BOUNDS FOR PARTIAL LIST COLOURINGS

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Abstract: Let G be a simple graph on n vertices with list chromatic number $\chi_l = s$. If each vertex of G is assigned a list of t colours Albertson, Grossman and Haas [1] asked how many of the vertices, $\lambda_{t,s}$, are necessarily colourable from these lists? They conjectured that $\lambda_{t,s} \geq tn/s$. Their work was extended by Chappell [2]. We improve the known lower bounds for $\lambda_{t,s}$.

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§1. Introduction.

Let G be a simple graph on n vertices. Suppose that each vertex $x \in V(G)$ is assigned a list, $l(x)$, of possible colours. A proper colouring c: $V(G) \rightarrow R$ is a *list colouring* of G if $c(x) \in l(x)$ for all $x \in V(G)$. The graph G is *s-choosable* if there exists a list colouring for every assignment of lists of $s = |l(x)|$ colours. The *list chromatic number*, or *choice number*, of G, χ_l, is the minimum s for which G is s-choosable. Clearly $\chi_l \geq \chi$, the chromatic number of G.

Albertson, Grossman and Haas [1] asked the following: if each vertex of G is

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assigned a list of $t \le s = \chi_l$ colours, how many of the vertices, $\lambda_{t,s}$, are necessarily colourable from these lists? They conjectured that $\lambda_{t,s} \ge tn/s$ and found bounds on $\lambda_{t,s}$ in different situations. Their work was extended by Chappell, [2]. In this note we establish some further inequalities involving $\lambda_{t,s}$.

§2. Inequalities.

A basic and important result in [1] is the following, a different proof of which we include for the sake of completeness.

Theorem 1 If G is a graph with n vertices and chromatic number χ and list chromatic number $s = \chi_l$, then

$$\lambda_{t,s} > \left(1-\left(\frac{\chi-1}{\chi}\right)^t\right)n.$$

Proof: Any proper χ-colouring of G partitions the vertices into independent sets $C_1, C_2, \ldots, C_\chi$. Consider (the set of) partitions of the set of available colours, $R = \{1, 2, \ldots, r\} = \cup l(x)$, into χ classes $R_1, R_2, \ldots, R_\chi$, where in any such partition a colour goes into a particular class with probability $1/\chi$. We now colour a vertex $x \in C_j$ with a colour from its list of t colours, $l(x)$, only if $l(x) \cap R_j \neq \phi$; such a vertex is not coloured with probability $(1 - 1/\chi)^t$ and the result follows (some partition of R is at least average).

Theorem 1, as pointed out in [1], is asymptotically correct for any given χ.

The chromatic number χ naturally plays an important role in the study of $\lambda_{t,s}$. For example, suppose that $s = pt + k$, $0 \le k < t$. As some colour class in a proper colouring has at least n/χ vertices, $\chi \le p$ implies that $\lambda_{t,s} \ge n/\chi \ge tn/s$ as required. When $t = 2$, as pointed out in [1], the lower bound in Theorem 1 exceeds $2/\chi_l$ whenever $\chi < \chi_l$, thus $\lambda_{2,s} \ge 2n/s$ as conjectured.

In a similar vein to Theorem 1 we have

Theorem 2 If G is a graph with n vertices and list chromatic number $s = \chi_l = tp$ then $\lambda_{t,s} \ge tn/s$.

Proof For each vertex $x \in V(G)$, we augment its list, $l(x) = \{x_1, x_2, \ldots, x_t\}$, to create a list of pt colours by adding p-1 copies of $l(x)$: $l^2(x) = \{x_1^2, x_2^2, \ldots, x_t^2\}, \ldots, l^p(x) = \{x_1^p, x_2^p, \ldots, x_t^p\}$. Since G is s-choosable we may find a proper colouring, φ, using the augmented lists. This colouring φ defines a partition of V(G) into list-coloured classes $L_1, L_2, \ldots, L_p$ according to the use of colours from the list $l(x)$ or its p-1 copies. One of these classes contains at least the average number of vertices, thus the t-lists on the vertices of G are sufficient to list colour at least n/p = tn/s vertices.

Corollary If G is a graph with n vertices and list chromatic number $s = \chi_l$ where $s = tp + k$, $0 < k < t$, then $\lambda_{t,s} \geq \dfrac{n}{\lceil s/t \rceil}$.

Proof Note that $\lceil s/t \rceil = p+1$. Using the same proof as in the above theorem except using copies of $l(x)$ to augment the lists to size (p+1)t, we are done since s-choosability implies (s+t-k)-choosability.

Notes:

1. It follows from the above arguments that $p\lambda_{t,s} \geq \lambda_{pt,s}$.
2. It is tempting to think that the above proof could be extended by adding only a list of size k as the last list that covers k/t times as many vertices as the lists of size t. However it is not too difficult to construct an example in which lists of size m do not cover more than lists of size m-1, i.e. we cannot assume that the function is strictly increasing.
3. Further to point 2, it is possible that the colours from the list of size k can be assigned to more than k/t times as many vertices as those from the lists of size t. Thus, since there are not the same number of copies of each colour, the averaging argument from the last sentence in the proof of Theorem 2 may fail.

In [2], Chappell showed in general that $\lambda_{t,s} > (6/7)tn/s$. By the Corollary we have, when $s = tp + k$, $0 < k < t$, that

$$\lambda_{t,s} \geq \frac{p+\frac{k}{t}}{p+1}\frac{tn}{s},$$

which is an improvement for $s > 6(t-k)$ (or simply, for $p \geq 6$). Chappell's main result is that $\lambda_{t,s} \geq qn$, where q is the unique positive root of the equation $1 - x - (1 - (1-x)/(s-t))^t = 0$. This may be improved slightly in certain cases by combining his proof with that of the above.

Theorem 3 If G is a graph with n vertices and list chromatic number $s = \chi_l$ where $s = tp + k$, $0 < k < t$, then $\lambda_{t,s} \geq qn$ where q is the unique positive root of $1 - x - (1-(1-px)/k)^t = 0$.

Proof For each vertex $x \in V(G)$ augment its list $l(x)$, $\{x_1, x_2, \ldots, x_t\}$ to create a list of size s by adding p-1 copies of $l(x)$: $l^2(x), \ldots, l^p(x)$, as before, and also by adding the following list of k new colours: $\{z_1, z_2, \ldots, z_k\}$. Since G is now choosable we may find a proper colouring, φ, using the augmented lists. This colouring φ defines a partition of V(G) into list-coloured classes $L_1, L_2, \ldots, L_{p+k}$ according to the use of colours from the list $l(x)$ or its p-1 copies, and k classes of independent vertices corresponding to the use of colours from the final set of k colours. Partition the original set of colours, $R = \{1, 2, \ldots, r\}$, randomly into p+k classes $R_1, R_2, \ldots, R_{p+k}$ where, for $1 \leq j \leq p$, prob(colour $i \in R_j$) = q, and, for $1 \leq j \leq k$, prob(colour $i \in R_{p+j}$) = (1- pq)/k. (Re)colour G as follows: a vertex in L_j, $1 \leq j \leq p$ is coloured by colour x_i, if it was coloured by x_i or by a copy of x_i under φ, and if also $x_i \in R_j$; a vertex in L_{p+j}, $1 \leq j \leq k$, is coloured with a colour from its list if its t-list intersects R_{p+j}. Otherwise a vertex is not coloured. In this scheme, we have prob($x \in L_j$, $1 \leq j \leq p$, is coloured) = q and prob($x \in L_{p+j}$ is not coloured, $1 \leq j \leq k$) = $1 - ((1-pq)/k)^t$. Then all vertices of V(G) are coloured with probability q if $f(q) = 1 - (1-(1-pq)/k)^t - q = 0$ and the result follows.

Note: the above f(q) is a decreasing function and, for $s = pt+k$,
$f(t/s) =$

$$1 - \frac{t}{s} - (1 - \frac{1}{s})^t = -\binom{t}{2}/s^2 + \binom{t}{3}/s^3 - \binom{t}{4}/s^4 + \ldots + (-1)^{t-1}\binom{t}{t}/s^t < 0.$$

The advantage over Chappell's result appears when p > 1. A typical implication of Theorem 3 is

Corollary For $s = 2p+1$, $\lambda_{2,s} > \dfrac{-1 + \sqrt{4p^2 + 1}}{2p^2} n$.

Thus for example $\lambda_{2,5}$ is bounded below by $(-1+17^{1/2})n/8$ instead of $(-13+189^{1/2})n/2)$ from Chappell's result (about .3904n instead of .3739n).

Note: We have learned (private communication) that Theorem 2 was also obtained by Chappel.

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On the Chromatic Number of the Complement of a Class of Line Graphs

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ABSTRACT

Let G be a graph, $\overline{G}$ its complement, $L(G)$ its line graph, and $\chi(G)$ its chromatic number. Then we have the following

THEOREM *Let G be a graph with n vertices. (i) If G is triangle free, then*

$$n-4 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n-2$$

(ii) If G is planar and every triangle bounds a disk, then

$$n-3 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n-2$$

KEYWORDS: chromatic number, line graph, planar graph, triangle-free graph, Kneser graph

1. PRELIMINARIES

Let G be a graph, $\overline{G}$ its complement, $L(G)$ its line graph, and $\chi(G)$ its chromatic number. A *nonedge* of G is an edge of $\overline{G}$. Two nonedges of G are *adjacent* in G if they are adjacent as edges of $\overline{G}$ (*i.e.*, their endpoints intersect). They are *nonadjacent* if their endpoints are disjoint. The *clique complex* $\Delta(G)$ of G is the simplicial complex on the vertex set of G whose simplices are the cliques of G.

Following [4] we make the following definitions. For any set system $\mathcal{S}$, $KG(\mathcal{S})$ denotes the *Kneser graph* of $\mathcal{S}$, namely the graph whose vertices are the elements of $\mathcal{S}$ and whose edges are pairs of nonintersecting sets. When $\mathcal{S} = \binom{[n]}{k}$, the set of all k subsets of an n set $[n] := \{1, 2, \dots, n\}$, we denote $KG(\mathcal{S})$ by $K_{n:k}$. $\mathrm{MIN}(\mathcal{S})$ is the system of all sets in $\mathcal{S}$ that are minimal with respect to inclusion. $\|K\|$ means the geometric realization of the simplicial complex K. $J \backslash K$ means the elements of J that are not in K.

The key result we need is Sarkaria's colouring/embedding theorem, which is a generalization of the Van Kampen-Flores theorem on the embeddability of simplices into $\mathbb{R}^d$. We recall the theorem in the form which we require:

1.1 THEOREM ([4,5,6,7,8]). *Let K be a subcomplex of the $n-1$ dimensional simplex σ^{n-1}, and let $\mathcal{S} := \mathrm{MIN}(\sigma^{n-1} \backslash K)$. If*

$$d \leq n - \chi(KG(\mathcal{S})) - 2$$

then for any continuous mapping $f : \|K\| \to \mathbb{R}^d$, the images of some two disjoint faces of K intersect.

2. THE THEOREM

We have the following

2.1 THEOREM. *Let G be a graph with n vertices. (i) If G is triangle free, then*

$$n - 4 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n - 2$$

(ii) If G is planar and every triangle bounds a disk, then

$$n - 3 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n - 2$$

REMARK. The upper bound of $n-2$ holds for any graph G, not just triangle free graphs.

Proof. A vertex of $\overline{L(\overline{G})}$ is a nonedge of G, and two vertices are adjacent in $\overline{L(\overline{G})}$ if the corresponding nonedges of G are nonadjacent in G. Let G be the empty graph on n vertices. Then $\overline{L(\overline{G})} = K_{n:2}$. By the Lovász-Kneser theorem [1,2,3,4] $\chi(K_{n:2}) = n - 2$. Adding an edge to G removes a vertex from $\overline{L(\overline{G})}$, which can only decrease its chromatic number. Hence, for any graph G, $\chi\left(\overline{L(\overline{G})}\right) \leq n - 2$.

Now let $\mathcal{S} = \text{MIN}(\sigma^{n-1} \backslash G)$, where G is viewed as a one-dimensional simplicial complex. If G is triangle free, the inclusion minimal sets of $\mathcal{S}$ all have size 2, and are precisely the edges of $\overline{G}$. Hence $KG(\mathcal{S})$ is the same thing as $\overline{L(\overline{G})}$. Every graph is embeddable in $\mathbb{R}^3$, so from Theorem 1.1 we conclude that

$$n - \chi\left(\overline{L(\overline{G})}\right) - 2 < 3$$

or

$$\chi\left(\overline{L(\overline{G})}\right) \geq n - 4 \quad .$$

This proves (i).

To prove the lower bound in (ii), suppose G is planar and every triangle bounds a disk. Then the simplicial complex obtained by adjoining to G all the faces bounded by triangles is homeomorphic to $\|\Delta(G)\|$. In particular, $\|\Delta(G)\|$ can be embedded in the plane. Now set $\mathcal{S} = \text{MIN}(\sigma^{n-1} \backslash \Delta(G))$. The inclusion minimal nonfaces of the clique complex $\Delta(G)$ are precisely the edges of $\overline{G}$, so once again $KG(\mathcal{S})$ is just $\overline{L(\overline{G})}$. As $\|\Delta(G)\|$ embeds in the plane,

$$n - \chi\left(\overline{L(\overline{G})}\right) - 2 < 2$$

so

$$\chi\left(\overline{L(\overline{G})}\right) \geq n - 3 \quad .$$

∎

3. Observations

We end with a few observations.

- The upper bound on $\chi\left(\overline{L(\overline{G})}\right)$, namely $n - 2$, is equivalent to the condition $d \geq 0$ in Theorem 1.1.
- The triangle free condition in (i) is necessary. For example, let G be $K_n - e$. Then $\overline{G}$ is a single edge, and $L(\overline{G})$ and $\overline{L(\overline{G})}$ are both a single point. As $\chi(\text{point}) = 1$, $\chi\left(\overline{L(\overline{G})}\right) < n - 4$ for any $n > 5$.

- To illustrate the theorem, let G be $K_{3,3}$, the complete bipartite graph on two sets of three vertices. Then $\overline{L(\overline{G})} = G$, and its chromatic number is $2 = 6 - 4$. Also, both bounds can be achieved in the planar case: $G = C_5$, the 5-cycle, satisfies $\overline{L(\overline{G})} = G$, and its chromatic number is $3 = 5 - 2$. On the other hand, if G is the 6-cycle plus an edge connecting two vertices a distance 3 apart on the cycle, then one can check that $\overline{L(\overline{G})}$ has chromatic number $3 = 6 - 3$.

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