B$ for $i = 1, 2$. Assume that w_i is the first vertex on P_i from y which belongs to $A \cup B$ for $i = 1, 2$, then $w_1 \neq w_2$. If w_1 and w_2 do not belong to the same set, then without loss of generality assume that $w_1 \in A$ and $w_2 \in B$ (the proof of the other case is similar) and suppose that z is the first vertex on P_3 from x such that $z \in V(P_1[y, w_1] \cup P_2(y, w_2]) \cup A \cup B$. If z belongs to $A \cup B$ (say $z \in A$), then there exists a path $P_5[z, w_2]$ containing only all vertices of S, and so $P = P_3[x, z] \cup P_5[z, w_2] \cup P_2[y, w_2]$ is what we require. If z is on $P_1[y, w_1] \cup P_2[y, w_2]$ (say $z \in V(P_1)$), then the path $P = P_3[x, z] \cup P_1[z, w_1]$ plus the path connecting w_1 and w_2 and containing only all vertices of S plus $P_2[y, w_2]$ is what we need.

Thus we have that $w_1, w_2 \in A$ or $w_1, w_2 \in B$ (say $w_1, w_2 \in A$). Then $|A| \geq 2$ and there is a p ath $P_4 = P_4[w, t]$ such that $t \in B$, $w \in A$ and $(V(P_4) - \{w, t\}) \cap S = \emptyset$. A similar argument to the above gives that $z \notin B$. We can assume that $w \notin \{w_1, w_2\}$ (since, if $w = w_1$, then we have an edge ww' on P_4 and an edge w_1w_1' on P_1 such that $w', w_1' \notin A$. Thus we obatin from $G[w, v, w', w_1'] \neq K_{1,3}$ that $w'w_1' \in E(G)$, and so there is a path $P_5 = P_5[y, t]$ connecting y and t. Replacing w_1 by t we obtain that w_2 and w_1 are not in the same set. Thus using a similar argument to the above we can obtain a path we require). Similarly, we can assume $z \neq w$ if $z \in A$. Hence $|A| \geq 3$. Furthermore, we have three vertex-disjoint paths whether z is in A or z is in $P_1[y, w_1] \cup P_2[y, w_2]$, and we can obtain a path we require. Therefore, we have proved the lemma.

Lemma 7. *Let C be a longest cycle in a 2-connected non-hamiltonian claw-free graph $G \notin J$ such that $H = G - C$ is connected. If H is hamiltonian connected, $|V(H)| \geq \delta - 3$ and $d_C(H) \geq 3$, then $|V(C)| \geq 4\delta - 2$.*

Proof. Suppose that the lemma is not true, then $|V(C)| \leq 4\delta - 3$. Let $d_C(H) = k$ and $N_C(H) = \{x_1, x_2, ..., x_k\}$, and let $x_1, x_2, ..., x_k$ be in that order around C and $S_i = C(x_i^+, x_{i+1}^-)$ for $i = 1, 2, ..., k$, where the subscripts of the x_i and S_i are reduced modulo k. Then $x_i^- x_i^+ \in E(G)$ by Lemma 1 (a). It is easy to check that for any $x_i \in N_C(H)$, $|C(x_i, x_{i+1})| \geq 3$ by the maximality of C. Let $M_G(V(C), V(H))$ denote a maximum matching in $E_G(V(C), V(H))$ and x_iHx_j denote a path connecting x_i and x_j $(i \neq j)$ and containing only all vertices of $V(H) \cup \{x_i, x_j\}$. Then $|M_G(V(C), V(H))| \leq 3$ since otherwise $|V(C)| \geq |M_G(V(C), V(H))|(\delta - 3 + 2) + |M_G(V(C), V(H))| \geq 4\delta$. Thus $2 \leq |M_G(V(C), V(H))| \leq 3$ since G is 2-connected. Let $h = |V(H)|$. Now consider two cases.

Case 1. $|M_G(V(C), V(H))| = 3$.

We can without loss of generality assume that $d_C(H) = |M_G(V(C), V(H))| =$

3 (the proof of the other case is similar to the following proof of Claim 1). Then $h \geq \delta - 2$, and by Lemma 1(b), $|S_i| \geq \delta - 2$ for $i = 1, 2, 3$. Let

$$W_1 = \{x_i \in N_C(H) : N_C(x_i) - (\{x_i^+, x_i^-\} \cup N_C(H)) \neq \emptyset\}.$$

Then we have the following fact.

Claim 1. $W_1 = \emptyset$, and $G[N(x_i^+) - (\{x_i, x_i^-\} \cup N_C(H))]$ and $G[N(x_i^-) - (\{x_i, x_i^+\} \cup N_C(H))]$ are complete subgraphs of G for $i = 1, 2, 3$.

Proof. Suppose, to the contrary, that $x_1 \in W_1$ and $u \in N(x_1) - (\{x_1^+, x_1^-\} \cup N_C(H))$. Without loss of generality assume that u is in S_1. By Lemma 2(a), we have $x_1^+ u, x_1^- u \in E(G)$. (Note that we use in the following proof the edge $x_1^- u$ instead of the edge $x_1 u$. From this, the following arguments give the proof of Claim 2 simultaneously.)

Consider the vertex u^-. Let u_2 be the neighbor of u^- closest to x_2 on $C(u^-, x_2^-)$. Then the cycle

$$C[u, u_2]C^-[u^-, x_1]Hx_2x_2^-C[x_2^+, x_1^-]u$$

gives $a = |C(u_2, x_2^-)| \geq h$. Let u_1 be the neighbor of u^- closest to x_3^+ on S_3 if $N(u^-) \cap S_3 \neq \emptyset$. Then, by Lemma 2(c), we have $b = |C(x_3^+, u_1)| \geq h$. Note that $N(u^-)$ and the set $\{x_2^-, x_2, x_2^+, x_3^-, x_3, x_3^+\}$ are disjoint, since otherwise, e.g., say $u^- x_3^- \in E(G)$, we have a longer cycle C' than C as follows.

$$C' = C[x_1, u^-]C^-[x_3^-, u]C^-[x_1^-, x_3]Hx_1.$$

Let u_3, u_4 be the neighbors of u^- closest to x_2^+ and x_3^- on S_2 if $N(u^-) \cap S_2 \neq \emptyset$, respectively. Then, by Lemma 2 (c)(d), we have $c = |C(x_2^+, u_3)| \geq h$ and $d = |C(u_4, x_3^-)| \geq h$. $N(u^-) \cup \{u^-\}$ is contained in $C[u_1, u_2] \cup C[u_3, u_4] = B$. Thus

$$|V(C)| \geq \begin{cases} |B| + a + b + c + d + |\{x_2^-, x_2, x_2^+, x_3^-, x_3, x_3^-\}| \geq 5\delta - 1 & \text{if } N(u^-) \cap S_2 \neq \emptyset, \\ |B| + a + b + |S_2| + |\{x_2^+, x_2, x_2^+, x_3^+, x_3, x_3^-\}| \geq 4\delta + 1 & \text{if } N(u^-) \cap S_2 = \emptyset. \end{cases}$$

This contradiction shows $W_1 = \emptyset$. Let z_1, z_2 be in $N_C(x_i^-) - (\{x_i, x_i^+\} \cup N_C(H))$. Since $G[x_i^-, x_i, z_1, z_2] \neq K_{1,3}$ and $N_C(H) = \{x_1, x_2, x_3\}$, $z_1 z_2 \in E(G)$. This completes the proof of Claim 1. □

Let

$$W_2 = \{x_i \in N_C(H) : N(x_i^-) \cap S_i \neq \emptyset \text{ or } N(x_i^+) \cap S_{i-1} \neq \emptyset\},$$

$$W_3 = \{x_i \in N_C(H) : N(x_i^-) \cap S_{i+1} \neq \emptyset \text{ or } N(x_i^+) \cap S_{i+1} \neq \emptyset\}.$$

Then a similar argument to Claim 1 shows the following claim.

Claim 2. $W_2 = \emptyset$.

In order to prove Case 1, we still need to verify the following three claims.

Claim 3. $W_3 = \emptyset$.

Proof. Otherwise, without loss of generality assume that $N(x_1^-) \cap S_2 \neq \emptyset$, and let $u_1, u_2 \in N(x_1^-) \cap S_2$ be the closest vertices to x_3^- and x_2^+, respectively. Then the cycle $C' = C[x_1, u_1]C^-[x_1^-, x_3^+]x_3^- x_3 H x_1$ gives $a = |C(u_1, x_3^-)| \geq h$. Let u_3 be the first vertex adjacent to x_1^- from x_3^+ on S_3. Then, by Claim 1, we have that $u_3 u_2 \in E(G)$ and $N(x_1^+) \cup \{x_1^+\}$ is contained in $C[u_2, u_1] \cup C[u_3, x_1^+] = B$. The cycle

$$C[u_3, x_2^-]x_2^+ x_2 H x_3 x_3^+ C^-[x_3^-, u_2]u_3$$

shows $b = |C(x_2^+, u_2)| + |C(x_3^+, u_3)| \geq h$. Hence $|V(C)| \geq a + b + |B| + |S_1| + |\{x_2, x_2^-, x_2^+, x_3^-, x_3\}| \geq 4\delta + 1$. This contradiction shows $N(x_1^-) \cap S_2 = \emptyset$. Similarly, we can prove the others. So the claim holds. □

From Claims 2 and 3, we obtain the following claim.

Claim 4. *$N(x_1^-) - \{x_1^-, x_1\}$ and $N(x_3^+) - \{x_3, x_3^-\}$ are contained in S_3, $N(x_1^+) - \{x_1^-, x_1\}$ and $N(x_2^-) - \{x_2, x_2^+\}$ are contained in S_1, and $N(x_3^-) - \{x_3^+, x_3\}$ and $N(x_2^+) - \{x_2, x_2^-\}$ are contained in S_2.*

Claim 5. $|S_i| \leq 2\delta - 8$ for $i = 1, 2, 3$.

Proof. Suppose, to the contrary, that e.g. $|S_1| \geq 2\delta - 7$. Then $|V(C)| \geq \sum_{i=1}^{3}(|S_i| + 3) \geq 4\delta - 2$, a contradiction. □

Claim 6. $E_G(S_i, S_j) = \emptyset$ for $i, j = 1, 2, 3$ and $i \neq j$.

Proof. Without loss of generality assume that $u \in S_1$ and $v \in S_2$ such that $uv \in E(G)$. Let $x \in N(x_1^+)$ be the closest vertex to x_2^- on S_1 and $x' \in N(x_2^-)$ the closest vertex to x_1^+ on S_1. Since $d_{S_1}(x_1^+) \geq \delta - 2$ and $d_{S_1}(x_2^-) \geq \delta - 2$, by Claim 5, we have $x' \in C(x_1^+, x)$. Hence for any vertex $w \in S_1$ we have either $w \in C(x_1^+, x)$ or $w \in C(x', x_2^-)$. It follows that we can without loss of generality assume that $u \in C(x_1^+, x)$. Let y be the neighbor of x_3^- closest to x_2^+ on S_2. Then, similarly, we can without loss of generality assume that $v \in C(y, x_3^-)$. Let u', u'' be the neighbors of x_1^+ on $C(x_1^+, x)$ such that $u' \in C(x_1^+, u)$, $u'' \in C(u, x)$ and $[C(u', u) \cup C(u, u'')] \cap N(x_1^+) = \emptyset$, and let v', v'' be the neighbors of x_3^- on $C(y, x_3^-)$ such that $v' \in C(y, v)$, $v'' \in C(v, x_3^-)$ and $[C(v', v) \cup C(v, v'')] \cap N(x_3^-) = \emptyset$. Then $N(x_1^+) \cup \{x_1^+\}$ is contained in $C[x_1^-, u'] \cup C[u'', x] \cup \{u\} = A$, and $N(x_3^-) \cup \{x_3^-\}$ is contained in $C[y, v'] \cup C[v'', x_3^+] \cup \{v\} = B$. The cycle

$$C' = C[x_1, u']C^-[x, u]C[v, x_3^-]C^-[v', x_2^+]x_2^- x_2 H x_3 C[x_3^+, x_1]$$

gives $a = |C(u', u)| + |C(x, x_2^+)| + |C(v', v)| \geq h$. Note that $u'x, v''y \in E(G)$ by Claim 1. Similarly, $b = |C(u, u'')| + |C(x_2^+, y)| + |C(v, v'')| \geq h$. Hence $|V(C)| \geq |A| + |B| + a + b + |S_3| + |\{x_2\}| \geq 4\delta - 1$. This contradiction shows the completion of the proof for Claim 6. □

We now complete the proof of Case 1.

By Claims 4 and 6, we have that the minimum degree δ_i of $G[S_i]$ is at least $\delta(G) - 2$ for $i = 1, 2, 3$. By Claim 5, we have $\delta_i \geq (|S_i|/2) + 2$. Hence $G[S_i \cup \{x_i^+, x_{i+1}^-\}]$ is hamiltonian connected. It follows that G belongs to J_2. This contradiction shows the completion of our proof for Case 1. □

Case 2. $|M_G(V(C), V(H))| = 2$.

Let r_1, r_2 belong to $N_H(N_C(H))$ such that $r_1 \neq r_2$ and $r_1x_1, r_2x_2 \in E(G)$. Obviously $h = |V(H)| \geq \delta - 1$. Since $d_C(H) \geq 3$, we can without loss of generality assume that $r_1x_3 \in E(G)$. Then we have the following claim.
Claim 7. $d_H(x_1) = d_H(x_3) = 1$.
Proof. Otherwise, e.g., $d_H(x_1) \geq 2$, then $x_1r_2 \in E(G)$. We easily prove by Lemma 1(b) that $|S_i| \geq h$ for $i = 1, 2, 3$, and $d_H(x_1) = 2$ since otherwise $|M_G(V(C), V(H))| = 3$. It follows that $d_C(x_1) \geq \delta - 2$. Let u_1, u_2 be the neighbors of x_1 closest to x_3^+ on S_3 and x_2^- on S_1, respectively. Then, by Lemma 1 (b), we have that $a = |C(u_2, x_2^-)| \geq h$ and $b = |C(x_3^+, u_1)| \geq h$. Let u_3 be the neighbor of x_1 closest to x_2^+ on S_2 if $N_C(x_1) \cap S_2 \neq \emptyset$. Then by Lemma 1(b) we have $d = |C(x_2^+, u_3)| \geq h$. $N_C(x_1) \cup \{x_1\}$ is contained in $C[u_1, u_2] \cup C[u_3, x_3^-] = B$. Note that $x_2^-, x_2^+, x_3^+ \notin N(x_1)$. Hence $|V(C)| \geq a+b+d+|B|+|\{x_2^-, x_2^+, x_3^+\}| \geq 4\delta - 1$. This contradiction shows the completion of our proof for Claim 7. □
In order to prove Case 2, we still need to establish the following two facts.
Claim 8. $x_1x_3 \in E(G)$, and $G[N_C(x_1)-N_C(r_1))]$ and $G[N_C(x_3)-N_C(r_1))]$ are complete subgraphs.
Proof. Since $G[r_1, r_1', x_1, x_3] \neq K_{1,3}$, $x_1x_3 \in E(G)$ by Claim 7, where $r_1' \neq r_2 \in V(H)$ and $r_1r_1' \in E(G)$. Similarly, $G[N_C(x_1) - (\{x_3\} \cup N_C(r_1))]$ and $G[N_C(x_3) - (\{x_1\} \cup N_C(r_1))]$ are complete subgraphs. □
Claim 9. $N_C(x_1) \cap N_C(x_3)$ is contained in $N_C(r_1)$.
Proof. Let $u \in N_C(x_1) \cap N_C(x_3)$ and $ur_1 \notin E(G)$. Then $d_{G-C}(u) = 0$. Without loss of generality assume that $u \in S_3$. Then, by Lemma 2(a), $x_1^+, x_1^-, x_3^-, x_3^+ \in N(u)$. If $u^-u^+ \in E(G)$, then the cycle

$$C[u^+, x_1^-]uC^-[x_3^-, x_1]r_1C[x_3, u^-]u^+$$

is of length $|V(C)| + 1$. This contradiction shows $u^+u^- \notin E(G)$.

Since $G[u, u^+, u^-, x_1^+] \neq K_{1,3}$, $u^+x_1^+ \in E(G)$ or $u^-x_1^+ \in E(G)$. It follows that G has a cycle C' of length $|V(C)| + 1$ as follows.

$$C' = \begin{cases} C[x_3^+, u^-]C[x_1^+, x_3^-]C[u, x_1]r_1x_3x_3^+ & \text{if } u^-x_1^+ \in E(G), \\ C[x_3, u]C^-[x_3^-, x_1^+]C[u^+, x_1]r_1x_3 & \text{if } u^+x_1^+ \in E(G). \end{cases}$$

This contradiction shows the completion of our proof for Claim 9. □
We now complete the proof of Case 2.
Let $u, v \in N_C(x_1) \cup N_C(x_3)$ be the closest vertices to x_2^- on S_1 and x_2^+ on S_2, respectively. Then $N_C(x_1) \cup N_C(x_3)$ is contained in $C[v, u] = B$. Note that if $x \in N_C(r_1) \cap N(x_1) \cap N(x_3)$ then $u^+, u^- \notin N(r_1) \cup N_C(x_1) \cup N_C(x_3)$ by Lemma 1(a). Hence $|B| \geq |N_C(x_1) \cup N_C(x_3)| + |N_C(r_1)| \geq |N_C(x_1)| + |N_C(x_3)| \geq 2\delta - 2$. By Lemma 1(b), we easily obtain $a = |C(u, x_2^-)| \geq h$ and $b = |C(x_2^+, v)| \geq h$. Hence $|V(C)| \geq a + b + |B| + |\{x_2^-, x_2^+\}| \geq 4\delta - 1$. This contradiction shows the completion of our proof for Case 2. Hence Lemma 7 is proved. □

3. Proof of Theorem 5

Assume that G is a non-hamiltonian graph of order n satisfying the conditions of Theorem 5 and G does not belong to J. By Theorem 8, there is a closed claw-free graph $cl(G)$ such that G is a spanning subgraph of $cl(G)$ and the length of a longest cycle in both G and $cl(G)$ is the same. For the convenience of our proof, without loss of generality assume that G is a closed claw-free graph. If this theorem is not true, let C be a longest cycle in G and H a component of $G-C$. Then, by Theorem 4, we have that $3\delta+2 \leq |V(C)| \leq 4\delta-3$ and so $\delta \geq 5$. Let $d_C(H)=k$ and $N_C(H)=\{x_1,x_2,...,x_k\}$, and let $x_1,x_2,...,x_k$ be in that order around C and $S_i = C(x_i^+, x_{i+1}^-)$ for $i=1,2,...,k$, where the subscripts of the x_i and S_i are reduced modulo k (these notations will be used after Claim 4). Then $x_i^- x_i^+ \in E(G)$ by Lemma 1 (a). It is easy to check that for any $x_i \in N_C(H)$, $|C(x_i, x_{i+1})| \geq 3$ by the maximality of C. By Lemma 4, we know that any component H of $G-C$ is 2-connected. We further have the following fact.

Claim 1. *If H is not hamiltonian connected, then H has at least one vertex v such that $d_C(v)=0$. In this case, $|V(H)| \geq d(v)+1$.*

Proof. Otherwise, we have $d_C(v) \geq 1$ for any vertex $v \in V(H)$. By Ore's theorem that a graph on n vertices is hamiltonian connected if $d(u)+d(v) \geq n+1$ for any pair of nonadjacent vertices u and v, we easily obtain that there is a longest path $P=P[a,b]$ in H such that

$3 \leq d_H(a)+d_H(b) \leq |V(P)| \leq |V(H)|$.

Let $p=|V(P)|$ and $h=|V(H)|$ and let Q be the set of ordered pairs (u_i,u_j) of distinct vertices of C such that either $u_i \in N(a)$ and $u_j \in N(b)$ or $u_i \in N(b)$ and $u_j \in N(a)$, and no vertices on $C(u_i,u_j)$ are adjacent to either a or b. Let Q' be the set of ordered pairs (u_i,u_j) of distinct vertices of C with $d_H(u_i)d_H(u_j) \neq 0$ such that no vertices on $C(u_i,u_j)$ are adjacent to any vertex of $V(H)$ and $|C(u_i^+,u_j^-)| \geq |V(P)|$.

By Lemma 1(2)(c), $|C(u_i,u_j)| \geq p+2$ for $(u_i,u_j) \in Q$. Let $q=|Q|$, $q'=|Q'|$ and $X=N_C(a)\cap N_C(b)$. Then $q' \geq q$. Note that, because C is maximal, if $u_i \in N_C(v)$ with $v \in V(H)$, then $u_i^+, u_i^- \notin N_C(v)$ and $u_i^+u_i^- \in E(G)$. Obviously, $|X|=x\neq 0$, otherwise, we obtain that

$|V(C)| \geq 2|V(P)|+3d_C(a)+3d_C(b) \geq 2(d(a)+d(b))+d_C(a)+d_C(b) \geq 4\delta+1$, a contradiction.

Note also that, because C is maximal, if $u_i \in N_C(v)$ with $v \in V(H)$, then $u_i^+, u_i^{++}, (u_i^{++})^+ \notin N_C(v)$. Note that if $x \geq 2$ then $q \geq x$. Thus

$$\begin{aligned} |V(C)| &\geq |N_C(\{a,b\})| + 3|N_C(\{a,b\}) - X| + q(p+2) \\ &\geq 4|N_C(a)\cup N_C(b)| + x(p-1) \\ &\geq 4(|N_C(a)|+|N_C(b)| - |N_C(a)\cap N_C(b)|) + x(p-1) \\ &\geq 4(d_C(a)+d_C(b)) + x(p-5) \\ &\geq 2(d(a)+d(b)) + 2(d_C(a)+d_C(b)) - 10 \geq 4\delta - 2 \end{aligned}$$

Thus $x=1$. This shows that $d_C(a)=d_C(b)=1$. Thus $|V(P)| = d_H(a)+d_H(b) \geq 2\delta-2$, and $ab \in E(G)$ since $G[x_1,a,b,x_1^+] \neq K_{1,3}$, where $x_1a \in E(G)$ and $x_1 \in V(C)$. Let $N_C(a)\cup N_C(b) = \{x_1\}$ and

$P = z_1(= a)z_2...z_p(= b)$. Then $N_C(z_2) = N_C(z_{p-1}) = \{x_1\}$ since otherwise, e.g., $z_2x_2 \in E(G)$ such that $x_2 \neq x_1 \in V(C)$, we have that $x_2^+x_2^- \in E(G)$ by Lemma 1(a) and $P[z_2, z_p]z_1$ is a path of order $V(P)|$ connecting z_1 and z_2. Thus, by Lemma 1(a), $a = |C(x_1^+, x_2^+)| \geq |V(P)|$ and $b = |C(x_2^+, x_1^+)| \geq |V(P)|$. So $|V(C)| \geq a+b+6 \geq 2(2\delta-2)+6 = 4\delta+2$, a contradiction. It follows that $G[z_1, z_2, z_{p-1}, z_p]$ is a clique by Lemma 2(a). Similarly, $x_1z_j \in E(G)$ and $N_C(z_j) = \{x_1\}$ for $j = 3, ..., p-2$. Thus, by Lemma 2(a), $G[V(P)]$ is a complete subgraph of H. Since G is 2-connected, there are two distinct independent edges r_1y_1, r_2y_2 between H and C such that $r_1, r_2 \in V(H)$ and $y_1, y_2 \in V(C)$. By Lemma 1(a), $y_i^+y_i^- \in E(G)$ for $i = 1, 2$. Since H is 2-connected and $G[V(P)]$ is complete, there is a path of order at least $|V(P)| \geq 2\delta - 2$ connecting r_1 and r_2 in H. Thus, by Lemma 1(a), we have that $a = |C(y_1^+, y_2^-)| \geq |V(P)|$ and $b = |C(y_1^+, y_2^-)| \geq |V(P)|$. So $|V(C)| \geq a+b+6 \geq 2(2\delta-2)+6 = 4\delta+2$, a contradiction. Thus Claim 1 is proved. □

Claim 2. *If H is hamiltonian connected, then H contains at least $\delta - 2$ vertices.*

Proof. Let $|V(H)| = h$. It is easy to check that $|V(H)| \geq 3$ by Lemma 1. If $3 \leq |V(H)| = h \leq \delta - 3$, then $\delta \geq 6$, and by Lemma 3 and taking $s = |V(H)| + 3 = h + 3$, we obtain $|V(C)| \geq (h+3)(\delta - h + 1)$.

Let $g(h) = (h+3)(\delta - h + 1)$. Obviously, $g(h)$ is a concave function and its minimum value occurs at the boundary. Since $g(3) \geq 6\delta - 9 \geq 4\delta + 3$ and $g(\delta - 3) = 4\delta$, $|V(C)| \geq min\{g(3), g(\delta - 3)\} \geq 4\delta$, a contradiction. Hence $h \geq \delta - 2$. □

From Claims 1-2 and Lemma 6, we know that for any component H of $G - C$, H has a path of order at least $\delta - 2$ connecting x and y for any two distinct vertices x and y in H with $d_C(x)d_C(y) \neq 0$ whether H is hamiltonian connected or not. Hence, without loss of generality, assume that H is hamiltonian connected. By Claim 2 and Lemma 7, we have that $d_C(H) = 2$ for any component H of $G - C$. We now introduce the following terminology.

Let H and F be two components of $G - C$, and r_1x_1 and r_2x_2 be two independent edges between H and C and $s_iy_i (i = 1, 2)$ two independent edges between F and C , where $r_1, r_2 \in V(H)$, $s_1, s_2 \in V(F)$ and $x_i, y_i \in V(C)$ for $i = 1, 2$. Let $j = 3 - i$. If $y_i \in C(x_1, x_2)$ and $y_j \in C(x_2, x_1)$, then we call that H and F have *crossed bridges*.

Claim 3. *If $G - C$ has at least two components H and F, then no pair of components H and F has crossed bridges*

Proof. Otherwise, let x_i, y_i be the same as the just previous segment for $i = 1, 2$ and x_1, y_1, x_2, y_2 be in that order around C. Then $x_i \neq y_j$ for $i, j = 1, 2$ since G is claw-free and 2-connected. By Claims 1-2 and Lemma 6, we know that there are a path (denoted by x_1Hx_2 or x_2Hx_1) of order at least δ connecting x_1 and x_2 in $G[V(F) \cup \{x_1, x_2\}]$ and a path (denoted by y_1Fy_2 or y_2Fy_1) of order at least δ connecting y_1 and y_2 in $G[V(H) \cup \{y_1, y_2\}]$. Note that if $x_1^+ = y_1$ then we easily prove that $x_2^- \neq y_1$ and $x_2^+ \neq y_2$. Without loss of generality assume that $x_1^+ \neq y_1$ and $x_1^- \neq y_2$. The cycle $C[y_1, x_2]Hx_1x_1^+C^-[x_1^-, y_2]Fy_1$ gives $a = |C(x_1^+, y_1)| + |C(x_2, y_2)| \geq 2(\delta - 2)$.

Similarly, $b = |C(y_1, x_2)| + |C(y_2, x_1^-)| \geq 2(\delta - 2)$. Hence we have $|V(C)| \geq a + b + 6 \geq 4\delta - 2$, a contradiction. □

Claim 4. *$G - C$ has only one component H.*

Proof. If $G - C$ has more than one components, let H and F be two components of $G - C$. Let M be a maximum matching in $E_G(H, C)$ and N a maximum matching in $E_G(F, C)$. Recall $d_C(H) = d_C(F) = 2$. Thus $|M| = 2$ and $|N| = 2$. By Claim 3, H and F have no crossed bridges. Let

$$M = \{r_i x_i \colon r_i \in H,\ x_i \in V(C),\ i = 1, 2\} \text{ and } N = \{s_i y_i \colon s_i \in F,\ y_i \in V(C),\ i = 1, 2\}.$$

Then $x_i^+ x_i^-, y_j^+ y_j^- \in E(G)$ and $y_j \neq x_i$ for $i = 1, 2$ and $j = 1, 2$. By Claim 3, without loss of generality assume that $y_j \in C(x_1, x_2)$ for $j = 1, 2$ (the proof of the other case is similar). Let x_1, y_1, y_2, x_2 are in that order around C. By Claim 2 and Lemma 6, we know that there exist a path of order $(h_1 + 2) \geq \delta$ connecting x_1 and x_2 in $G[V(H) \cup \{x_1, x_2\}]$ and a path of order $(h_2 + 2)$ at least δ connecting y_1 and y_2 in $G[V(F) \cup \{y_1, y_2\}]$. We denote such two paths by $x_1 H x_2$ (or $x_2 H x_1$) and $y_1 F y_2$ (or $y_2 F y_1$), respectively. We further have the following fact.

Claim 4.1. $E_G(C(y_1, y_2), C(x_2, x_1)) = \emptyset$.

Proof. Otherwise, let $f \in C(y_1, y_2)$ and $g \in C(x_2, x_1)$ such that $fg \in E(G)$. Then the cycle

$$C[y_2, x_2^-] x_2^+ x_2 H x_1 x_1^+ C^-[x_1^-, g] C^-[f, y_1] F y_2$$

gives $a' = |C(f, y_2)| + |C(x_2^+, g)| + |C(x_1^+, y_1)| \geq h_1 + h_2$. Similarly, $b' = |C(y_1, f)| + |C(g, x_1^-)| + |C(y_2, x_2^-)| \geq h_1 + h_2$. Hence $|V(C)| \geq a' + b' + |\{x_1^+, x_1, x_1^-, x_2^+, x_2, x_2^-, f, g\}| \geq 4\delta$. This contradiction shows that Claim 4.1 is true. □

Let

$$S = \{x_i \in \{x_1, x_2\} \colon N_C(x_i) - \{x_i^+, x_i^-\} \cup \{x_1, x_2\} \neq \emptyset\},$$

$$T = \{y_j \in \{y_1, y_2\} \colon N_C(y_j) - \{y_j^+, y_j^-\} \cup \{y_1, y_2\} \neq \emptyset\}.$$

Then we have the following fact.

Claim 4.2. $S = T = \emptyset$.

Proof. Let $u \in N_C(x_1) - \{x_1^+, x_1^-, x_2\}$. Then $ux_1^+, ux_1^- \in E(G)$ by Lemma 2(a). By Lemma 2(c)(d), we easily prove that

$$a = |C(y_1^+, y_2^-)| \geq h_2.$$

In order to prove Claim 4.2, we first verify the following six facts. Note that $|C(x_2, x_1)| \geq h_1 + 2$ by Lemma 1.

Fact 1. u does not belong to $C(y_1, y_2)$.

Otherwise, the cycle $C[x_2, x_1^-] C[u, y_2] F C^-[y_1, x_1] H x_2$ gives $a' = |C(y_1, u)| + |C(y_2, x_2)| \geq h_1 + h_2$. Similarly, $b = |C(x_1, y_1)| + |C(u, y_2)| \geq h_2$. Hence $|V(C)| \geq a' + b + |C(x_2, x_1)| + |\{x_2, x_1, y_1, y_2, u\}| \geq 4\delta - 1$. This contradiction shows that Fact 1 is true.

Obviously, $u \notin \{y_1, y_2\}$ by Lemma 2(a) and Fact 1. Without loss of generality assume that $u \in C(x_1, y_1)$, and consider u^-. A similar proof to Fact 1 shows $N(u^-) \cap C(y_1, y_2) = \emptyset$. Let w_1, w_2 be the neighbors of u^- closest to y_1 on $C(u, y_1)$ and x_2^+ on $C(x_2, x_1)$, respectively. Then we have the second fact as follows.

Fact 2. $N_C(u^-) \cap C(y_2, x_2) = \emptyset$ and $N_C(u^-) \cup \{u^-\}$ is contained in $C[w_2, w_1] = B$.

Otherwise, let w_3 be the neighbor of u^- closest to x_2^- on $C(y_2, x_2)$. Then, by Lemma 2(c)(d), we have that $a' = |C(w_3, x_2^-)| \geq h_1$ and $b = |C(x_2^+, w_2)| \geq h_1$, and $N_C(u^-) \cup \{u^-\}$ is contained in $C[w_2, w_1] \cup C[y_2^+, w_3] = B'$.

Obviously, $d_{G-C}(u^-) = 0$. Otherwise, let $d_{H_1}(u^-) \neq 0$ for some component H_1 of $G - C$. Then, by Claims 2-3 and Lemma 6, we easily obtain that $N_C(H_1)$ is contained in $C[x_1, y_1]$ and so $b' = |C(x_1, y_1)| \geq \delta - 2$. Thus $|V(C)| \geq a' + b' + a + |C[x_2, x_1]| + |\{x_2^-, x_1^+\}| \geq 4\delta - 2$, a contradiction.

Thus $|B'| \geq \delta + 1$. So we have $|V(C)| \geq |B'| + a' + b + a + |\{x_2, x_2^-, x_2^+\}| \geq 4\delta - 2$. This contradiction shows $N_C(u^-) \cap C(y_2, x_2) = \emptyset$. So $N_C(u^-) \cup \{u^-\}$ is contained in $C[w_2, w_1] = B$.

Fact 3. *$x_2 \notin S$, and $G[N(x_2^-) - \{x_2^+, x_2\}]$ and $G[N(x_2^+) - \{x_2^-, x_2\}]$ are complete subgraphs.*

Otherwise, let $v \in N_C(x_2) - \{x_2^+, x_2^-, x_1\}$. Then, by Lemma 2(c)(d) and a similar argument to Facts 1-2, we obtain $v \in C(y_2, w_2)$. Without loss of generality assume that $v \in C(y_2, x_2^-)$, then we can similarly prove that $N_C(v^+) \cup \{v^+\}$ is contained in $C(y_2, w_2)$ by Lemma 2(c)(d). Let v' be the neighbor of v^+ closest to w_2 on $C[x_2, w_2]$. Then $N_C(v^+) \cup \{v^+\}$ is contained in $C[y_2, v'] = D$. By Lemma 2(b-d), we easily prove that $v' \in C(x_2, w_2)$ and $b = |C(v', w_2)| \geq h_1$. Note that if $d_{G-C}(v^+) \neq 0$, then, by Claims 2-3 and Lemma 6, we have $c = |C(y_2, x_2)| \geq \delta - 2$, and if $d_{G-C}(u^-) \neq 0$, then $d = |C(x_1, y_1)| \geq \delta - 2$. Recall $a = |C(y_1^+, y_2^-)| \geq h_2$. Thus

$$|V(C)| \geq \begin{cases} |B| + |D| + a + b + |\{y_1, y_2\}| \geq 4\delta & \text{if } d_{G-C}(v^+) = 0 \text{ and } d_{G-C}(u^-) = 0 \\ |B| + b + a + c + |\{y_1, y_2, x_2, x_2^+\}| \geq 4\delta - 1 & \text{if } d_{G-C}(v^+) \neq 0 \text{ and } d_{G-C}(u^-) = 0 \\ |D| + a + b + d + |\{y_1, y_2, x_1, x_1^-\}| \geq 4\delta - 1 & \text{if } d_{G-C}(v^+) = 0 \text{ and } d_{G-C}(u^-) \neq 0 \\ a + b + c + d + |\{x_1, x_2, y_1, y_2, x_1^-, x_2^+\}| \geq 4\delta - 2 & \text{if } d_{G-C}(v^+) \neq 0 \text{ and } d_{G-C}(u^-) \neq 0, \end{cases}$$

a contradiction. Obviously, $G[N(x_2^-) - \{x_2^+, x_2\}]$ and $G[N(x_2^+) - \{x_2^-, x_2\}]$ are complete subgraphs since G is claw-free. Thus Fact 3 is true.

Similarly, by Lemma 2(c)(d), we have

Fact 4. $N(x_2^+) \cap C(x_1, x_2) = N(x_2^-) \cap C(x_2, x_1) = \emptyset$.

Fact 5. $N(x_2^-) \cap C(x_1, w_1) = \emptyset$.

Otherwise, let $f \in N(x_2^-) \cap C(u, w_1)$ (the proof of the other cases is similar) and w_1' be the neighbor of u^- closest to f on $C[f, w_1]$. Then $N_C(u^-) \cup \{u^-\}$ is contained in $C[w_2, f] \cup C[w_1', w_1] = B'$. The cycle

$$C[w_1', x_2^-]C^-[f, u]C^-[x_1^-, x_2]HC[x_1, u^-]w_1'$$

gives $a' = |C(f, w_1')| \geq h_1$. By Claims 2-3 and 4.1 and a similar argument to Fact 2, we easily prove $N_{G-C}(u^-) = \emptyset$. It follows that $|V(C)| \geq$

$|B'| + a' + a + |C(x_2, x_1)| + |\{x_2, x_2^-, x_2^+\}| \geq 4\delta - 2$. This contradiction shows that Fact 5 is true.
Fact 6. $N(x_2^-) \cap C(y_1, y_2) = \emptyset$.
Otherwise, let z_1, z_2 be the neighbors of x_2^- closest to y_1 on $C(y_1, y_2)$ and to y_2 on $C(y_2, x_2^-)$, respectively. Then $z_1 z_2 \in E(G)$ and $N(x_2^-) \cup \{x_2^-\}$ is contained in $C[z_1, y_2] \cup C[z_2, x_2^+] \cup C(w_1, y_1) = D$. Note that $d_{G-C}(x_2^-) = 0$ since $x_2 \notin S$. The cycle $C[x_2, y_1^-] y_1^+ y_1 F y_2 y_2^+ C^-[y_2^-, z_1] C[z_2, x_2]$ gives $d = |C(y_1^+, z_1)| + |C(y_2^+, z_2)| \geq h_2$. A similar argument to Fact 2 gives $d_{G-C}(u^-) = 0$. Hence $|V(C)| \geq d + |D| + |B| + |C(x_2^+, w_2)| \geq 4\delta$. This contradiction shows that Fact 6 is true.
Now we complete the proof of Claim 4.2.
It follows from Facts 3-6 that $N(x_2^-) \cup \{x_2^-\}$ is contained in $C[y_2, x_2^+] \cup C(w_1, y_1) = D$. A similar argument to Fact 6 gives a contradiction. Hence $x_1 \notin S$ and similarly, $x_2 \notin S$ and $T = \emptyset$. Thus Claim 4.2 is proved. □
A similar argument to Claim 4.2 shows the following fact.
Claim 4.3. $(N(z_1^-) - \{z_1^+, z_1, z_2\}) \cap C(z_1, z_2) = \emptyset$, $(N(z_1^+) - \{z_1^-, z_1, z_2\}) \cap C(z_2, z_1) = \emptyset$, $(N(z_2^-) - \{z_2^+, z_1, z_2\}) \cap C(z_2, z_1) = \emptyset$, $(N(z_2^+) - \{z_2^-, z_1, z_2\}) \cap C(z_1, z_2) = \emptyset$, where $z = x$ and y.
Let f be the neighbor of x_1^- closest to x_2^+ on $C(x_2, x_1)$ and g the neighbor of x_2^+ closest to x_1^- on $C(x_2, x_1)$. Then we have the following fact. Note that $G[N(z_i^+) - \{z_1, z_i^-\}]$ and $G[N(z_i^-) - \{z_i^+, z_i\}]$ are complete subgraphs for $i = 1, 2$ and $z = x$ and y since G is claw-free. By Claim 4.2, we still have that $d_{G-C}(z_i^+) = d_{G-C}(z_i^-) = 0$ for $i = 1, 3$ and $z = x$ and y.
Claim 4.4. *f must belong to $C(x_2^+, g)$.*
Proof. Otherwise, we have $a' = |C(x_2, x_1)| \geq d(x_2^+) - 1 + d(x_1^-) - 2 \geq 2\delta - 3$. A similar argument to Fact 6 shows $N(x_2^-) \cap C(y_1, y_2) = \emptyset$. Hence $N(x_2^-) \cup \{x_2^-\}$ is contained $C[x_1, y_1] \cup C[y_2, x_2^+] = D$. It follows that $|V(C)| \geq a' + |D| + a + |\{y_1^+, y_2^-, x_1\}| \geq 4\delta - 1$, a contradiction. □
Claim 4.5. $E_G(C(x_1, x_2), C(x_2, x_1)) = \emptyset$.
Proof. Otherwise, let $a' \in C(x_1, x_2)$ and $b' \in C(x_2, x_1)$ such that $a'b' \in E(G)$. Then, by Claim 4.4, we can without loss of generality assume that $b' \in C(f, x_1^-)$, and we have by Claim 4.1 that $a' \notin C(y_1, y_2)$. Let f' and f'' be the neighbors of x_1^- closest to b' on $C(f, b')$ and $C(b', x_1^-)$, respectively. Then we have that $N(x_1^-) \cup \{x_1^-\}$ is contained in $C[f, f'] \cup C[f'', x_1^+] \cup \{b'\} = B'$ and $f'f'' \in E(G)$. Let u be the neighbor of x_1^+ closest to x_2^- on $C(x_1, x_2)$ and v the neighbor of x_2^- closest to x_1^+ on $C(x_1, x_2)$. Then we have the following fact.

(1). $u \in C[x_1^+, a']$ and $v \in C[a', x_2^-]$. Suppose that $u \in C(a', x_2)$ and u', u'' are the neighbors of x_1^+ closest to a' on $C(a', u)$ and $C(x_1^+, u'')$, respectively. Then $uu'', u'u'' \in E(G)$ and $N(x_1^+) \cup \{x_1^+\}$ is contained in $C[x_1^-, u''] \cup C[u', u] \cup \{a'\} = D$. The cycle

$$C[x_2^+, f']C^-[x_1^-, b']C[a', u]C^-[u'', x_1]Hx_2x_2^-x_2^+$$

gives $d = |C(u, x_2^-)| + |C(u'', a')| + |C(f', b')| \geq h_1$. Similarly, $d' =$

$|C(b', f'')| + |C(x_2^+, f)| + |C(a', u')| \geq h_1$. So we have that $|V(C)| \geq d + d' + |B'| + |D| - 2 + |\{x_2, x_2^+, x_2^-\}| \geq 4\delta - 1$. This contradiction shows $u \in C[x_1^+, a']$ and similarly, $v \in C[a', x_2^-]$.

(2). A similar argument to Fact 6 shows that $v \notin C(y_1, y_2)$.

We now complete the proof of Claim 4.5.

Note first that $C(y_1, y_2)$ is contained in $C(a', y_2)$ since $a' \notin C(y_1, y_2)$. We obtain from (1) and (2) that $d = |C[x_1^+, x_2^-]| = |C[x_1^-, a']| + |C(a', y_2)| + |C[y_2, x_2^-]| \geq d(x_1^+) - |\{x_1, x_1^-\}| + |C(a', y_2)| + d(x_2^-) - |\{x_2, x_2^+\}| \geq \delta - 2 + (\delta - 2) + 2 + \delta - 2 = 3\delta - 4$. Thus $|V(C)| \geq |C(x_2, x_1)| + d + |\{x_1, x_1^-, x_2^+, x_2\}| \geq 4\delta - 2$. This contradiction shows that Claim 4.5 is proved. □

Now we complete the proof of Claim 4.

As a consequence of Claims 4.1-4.5, G belongs to J_1, and so the proof of Claim 4 is completed. □

By Claims 1-2 and 4 and Lemma 6, we can without loss of generality assume that H is hamiltonian connected since there is a path of order at least $\delta - 2$ connecting any pair of vertices x and y in H with $d_C(x)d_C(y) \neq 0$ whether H is hamiltonian connected or not. Let $M_G(V(C), V(H))$ denote a maximum matching in $E_G(V(C), V(H))$ and $x_i H x_j$ denote a path connecting x_i and x_j $(i \neq j)$ containing all vertices of H. Then we easily prove that $2 \leq |M_G(V(C), V(H))| \leq 3$ since G is 2-connected. By Lemma 7, we obtain $d_C(H) = |M_G(V(C), V(H))| = 2$. Let $h = |V(H)|$, and r_1, r_2 belong to $N_H(N_C(H))$ such that $r_1 \neq r_2$ and $r_1x_1, r_2x_2 \in E(G)$. Obviously $h = |V(H)| \geq \delta - 1$. Let W_1 and W_2 be the same as in the proof of Lemma 7. From Lemma 2(a), we know that if $W_1 \neq \emptyset$ then $W_2 \neq \emptyset$. We further have the following claim.

Claim 5. $|W_1| \leq 1$ and $|W_2| \leq 1$. If $|W_1| = 1$, then $W_1 = W_2$, and if $|W_2| = 1$, then $W_1 = \emptyset$ or $W_1 = W_2$.

Proof. Suppose that $|W_1| = 2$. Let $u \in N_C(x_1) - \{x_1^+, x_1^-, x_2\}$ and $v \in N_C(x_2) - \{x_2^-, x_2^+, x_1\}$. Without loss of generality assume that $u, v \in S_1$ (and the proof of other cases is similar). By Lemma 2(b), we have $u \neq v$. We further have $v \notin C(x_1^+, u)$, since otherwise, by Lemma 2(d), $a = |C(v, u)| \geq h$, $b = |C(u, x_2^-)| \geq h$ and $c = |C(x_1^+, v)| \geq h$. Hence $|V(C)| \geq a + b + c + |S_2| + |\{x_1^-, x_1, x_1^+, x_2^-, x_2, x_2^+\}| \geq 4\delta + 1$, a contradiction.

Consider the vertices u^- and v^+. Let x, z be the neighbors of u^- closest to x_2^- on $C[u, x_2^-]$ and x_2^+ on S_2 (if $N(u^-) \cap S_2 \neq \emptyset$), respectively, and let y, w be the neighbors of v^+ closest to x_1^+ on $C[x_1^+, v]$ and x_1^- on S_2 (if $N(v^+) \cap S_2 \neq \emptyset$), respectively. Then, by Lemma 2(c)(d), we have $x \neq y$ and $z \neq w$. Similarly to the argument at the end of the previous paragraph, we have that $x, y \in C[u, v]$, $z \in C(w, x_1^-)$ and $x \in C(x_1^+, y)$. Hence $N(u^-) \cup \{u^-\}$ is contained in $C[z, x] = A$, and $N(v^+) \cup \{v^+\}$ is contained in $C[y, w] = B$. So $|A| \geq d(u^-) + 1$ and $|B| \geq d(v^+) + 1$. By Lemma 2(d), we have $a = |C(x, y)| \geq h$. By Lemma 2(c), we have $b = |C(w, z)| \geq h$. Hence we obtain $|V(C)| \geq a + b + |A| + |B| \geq d(u^-) + 1 + d(v^+) + 1 + 2h \geq 4\delta - 1$. This contradiction shows $|W_1| \leq 1$. Similarly, $|W_2| \leq 1$ and if $|W_1| = 1$, then $W_1 = W_2$, and if $|W_2| = 1$, then $W_1 = \emptyset$ or $W_1 = W_2$. Hence we have completed the proof of Claim 5. □

Claim 6. $W_1 = \emptyset$ and $W_2 = \emptyset$.

Proof. Without loss of generality assume that $x_1 \in W_1$, and u is in $N(x_1) - \{x_1^-, x_1^+, x_2\}$. By Lemma 2(a), $x_1^+u, x_1^-u \in E(G)$. By Claim 5, we have $W_2 = \{x_1\}$. By symmetry, assume that $u \in S_1$. Let v be the neighbor of x_2^+ closest to x_1^- on S_2 and w the neighbor of x_2^- closest to x_1^+ on S_1, and let x, y be the neighbors of u^- closest to x_2^+ on S_2 (if $N(u^-) \cap S_2 \neq \emptyset$) and x_2^- on S_1, respectively. Then , by Claim 5, $N(x_2^-) \cup \{x_2^-\}$ is contained in $C[w, x_2^+]$, and $N(x_2^+) \cup \{x_2^+\}$ is contained in $C[x_2^-, v]$. Moreover, we have from Lemma 2(c)(d) that $|C(x_2^+, x)| \geq h$ and $|C(y, x_2^-)| \geq h$. In order to prove this claim, we first need to verify the following four facts.

Claim 6.1. $u \notin C(w, x_2^-)$.

Proof. Otherwise, we have the following five facts.

(a). $N(x_2^-) \cap C(x_1, u) = \{w\}$.

Otherwise, let $w_1 \in N(x_2^-) \cap C(w, u)$ be the closest vertex to u. Then the cycle

$$C' = C[x_1^+, w_1]C^-[x_2^-, u]C^-[x_1^-, x_2]Hx_1x_1^+$$

gives $a = |C(w_1, u)| \geq h$. Let w_2, w_3 be the neighbors of x_2^- closest to w and u on $C[w, w_1]$ and $C(u, x_2^-)$, respectively. Then, since $G[N(x_2^-) - \{x_2, x_2^+\}]$ is a complete subgraph of G, w_2w_3 is an edge of G. $N(x_2^-) \cup \{x_2^-\}$ is contained in $C[w_2, w_1] \cup C[w_3, x_2^+] \cup \{w, u\} = B$. The cycle

$$C[x_2^+, x_1^-]C[x_1^+, w]C^-[x_2^-, w_3]C[w_2, u]x_1Hx_2x_2^+$$

gives $b = |C(w, w_2)| + |C(u, w_3)| \geq h$. Hence $|V(C)| \geq a + b + |B| + |S_2| \geq 4\delta - 2$. This contradiction shows that (a) is true.

(b). $N(x_1) \cap C(w, x_2^-) = \{u\}$ and $u^+u^- \in E(G)$.

Let $u_1 \in N(x_1) \cap C(w, x_2^-)$ and $u_1 \neq u$. Then, by Lemma 2(a), we have $x_1^-u_1 \in E(G)$. Without loss of generality assume that u_1 belongs to $C(u, x_2^-)$. By Lemma 1(b), we have $c = |C(u_1, x_2^-)| \geq h$. Similarly to (a), we have $a = |C(w, u)| \geq h$. The cycle

$$C[x_2^+, x_1^-]C[u_1, x_2^-]C[w, u]x_1^+x_1Hx_2x_2^+$$

gives $d = |C(x_1^+, w)| + |C(u, u_1)| \geq h$. Hence $|V(C)| \geq a + c + d + |S_2| + |\{x_1^-, x_1, x_1^+, x_2^-, x_2, x_2^+\}| \geq 4\delta$, a contradiction. Since $G[u, u^+, u^-, x_1] \neq K_{1,3}$, $u^+u^- \in E(G)$. Hence (b) is true.

Let f be the neighbor of x_1^- closest to x_2^+ on S_2. Then we have the following fact.

(c). f belongs to $C(x_2^+, v)$. Hence we have that $p \in C(x_2^+, v)$ or $p \in C(f, x_1^-)$ for any vertex $p \in S_2$.

Otherwise, let u_3, u_4 be the neighbors of x_1^- closest to w and x_2^- on $C(w, x_2^-)$ if $N(x_1^-) \cap C(w, x_2^-) \neq \emptyset$. Then $N(x_1^-) \cup \{x_1^-\}$ is contained in $C[f, w] \cup$

$C[u_3, u_4] = B$. By Lemma 1(b), we have $a_1 = |C(u_4, x_2^-)| \geq h$. Recall $a_2 = |C(w, u_3)| \geq h$, which implies that $w^+ \notin N(x_1^-)$. By the maximality of C, we easily prove that $x_1^- x_2^- \notin E(G)$. Note that $w \notin N(x_1^-)$ since otherwise we have from (a) and (b) that $G[w, w^+, x_1^-, x_2^-] = K_{1,3}$. Hence $|B| \geq d(x_1^-) + 1 + 1 \geq \delta + 2$. Obviously $N(x_2^+) \cup \{x_2^+\}$ is contained in $C[x_2^-, v] = A$. Hence $|V(C)| \geq a_1 + a_2 + |A| + |B| \geq 4\delta$. This contradiction shows that (c) is true.

(d). $E_G(S_2, S_1) = \emptyset$.

Suppose, for example, that $f_1 \in C(v, x_1^-)$ and $f_2 \in C(x_1^+, w)$ such that $f_1 f_2 \in E(G)$. Let v' be the neighbor of x_2^+ closest to f on $C[f, v]$ and u' the neighbor of x_2^- closest to u on $C(u, x_2^-)$. Then, $wu' \in E(G)$, and $N(x_2^-) \cup \{x_2^-\}$ is contained in $C[u', x_2^+] \cup \{u, w\} = B$. $N(x_2^+) \cup \{x_2^+\}$ is contained in $C[x_2^-, f] \cup C[v', v] = D$. The cycle

$$C[x_1, f_2]C^-[f_1, v']C[x_2^+, f]x_1^- C^-[u, w]C[u', x_2]Hx_1$$

gives $c = |C(f_2, w)| + |C(u, u')| + |C(f, v')| + |C(f_1, x_1^-)| \geq h$. Recall $ux_1^- \in E(G)$. The cycle

$$C[u, x_2^-]C^-[w, x_1]Hx_2C[x_2^+, x_1^-]u$$

shows $b = |C(w, u)| \geq h$. Hence $|V(C)| \geq |B| + |D| - 3 + c + b + |\{f_1, f_2\}| \geq 4\delta - 1$. This contradiction shows $E_G(C(v, x_1^-), C(x_1^+, w)) = \emptyset$. Similarly, we can prove other cases. So (d) is true.

Since G is claw-free, by (b) and (d), we have the following fact.

(e). $G[N_S(u)]$ is a complete subgraph of G, where $S = C(w, x_2^-)$.

Now we complete the proof of Claim 6.1.

Let us assume that u_1, u_2, u_3 are the neighbors of u closest to x_2^- on $C(u, x_2^-)$, to w on $C(w, u)$ and to w on $C(x_1^+, w)$ if $N(u) \cap C(x_1^+, w) \neq \emptyset$, respectively. Then, by (d), $N(u) \cup \{u\}$ is contained in $C[x_1^-, u_3] \cup C[u_2, u_1] \cup \{w\} = B'$. The cycle $C[x_2^+, x_1^-]uC^-[u_1, u^+]C[u^-, x_1]Hx_2x_2^-x_2^+$ gives $a = |C(u_1, x_2^+)| \geq h$. The cycle $C[x_2^+, x_1^-]uC[u_2, u^-]C[u^+, x_2^-]C^-[w, x_1]Hx_2x_2^-x_2^+$ gives $b = |C(w, u_2)| \geq h$. Hence $|V(C)| \geq a + b + |B'| + |S_2| + 2 \geq 4\delta - 2$. This contradiction shows that Claim 6.1 is proved. □

Claim 6.2. We can assume that $u \neq w$.

Proof. By Lemma 1(a), we have that $x_2^- x_1 \notin E(G)$. If $u = w$, since $G[u, u^-, x_2^-, x_1] \neq K_{1,3}$, $u^- x_1 \in E(G)$. Replacing u by u^-, we obtain that we can assume that $u \neq w$. □

Recall that x, y are the neighbors of u^- closest to x_2^+ on S_2 and x_2^- on S_1, respectively. Then we have the following fact.

Claim 6.3. $y \notin C(w, x_2^-)$ and $x \notin C(x_2, v)$.

Proof. Let $y \in C(w, x_2^-)$ be the closest vertex to x_2^- and y' be the neighbor of u^- closest to w on $C(w, x_2^-)$. Then $N(u^-) \cup \{u^-\}$ is contained in $C[x, w] \cup C[y', y] = B$. By Lemma 2(d), we have $a = |C(y, x_2^-)| \geq h$ and $b = |C(x_2^+, x)| \geq h$. The cycle $C' = C[y', x_2^-]C^-[w, u]C^-[x_1^-, x_2]HC[x_1, u^-]y'$

gives $d = |C(w, y')| \geq h$. Hence $|V(C)| \geq a+b+d+|B|+|\{x_2^+, x_2^-\}| \geq 4\delta$. This contradiction shows $y \notin C(w, x_2^-)$. Similarly, $x \notin C(x_2^+, v)$. □

Claim 6.4. $E_G(C(w, x_2^-), C(x_2^+, y)) = \emptyset$, and $E_G(C(x_2^+, v), C(x, x_2^-)) = \emptyset$.

Proof. Let $f \in C(w, x_2^-)$ and $g \in C(x_2^+, v)$ such that $fg \in E(G)$ and let f' be the neighbor of x_2^- closest to f on $C(w, f)$ and g' the neighbor of x_2^+ closest to g on $C(x_2^+, g)$. Then $(N(x_2^-) - \{x_2, x_2^+\}) \cup \{x_2^-\}$ is contained in $C[w, f'] \cup C[f, x_2^-] = A$, and $(N(x_2^+) - \{x_2, x_2^-\}) \cup \{x_2^+\}$ is contained in $C[x_2^+, g'] \cup C[g, v] = B$. Note that $vg' \in E(G)$ since $G[N(x_2^+) - \{x_2, x_2^-\}]$ is a clique, and $|C(x, y)| \geq d(u^-) - 1$. The cycle

$$C[x_2^+, g']C^-[v, g]C[f, x_2^-]C^-[f', u]C^-[x_1^-, x]C^-[u^-, x_1]Hx_2x_2^+$$

shows $a = |C(g', g)| + |C(v, x)| + |C(f', f)| \geq h$. Hence $|V(C)| \geq a + |B| + |A| + |C[x, y]| \geq 4\delta - 2$. This contradiction shows $E_G(C(w, x_2^-), C(x_2^+, v)) = \emptyset$. Similarly, $E_G(C(w, x_2^-), C(x, y) \cup C(u, w)) = \emptyset$ and $E_G(C(x_2^+, v), C(x, y) \cup C(v, x)) = \emptyset$.

Assume that $vf \in E(G)$ such that $f \in C(w, x_2^-)$. Note that $u^-x_2^+ \notin E(G)$. Obviously, $x \neq v$ since otherwise $G[v, u^-, x_2^+, f] = K_{1,3}$. The cycle

$$C[x_2, v]C[f, x_2^-]C^-[f', u]C^-[x_1^-, x]C^-[u^-, x_1]Hx_2 \ .$$

gives $d = |C(v, x)| + |C(f', f)| \geq h$. Similarly, we can easily get a contradiction. So $N(v)$ and $C(w, x_2^-)$ are disjoint.

If $xf \in E(G)$ such that $f \in C(w, x_2^-)$, let x' be the neighbor of u^- closest to x on $C(x, x_1)$. Then replacing x by x', we can similarly get a contradiction. Thus $N(x)$ and $C(w, x_2^-)$ are disjoint. Similarly, $(N(y) \cup N(w)) \cap C(x_2^+, v) = \emptyset$. Thus Claim 6.4 is proved. □

Now we complete the proof of Claim 6.

By a similar argument to Claim 6.4, we easily obtain the following facts.

(a) If there are vertices $v_1 \in C(v, x)$ and $v_1' \in C(x_2^+, v)$ such that $v_1v_1' \in E(G)$, then
$E_G(C(x_2^+, v_1), C(x, x_2^-)) = \emptyset$.

(b) If there are vertices $w_1 \in C(y, w)$ and $w_1' \in C(w, x_2^-)$ such that $w_1w_1' \in E(G)$, then
$E_G(w_1, x_2^-), C(x_2, y)) = \emptyset$.

(c) If there are vertices $u_1 \in C(v, x)$ and $u_1' \in C(x, y)$ such that $u_1u_1' \in E(G)$, then
$E_G(C(x_2, v), C(u_1, y)) = \emptyset$.

(d) If there are vertices $y_1 \in C(y, w)$ and $y_1' \in C(x, y)$ such that $y_1y_1' \in E(G)$, then
$E_G(C(x, y_1), C(w, x_2^-)) = \emptyset$.

(e) If the conditions of (a)-(d) hold simultaneously, then a similar argument to Claim 6.3 shows $v_1, w_1 \notin C(u_1, y_1)$, and $E_G(C(w_1, x_2^-), C(x_2^+, y_1)) = \emptyset$, and $E_G(C(x_2^+, v_1), C(u_1, x_2^-)) = \emptyset$.

Thus we know from (a)-(e) that there is a vertex v_0 on $C(v, x)$ and

$w_0 \in C(y,w)$ such that $N(C(x_2^+,v_0))$ is contained in $C[x_2^+,v_0] = S_1'$, $N(C(w_0,x_2^-))$ is contained in $C[w_0,x_2^-] = S_2'$, and $N(C(v_0,w_0))$ is contained in $C[v_0,w_0] = S_3'$. Let $G_1 = G[S_1']$, $G_2 = G[S_2']$, $G_3 = G[V(H) \cup \{x_2\}]$ and $G_4 = G[S_3']$. It follows that $G \in J_3$, a contradiction. Thus $W_1 = \emptyset$. Similarly, $W_2 = \emptyset$. This completes the proof of Claim 6. □
By Claim 6, we have the following fact.
Claim 7. $N(x_1^+) - \{x_1, x_1^-\}$ and $N(x_2^-) - \{x_2, x_2^+\}$ are contained in S_1, $N(x_2^+) - \{x_2, x_2^-\}$ and $N(x_1^-) - \{x_1, x_1^+\}$ are contained in S_2.
Claim 8. $|S_1| \le 2\delta - 8$ or $|S_2| \le 2\delta - 8$ (say $|S_1| \le 2\delta - 8$).
Proof. Otherwise, we have $|V(C)| \ge \sum_{i=1}^{2} |S_i| + 6 \ge 4\delta - 2$, a contradiction.□
Since we have assumed that G is a closed claw-free graph, $G[N(x_i^+) - \{x_i, x_i^-\}]$ and $G[N(x_i^-) - \{x_i, x_i^+\}]$ are complete subgraphs for $i = 1, 2$. By Claim 7, we have $d_{S_1}(x_1^+) \ge \delta - 2$ and $d_{S_1}(x_2^-) \ge \delta - 2$, $d_{S_2}(x_2^+) \ge \delta - 2$ and $d_{S_2}(x_1^-) \ge \delta - 2$. Let $f \in N(x_1^+)$ and $g \in N(x_2^-)$ be the closest vertices to x_2^- and x_1^+ on S_1, respectively. Then, by Claim 8, we have that $f \in C(g, x_2^-)$. Thus, for any vertex $x \in S_1$, we have $x \in C(x_1^+, f)$ or $x \in C(g, x_2^-)$. Let $r \in N(x_1^-)$ and $s \in N(x_2^+)$ be the closest vertices to x_2^+ and x_1^- on S_2, respectively. Then we have the following fact.
Claim 9. $E_G(S_1, C(x_2^+, s) \cup C(r, x_1^-)) = \emptyset$.
Proof. Otherwise, let $y \in C(g, x_2^-)$ and $x \in C(x_2^+, s)$ such that $xy \in E(G)$ and let x', x'' be the neighbors of x_2^+ closest to x on $C(x_2^+, x)$ and $C(x, s)$, respectively, and y', y'' the neighbors of x_2^- closest to y on $C(g, y)$ and $C(y, x_2^-)$, respectively. Then, by Claim 6, $x's \in E(G)$ and $y''g \in E(G)$. $(N(x_2^-) - \{x_2, x_2^+\}) \cup \{x_2^-\}$ is contained in $C[g, y'] \cup C[y'', x_2^-] \cup \{y\} = A$, and $(N(x_2^+) - \{x_2, x_2^-\}) \cup \{x_2^+\}$ is contained in $C[x_2^+, x'] \cup C[x'', s] \cup \{x\} = B$. The cycle $C[x_2, x']C^-[s, x]C[y, x_2^-]C^-[y', x_1^+]x_1^- x_1 H x_2$ shows $a = |C(x', x)| + |C(s, x_1^-)| + |C(y', y)| \ge h$. Similarly, $b = |C(x, x'')| + |C(y, y'')| + |C(x_1^+, g)| \ge h$. Similarly, $|V(C)| \ge |A| + |B| + a + b + |\{x_2, x_1, x_1^-, x_1^+\}| \ge 4\delta$, a contradiction. Similarly, we can prove $E_G(S_1, C(r, x_1^-)) = \emptyset$ and $E_G(C[x_1^+, g], C(x_2^+, s) \cup C(r, x_1^-)) = \emptyset$. So Claim 9 is proved. □
We now complete the proof of Theorem 5.
If $E_G(S_1, S_2) = \emptyset$, then G belongs to J_1. Hence $E_G(S_1, S_2) \ne \emptyset$. It follows from Claim 9 that $r \notin C(x_2^+, s)$. Let $z \in S_1$ and $w \in S_2$ such that $zw \in E(G)$. Then, from Claim 9, we have $w \in C[s, r]$. We have $r \ne s$ since otherwise $r = s = w$ and $G[w, x_1^-, x_2^+, z] = K_{1,3}$. By a similar argument to Claim 9, we have $E_G(C(r, x_1^-), C(x_2^+, w)) = \emptyset$ and $E_G(C(x_2^+, s), C(w, x_1^-)) = \emptyset$. We further have the following three properties using a proof similar to that of Claim 9.

(a) If there is a vertex $r_1 \in C(w, r)$ and a vertex $r_1' \in C(r, x_1^-)$ such that $r_1 r_1' \in E(G)$, then
$E_G(C(r_1, x_1^-), C(x_2^+, w)) = \emptyset$.

(b) If there is a vertex $s_1 \in C(s, w)$ and a vertex $s_1' \in C(x_2^+, s)$ such that $s_1 s_1' \in E(G)$, then

$E_G(C(x_2^+, s_1), C(w, x_1^-)) = \emptyset$.

(c) If the conditions of (a) and (b) are satisfied simultaneously, then

$$E_G(C(x_2^+, s_1), C(r_1, x_1^-)) = \emptyset.$$

Let $M_G(S_1, C(s,r))$ denote a maximum matching in $E_G(S_1, C(s,r))$. Then we have the following fact.

(d) $|M_G(S_1, C(s,r))| = 1$. Otherwise, let $w_1z_1 \in E(G)$ such that $w_1 \in C(w,r)$ and $z_1 \in C(x_1^+, z)$. Without loss of generality assume that $z_1, z \in C(g, x_2^-)$ and $C(z_1, z) \cap N(x_2^-) = \emptyset$. The cycle

$$C[x_2, w]C[z, x_2^-]C[g, z_1]C[w_1, x_1^-]x_1^+x_1Hx_2$$

gives $a = |C(z_1,z)| + |C(w,w_1)| + |C(x_1^+, g)| \geq h$. Hence $|V(C)| \geq a + d(x_2^+) + d(x_1^-) + 1 + d(x_2^-) - 1 \geq 4\delta - 1$, a contradiction.

From (a)-(c), we can find a vertex $r_0 \in C(w, x_1^-)$ and a vertex $s_0 \in C(x_2^+, w)$ such that r_0, s_0 are the closest possible to w, $N(C(r_0, x_1^-))$ is contained in $C[r_0, x_1^-]$ and $N(C(x_2^+, s_0))$ is contained in $C[x_2^+, s_0]$. Since G is claw-free, we easily prove that $r_0 = w = s_0$ can not occur. Let $S = C[s_0, r_0]$, then we have the following fact.

(e) $2 \leq |S| \leq 3$. Otherwise, for example, $w \neq s_0^+$. Consider w^-. Then $N(w^-)$ and $C[x_2^+, s_0) \cup C(r_0, x_1^-)$ are disjoint by the defintions of r_0 and s_0. We have from (d) that $|N_{S_1}(w^-)| \leq 1$. Hence $d_S(w^-) \geq \delta - 1$. It follows that $|S| \geq \delta - 1$. So $|V(C)| \geq |S| + d(x_2^+) + d(x_1^-) + 1 + |S_1| \geq 4\delta - 1$. This contradiction shows that $2 \leq |S| \leq 3$.

If $|S| = 2$, without loss of generality assume that $w = s_0$. Since $G[w, r_0, z, w^-] \neq K_{1,3}$, $r_0z \in E(G)$. By (d), we have $d_{S_1}(w) = 1$. Let $G_1 = G[C[x_2^+, s_0]]$, $G_2 = G[C[r_0, x_1^-]]$, $G_3 = G[V(H) \cup \{x_1, x_2\}]$ and $G_4 = G[C[x_1^+, x_2^-]]$. Since $d_{S_1}(x) \geq \delta - 2 \geq (|S_1| + 2)/2 + 1$ for any vertex $x \in S_1$, G_4 is hamiltonian connected. It is easy to check that G_i is hamiltonian connected since $|V(G_1)| + |V(G_2)| \leq 3\delta - 2$ for $i = 1, 2$. Hence we have that $G \in J_2$, a contradiction.

Thus we have $|S| = 3$, namely, $S = \{r_0, w, s_0\}$. Then $r_0z, s_0z \notin E(G)$ (since otherwise, e.g., $r_0z \in E(G)$, by (d), we have $d_{S_1}(w) = 1$. It follows from the definitions of r_0 and s_0 that $d(w) = 3$, a contradiction). Since $G[w, r_0, s_0, z] \neq K_{1,3}$, $r_0s_0 \in E(G)$. It is easy to see that $d_{S_2}(w) = 2$. Hence $d_{S_1}(w) \geq \delta - 2$. Let $G_1 = G[C[x_2^+, s_0]]$, $G_2 = G[C[r_0, x_1^-]]$, $G_3 = G[V(H) \cup \{x_1, x_2\}]$ and $G_4 = G[S_1 \cup \{x_1^+, x_2^-, w\}]$. Then a similar argument to the preceding paragraph shows $G \in J_2$. With this contradiction, the proof of Theorem 5 is completed.

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