On the linear structure and clique-width of bipartite permutation graphs

A. Brandstädt* and V.V. Lozin†

Abstract

Bipartite permutation graphs have several nice characterizations in terms of vertex ordering. Besides, as AT-free graphs, they have a linear structure in the sense that any connected bipartite permutation graph has a dominating path. In the present paper, we elaborate the linear structure of bipartite permutation graphs by showing that any connected graph in the class can be stretched into a "path" with "edges" being chain graphs. A particular consequence from the obtained characterization is that the clique-width of bipartite permutation graphs is unbounded, which refines a recent result of Golumbic and Rotics for permutation graphs.

Keywords: Bipartite permutation graphs; Clique-width

1 Introduction

In this note we consider the intersection of two well studied subclasses of perfect graphs, namely bipartite and permutation graphs. Several nice characterizations of bipartite permutation graphs in terms of vertex ordering have been proposed in [17, 15] and have been exploited extensively in algorithmic applications (see e.g. [2, 3, 4, 13, 15, 16, 18, 19]). Independently, the same class of graphs has been introduced in [6] under the name of bipartite tolerance graphs. The fact that two the classes coincide was proved in [1]. Furthermore, easy arguments based on Gallai's list of minimal incomparability graphs [9] lead to the conclusion that the class of bipartite permutation graphs is exactly the class of bipartite graphs without asteroidal triples. A remarkable property of asteroidal triple-free graphs is that they have a linear structure in the sense that any connected graph in

*Universität Rostock FB Informatik, Albert-Einstein-Str. 21, D 18051 Rostock, Germany. E-mail: ab@informatik.uni-rostock.de

†RUTCOR, Rutgers University, 640 Bartholomew Rd. Piscataway NJ 08854-8003 USA. E-mail: lozin@rutcor.rutgers.edu

this class has a dominating path [5]. In the present paper we analyze the proposition about the linear structure of bipartite permutation graphs in more detail. Particularly, we show that every connected bipartite permutation graph can be stretched into a "path" with "edges" being chain graphs (the exact formulation and the proof of the result is given in Section 2).

The class of chain graphs appeared in the literature under different names such as difference graphs [12], bi-split graphs [8] or nonseparable bipartite graphs [7]. In terms of forbidden induced subgraphs it can be characterized as $2K_2$-free bipartite graphs, where a $2K_2$ is the disjoint union of two edges. From this characterization one can easily derive the property that vertices in each part of a chain graph can be linearly ordered under inclusion of their neighborhoods. We call a vertex ordering "$<$" *increasing* if $x < y$ implies $N_G(x) \subseteq N_G(y)$, and *decreasing* if $x < y$ implies $N_G(y) \subseteq N_G(x)$, where $N_G(x)$ denotes the neighborhood of a vertex x in graph G. Another useful property of chain graphs is that they are of bounded clique-width, which implies linear time algorithms for a number of graph problems NP-hard in general graphs. This follows from the fact that chain graphs are distance-hereditary since they are (house, hole, domino, gem)-free and thus according to a recent result of Golumbic and Rotics [10] have clique-width at most 3. The same holds for any tree and hence for any path. Surprisingly, the clique-width of "paths" composed from chain graphs, i.e. of bipartite permutation graphs, is not bounded by any constant. This proposition, proved in Section 3 of the paper, strengthens a recent result of Golumbic and Rotics for permutation graphs [10]. Unboundedness of the clique-width is in a dissonance with the fact that numerous hard problems, restricted to bipartite permutation graphs, have efficient solutions. This suggests the idea to look for a more powerful concept than the clique-width. We settle this idea as an open problem in the concluding part of the paper along with some other questions for prospective research.

2 The structure of bipartite permutation graphs

To derive the structural characterization, we use an ordinary intersection model for permutation graphs. With each vertex v of a permutation graph G we associate a line segment $S(v)$ with endpoints $x(v)$ and $y(v)$ being on two parallel lines X and Y, respectively. Two vertices v and u are adjacent in G if and only if $S(v)$ and $S(u)$ cross each other. Assuming that X and Y are horizontal, we write $x(v) < x(u)$ to indicate that $x(u)$ is to the right of $x(v)$. If both $x(v) < x(u)$ and $y(v) < y(u)$, we write $S(v) < S(u)$ and say that $S(u)$ is to the right of $S(v)$.

Theorem 1 *A connected graph G is bipartite permutation if and only if the vertex set of G can be partitioned into stable sets $D_0, D_1, \ldots, D_q$ in such a way that*

(a) any two vertices in non-consecutive sets are non-adjacent,

(b) any two consecutive sets D_{j-1} and D_j induce a chain graph, denoted G_j,

(c) for each $j = 1, 2, \ldots, q-1$, there is an ordering of vertices in set D_j, which is decreasing in G_j and increasing in G_{j+1}.

Proof. Necessity. Let G be a connected bipartite permutation graph and M an intersection model for G. We denote by p_0 a vertex in G with leftmost line segment in M, i.e. $S(p_0) < S(v)$ for each vertex v non-adjacent to p_0.

Define D_j to be the subset of vertices at distance j from p_0. In particular, $D_0 = \{p_0\}$. Since G is bipartite, D_j is a stable set for each j. Therefore, the line segments of M corresponding to the vertices of D_j are mutually non-crossing. For each $j > 0$, let p_j denote the vertex in D_j with rightmost line segment in M.

We will prove that partition $D_0 \cup D_1 \cup \ldots \cup D_q$ satisfies all three conditions of the theorem.

Condition (a) is due to the definition of the partition. Condition (b) will be proved by induction. Moreover, we will show by induction on j that

(1) G_j is $2K_2$-free,

(2) p_{j-1} is adjacent to each vertex in D_j,

(3) for every $v \in D_i$ with $i > j$, $S(p_{j-1}) < S(v)$.

For $j = 1$, statements (1), (2), (3) are obvious. To make the inductive step, we assume by contradiction that vertices $x_1, x_2 \in D_{j-1}$ and $y_1, y_2 \in D_j$ induce in G_j a $2K_2$ with edges (x_1, y_1) and (x_2, y_2). By the inductive hypothesis, both $S(x_1)$ and $S(x_2)$ intersect $S(p_{j-2})$, and both $S(y_1)$ and $S(y_2)$ are to the right of $S(p_{j-2})$. Taking into account that $S(x_1)$ and $S(x_2)$ are non-crossing, we must conclude that either $S(y_1)$ or $S(y_2)$ intersects both $S(x_1)$ and $S(x_2)$, which contradicts the assumption. Hence, statement (1) is correct. To prove (2) and (3), consider a vertex $v \in D_i$, $i \geq j$, non-adjacent to p_{j-1}. Since $S(p_{j-2})$ crosses $S(p_{j-1})$, and $S(p_{j-2}) < S(v)$, we conclude that $S(p_{j-1}) < S(v)$, which proves (3). Furthermore, from $S(p_{j-1}) < S(v)$ it follows that v does not have neighbors in D_{j-1} because of the choice of p_{j-1}. Consequently, $v \notin D_j$ and hence (2) is valid.

To prove (c), we will show that for every pair of vertices u and v in D_j, $N_{G_j}(u) \subset N_{G_j}(v)$ implies $N_{G_{j+1}}(v) \subseteq N_{G_{j+1}}(u)$. Assume the contrary:

$s \in N_{G_j}(v) - N_{G_j}(u)$ and $t \in N_{G_{j+1}}(v) - N_{G_{j+1}}(u)$. From (2) we deduce that $s \neq p_{j-1}$. Consequently, $j > 1$. Due to the choice of p_{j-1} we have $S(s) < S(p_{j-1})$, and from (3) we obtain $S(p_{j-2}) < S(u)$ and $S(p_{j-2}) < S(v)$. Therefore, $S(v) < S(u)$ by geometrical considerations. But now, geometrical arguments lead us to conclusion that $(t, v) \in E(G)$ implies $(t, u) \in E(G)$. This contradiction proves condition (c).

Sufficiency. Consider a graph G with a partition of vertices into stable sets $D_0, D_1, \ldots, D_q$ satisfying conditions (a), (b), (c). We assume that vertices of

$$D_j = \{v_{j,1}, v_{j,2}, \ldots, v_{j,k_j}\}$$

are listed in the order that agrees with (c). Obviously G is bipartite. To prove that G is a permutation graph, we will construct an intersection model M for it as follows. Each subset D_j will be represented in M by a series of parallel line segments in such a way that $S(v_{j,i}) < S(v_{j,k})$ whenever $i < k$. For $j = 0$, there are no other restrictions. For $j > 0$, we proceed inductively distinguishing between odd and even cases. Let j be odd. For every vertex $u \in D_j$ with neighborhood

$$N_{G_j}(u) = \{v_{j-1,s}, v_{j-1,s+1}, \ldots, v_{j-1,k_{j-1}}\},$$

we place $x(u)$ between $x(v_{j-1,s})$ and the nearest point to the left of $x(v_{j-1,s})$ (if any). The other end of $S(u)$ is placed on the line Y in such a way that $y(u) > y(v)$ for each $v \in D_{j-1}$. For an even j, we switch between X and Y in the description above. It is an easy exercise to verify that the constructed model represents graph G. ■

3 Clique-width of bipartite permutation graphs is unbounded

In the present section we use the obtained characterization for bipartite permutation graphs in order to prove that the clique-width of these graphs is unbounded. The clique-width of a graph G, denoted $cwd(G)$, is defined as the minimum number of labels needed to construct G, using the four graph operations: creation of a new vertex v with label i (denoted $i(v)$), disjoint union of two labeled graphs G and H (denoted $G \oplus H$), connecting vertices with specified labels i and j (denoted $\eta_{i,j}$) and renaming labels (denoted ρ). Every graph can be defined by an algebraic expression using the four operations above. For instance, the graph consisting of 2 isolated vertices x and y can be defined by expression $1(x) \oplus 1(y)$, and the graph consisting of two adjacent vertices x and y can be defined by expression $\eta_{1,2}(1(x) \oplus 2(y))$.

With any graph G and an algebraic expression T, defining G, we associate a tree, denoted $tree(T)$, whose leaves are the vertices of G, and the internal nodes correspond to operations $\oplus$, η and ρ in T. Given a node a in $tree(T)$, we denote by $tree(a,T)$ the subtree of $tree(T)$ rooted at a. The label of a vertex v of graph G at the node a of $tree(T)$ is defined as the label that v has immediately before the operation a is applied.

In [10], it has been proved that classes of unit interval graphs, permutation graphs and square grids are not of bounded clique-width. In the present paper we strengthen the result for permutation graphs by showing that the clique-width is unbounded even for bipartite permutation graphs. Our approach is quite similar to the idea used in [10] for square grids.

Consider an $n \times n$ grid G, i.e. a graph with vertex set $\{v_{i,j} : 1 \leq i,j \leq n\}$ and edge set $\{(v_{i,j}, v_{k,l}) : (k = i \text{ and } l = j{+}1) \text{ or } (k = i{+}1 \text{ and } l = j)\}$. We shall say that vertex $v_{i,j}$ belongs to row i and column j. Let us construct graph H_n from grid G by adding edges $(v_{i,j}, v_{k,l})$ whenever $i < k$, $j < l$ and $k - i = l - j + 1$.

Lemma 2 *H_n is a bipartite permutation graph.*

Proof. Partition the vertices of H_n into subsets

$$D_{-n+1}, D_{-n+2}, \ldots, D_{-1}, D_0, D_1, \ldots, D_{n-1}$$

defined as follows. D_0 consists of the vertices on the main diagonal of H_n, i.e. the vertices of form $v_{i,i}$. For $j > 0$, $D_j = \{v_{i,i+j} : i = 1, \ldots, n - j\}$ and D_{-j} is the vertex subset symmetric to D_j with respect to the main diagonal. It is not hard to verify that the defined partition satisfies all three conditions of Theorem 1. Therefore, H_n is a bipartite permutation graph. ∎

Let D_p be an arbitrary vertex subset of graph H_n as defined in Lemma 2. In the subsequent lemma the following two observations will be useful:

(1') if a vertex $v_{i,j}$ belongs to D_p, then it has at most one neighbor $v_{k,l} \in D_{p+1}$ with $l > j$;

(1") if a vertex $v_{i,j}$ belongs to D_p, then it has no neighbor $v_{k,l} \in D_{p-1}$ with $k < i$.

Lemma 3 *$cwd(H_n) \geq n/6$.*

Proof. Let T be an algebraic expression defining H_n and a the lowest $\oplus$ node in $tree(T)$ such that the graph defined by $tree(a,T)$ contains a full row and a full column of H_n. We denote by b and c the two sons of a in $tree(T)$. Let us color the vertices of H_n in $tree(b,T)$ and $tree(c,T)$ by red and blue, respectively, and all the other vertices by white. We denote the color of a vertex u by $color(u)$ and say that u is non-white (respectively, non-red, non-blue) if $color(u)$ is different from white (respectively, red, blue).

Let u be a red vertex and v a non-white vertex. Assume that there is a non-red vertex w which is adjacent to u but not to v. Since w is non-red, the edge (w, u) appears in H_n under some operation of type η located outside of $tree(a, T)$. Therefore, to avoid introducing edge (w, v) under the same operation, vertices u and v must have different labels at node a. The same can be proved if u is a blue vertex and w is non-blue. We summarize the above arguments as observation

(2) If u is red (respectively, blue), v is non-white and there is a non-red (respectively, non-blue) vertex w which is adjacent to u but not to v, then u and v must have different labels at node a.

Due to the choice of a, H_n contains a row and a column with no white vertex. Let us denote a row without white vertices by r and assume first that there is neither blue nor red column in H_n. Then every column j must have a vertex with different color than that of $v_{r,j}$. We denote by $w_{i,j}$ a nearest to $v_{r,j}$ vertex in the same column with $color(w_{i,j}) \neq color(v_{r,j})$. We then let u_j denote $w_{i+1,j}$ if $i < r$ or $w_{i-1,j}$ if $i > r$. Notice that u_j is non-white and $w_{i,j}$ is adjacent to u_j. Our goal is to show that vertices in set $Q = \{u_j \ : \ j = 1, \ldots, n\}$ are labeled with at least $n/6$ labels at a. To this end, we will prove that at most 6 vertices in Q can have the same label at node a. Suppose by contradiction that vertices

$$Q' = \{u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}, u_{j_5}, u_{j_6}, u_{j_7}\} \subseteq Q$$

have the same label at a. We assume that $i < k$ implies $j_i < j_k$. Recall that w_{i_k,j_k} denotes a vertex adjacent to u_{j_k} with $color(w_{i_k,j_k}) \neq color(u_{j_k})$. We call w_{i_k,j_k} the vertex corresponding to u_{j_k}, and denote the set of vertices corresponding to the vertices in Q' by W'. Obviously, different vertices in Q' have different corresponding vertices in W'.

Assume a vertex u_i in Q' is not adjacent to a vertex $w_{k,j}$ in W'. Clearly $i \neq j$ and hence vertices u_j, u_i and $w_{k,j}$ form a triple satisfying conditions of observation (2). Due to this observation, vertices u_j and u_i have different labels at a that contradicts our assumption. Therefore, to avoid the contradiction, we must assume that each vertex in Q' is adjacent to each vertex in W'. We let vertex w_{i_1,j_1} (i.e. the vertex of W' in the leftmost column) belong to subset D_p defined in the preceding lemma. Vertex w_{i_1,j_1} has six neighbors $u_j \in Q'$ with $j > j_1$. Taking into account observation (1'), we conclude that at least five vertices of Q' belong to D_{p-1}. Their corresponding neighbors in set W' are hence in D_p or in D_{p-2}. Assume D_{p-2}

contains at least three vertices in W'. Then the vertex $w_{i,j} \in D_{p-2} \cap W'$ with the minimum index j has at least two neighbors $u_s, u_t \in D_{p-1} \cap Q'$ with $j < s, t$, contradicting observation (1'). Hence D_{p-2} contains at most 2 vertices in W'. Consequently at least 3 vertices of W' are in D_p. But now a vertex $u_j \in D_{p-1} \cap Q'$ with the minimum index j has at least two neighbors $w(i,s), w(k,t) \in D_p \cap W'$ with $j < s, t$. This contradiction proves that at most 6 vertices in Q can have the same label. Consequently, the vertices in Q are labeled with at least $n/6$ labels at node a.

The above conclusion has been obtained under assumption that there is neither blue nor red column in H_n. Let now H_n contain a blue column. Then obviously it cannot contain a red row. Moreover, it cannot contain a blue row due to the choice of a. Hence, we may apply observation (1") to prove the lemma by arguments similar to those developed above. The case when H_n contains a red column is analyzed by analogy. ■

A natural consequence from Lemmas 2 and 3 is

Theorem 4 *The clique-width of bipartite permutation graphs is unbounded.*

4 Concluding remarks and open problems

In the present paper we have proven that any connected bipartite permutation graph can be stretched into a "path" with "edges" being chain graphs. This result brought us to the conclusion that the clique-width of bipartite permutation graphs is unbounded. In conjunction with the fact that many algorithmic problems have efficient solutions for bipartite permutation graphs, the latter result is a challenge to generalize the notion of clique-width. We consider the question of finding such a generalization as an open problem. A simple but important observation in this direction is that any vertex of a bipartite permutation graph belongs to at most two chain graphs. In addition, we conjecture that for any hereditary subclass of bipartite permutation graphs, the clique-width is bounded. Along with the question raised above, the obtained results motivate us to study the following problems.

Kloks et al. proposed in [14] a polynomial time algorithm to compute the bandwidth of chain graphs and asked whether the result can be extended to the class of bipartite permutation graphs. The question is still open, and we believe that the characterization proposed in the present paper can help in finding the answer.

It is well known that chain graphs are closely related to threshold graphs. Namely, replacing one of the parts of a chain graph with a clique, we obtain a threshold graph and vice versa (see e.g. [12]). Using this relation one can easily obtain universal chain graphs from universal threshold graphs described in [11]. Furthermore, the obtained characterization for bipartite permutation graphs provides a simple way to construct universal bipartite permutation graphs from universal chain graphs. However, the number and the size of universal chain graphs needed to obtain a minimum universal bipartite permutation graph is the topic for a special research.

References

[1] A. Brandstädt, J. Spinrad and L Stewart, Bipartite permutation graphs are bipartite tolerance graphs, Congr. Numer. 58 (1987) 165-164.

[2] L. Chen, J.Y. Jiang and M.T. Nyeu, Provably fastest parallel algorithms for bipartite permutation graphs, Parallel Process. Lett. 9 (1999) 385-390.

[3] L. Chen, Solving the shortest path problem on bipartite permutation graphs efficiently, Inform. Process. Lett. 55 (1995) 259-264.

[4] L. Chen and Y. Yesha, Efficient parallel algorithms for bipartite permutation graphs, Networks 23 (1993) 29-39.

[5] D.G. Corneil, S. Olariu and L. Stewart, Asteroidal triple-free graphs, SIAM J. Discrete Math. 10 (1997) 399-430.

[6] U. Derigs, O. Goecke and R. Schrader, Bisimplicial edges, Gaussian elimination and matchings in bipartite graphs, Inter. Workshop on Graph-Theoretic Concepts in Comp. Sci. Trauner Verlag, Berlin, W. Germany (1984) 79-87.

[7] G. Ding, Covering the edges with consecutive sets, J. Graph Theory 15 (1991) 559-562.

[8] H. Frost, M. Jacobson, J. Kabell, F.R. Morris, Bipartite analogues of split graphs and related topics, ARS Combinatorica, 29 (1990) 283-288.

[9] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hung. 18 (1967) 25-66.

[10] M.C. Golumbic and U. Rotics, On the clique-width of some perfect graph classes, Internat. J. Found. Comput. Sci. 11 (2000) 423–443.

[11] P.L. Hammer and K. Kelmans, On universal threshold graphs, Combinatorics, probability and computing 3 (1994) 327-344.

[12] P.L. Hammer, U.N. Peled, and X. Sun, Difference graphs. Discrete Appl. Math. 28 (1990) 35-44.

[13] H. Kim and A.P. Sprague, Optimal parallel matchings on bipartite permutation graphs, Parallel Algorithms Appl. 10 (1997) 165-176.

[14] T. Kloks, D. Kratsch and H. Müller, Bandwidth of chain graphs, Inform. Process. Lett. 68 (1998) 313-315.

[15] T.-H. Lai and S.-S. Wei, Bipartite permutation graphs with application to the minimum buffer size problem, Discrete Appl. Math. 74 (1997) 33-55.

[16] C.L. Lu and C. Y. Tang, Solving the weighted efficient edge domination problem on bipartite permutation graphs, Discrete Appl. Math. 87 (1998) 203-211.

[17] J. Spinrad, A. Brandstädt and L. Stewart, Bipartite permutation graphs, Discrete Appl. Math. (1987) 68-77.

[18] R. Srikant, R. Sundaram, K. Sher Singh and C. Pandu Rangan, Optimal path cover problem on block graphs and bipartite permutation graphs, Theoret. Comput. Sci. 115 (1993) 351-357.

[19] A. Srinivasan, K. Mudhukar, P. Nagavamsi, C. Pandu Rangan and M.-S. Chang, Edge domination on bipartite permutation graphs and cotriangulated graphs, Inform. Process. Lett. 56 (1995) 165-171.