EIGENVALUES OF THE DISCRETE p-LAPLACIAN FOR GRAPHS

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ABSTRACT. We study the discrete version of the p-Laplacian operator $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ and we give some estimates of its smallest positive eigenvalue. In earlier papers eigenvalues of the discrete Laplacian have been considered. We shall here study more general means. We shall also in particular study the case when the graph is complete. We give an estimated of the smallest positive eigenvalue of the p-Laplacian when the graph is a subgraph of $\mathbb{Z}^d$; in this context we give all eigenvalues of the p-Laplacian when the graph is complete.

1. INTRODUCTION

The eigenvalues of the Laplacian have many physical interpretations, for example as the frequencies of vibration of a membrane, as the rates of decay for the heat equation (or mass diffusion). However, the eigenvalues of the p-Laplacian can be calculated explicitly only for a few special graphs, most notably for regular graphs. In this paper we study the discrete p-Laplacian, which is the analog of the operator $\mathrm{div}(|\nabla\cdot|^{p-2}\nabla\cdot)$. We will give sharp estimates of the smallest eigenvalue of the p-Laplacian involving Sobolev inequalities. A more familiar operator in the field of spectral geometry is the Laplacian operator, Δ, which appears so prominently in the heat equation and the wave equation. The analog on Γ, $-\Delta$, can be defined as the differential operator associated to the standard Dirichlet form

$$Q(u) = \frac{1}{2}\sum_{x\sim y} |u(x) - u(y)|^2.$$

In this way we define the Laplacian operator on Γ by

$$\Delta\phi(x) = \frac{1}{d(x)}\sum_{x\sim y}[\phi(x) - \phi(y)]$$

where ϕ is a function on the graph Γ. The basic idea of generalizing the discrete Laplacian is the following: instead of the uniform mass on Q, we assign to each edge $[x,y]$ a weight $c[x,y]$. We then obtain a new quadratic form Q_c defined by

$$Q_c(u) = \frac{1}{2}\sum_{x\sim y} c[x,y]|u(x) - u(y)|^2.$$

Date: February 27, 2001.
1991 *Mathematics Subject Classification.* Primary 31C40, 31C45, 60J50; Secondary 31C35, 31B35.
Key words and phrases. Dirichlet form, Eigenfunction, Eigenvalue, Graphs, Isoperimetric constant, Isoperimetric inequalities, p-Laplacian, Trees .

The analog of Q_c in the continuous case on $\mathbb{R}^n$ is

$$Q_w(u) = \int_\Omega w(x)|\nabla u(x)|^2 \mathrm{d}x,$$

where Ω is an open set of $\mathbb{R}^n$; and the quadratic differential operator induced by Q_w is denoted by $-\mathrm{div}(w\nabla u)$. The concept of the p-Laplacian Δ_p arises as a natural generalization of the Laplacian when we are interested in the discrete analogue case of the operator $-\mathrm{div}(w|\nabla \cdot|^{p-2}\nabla\cdot)$. Since it is natural to consider a Dirichlet form that decays with rate p (rather than the square), so the relevant form is the *p-Dirichlet form* of a function u on Γ and defined by

$$|u|_p^p = \frac{1}{2}\sum_{x\sim y} c[x,y]^{p-1}|u(x) - u(y)|^p.$$

A wealth of beautiful discrete Dirichlet form ($p = 2$) is due to [BD58]. Later development in the linear case are exhaustively [NY76], [NY77] and [KY84]. More detailed information is in [Kai92]. Recently, work has been done to extend the principle to finite graphs, as well as to investigate nonlinear extensions of the classical principle [Lin93], [Lin90] and [PS51]. In particular, the linear situation. which involves Markov chains, is directly related to many applications and research. Here we extend the principle to a graph setting. Note that finite graphs are good illustration of finite gradient objects associated to numerical solutions of differential equations.

Another theme in this paper is that of functional analysis viewpoints on a graph. We begin to develop some of these ideas with respect to finite electrical networks.

The original motivation was to further investigate the nonlinear version of the p-Laplacian. The work presented in this paper represents the second half of the following investigation: we have established the type of discretized Wiener test associated to the solution of the Dirichlet problem relatively to the p-Laplacian on infinite trees [Amg98].

There are already many extension of the Laplacian in the literature; we shall emphasize the discrete linear case of recent interest has been the work of [CDS80], [Chu97], [DSC96b],[DS91] and [CRS97]. In the nonlinear discrete case [NY76] the p-Laplacian and some other problems were studied. Recently the Harnack inequalities for p-Laplacian on graphs was studied in [HS97]. It is not uninteresting to see how our approach applies to results already known. However the true value of a new method must be judged by its ability to generate new results in important cases.

The structure of this paper is as follows. Section 2 develops the basic of analytical tools of eigenvalues and electrical networks. Section 3 introduce the largest eigenvalue of the p-Laplacian, the mean result of this section is Theorem 2 which has an analog in the continuous case. This section contains also some examples which include the tree and regular graphs.

Section 4 discusses isoperimetric inequalities and Cheeger's constant, this section contains precise statments of the main results and contains a version of the argument mentioned above. It also contains a proof of lemma 3, which requires very little computation. Lemma 3 will be useful in other contexts. Co-area formula is used to give upper and lower bounds for the smallest eigenvalue of the p-Laplacian. Section 5 treat normal "extension" such as d-dimensional isoperimetric inequalities and Sobolev inequalities.

Section 6 contains the Log-Sobolev inequality and an analog of Theorem 2 for the smallest eigenvalue of the p-Laplacian for some particular graphs.

Section 7 and 8 introduce two types of graphs for which we give the upper bound for the first eigenvalue of the p-Laplacian that could be seen as an generalization of the linear case.

Section 9 contains the values of the eingenvalues in the special case where the graph is complete. The reason for separating this from the general case is that in the simple complete graph we have reasonably accurate values of all eigenvalues and eigenfunctions.

2. Eigenvalue and eigenfunction

All the work is in obtaining the estimates on λ_p, in particular the results of this section will be useful in other section of this paper. Let $G = (V, E)$ be a finite graph and $C[u,v]$ denote the *conductance* of the edge $[u,v]$ with $C[u,v] = C[v,u]$. We first define the p-Laplacian for graphs without loops and multiple edges as in [Yam77] and [Yam86].

Let $\phi_p(\cdot)$ be the real function on the real line defined by

$$\phi_p(t) = |t|^{p-1}\text{sign}(t).$$

For a real function u on V and $1 < p < \infty$, we define as in [Yam86] the p-Laplacian $\Delta_p u$ of u to be the real function on V by

$$\Delta_p u(x) = \frac{1}{C[x]} \sum_{y\sim x} C[x,y]^{p-1}\phi_p[(u(x) - u(y))];$$

where $C[x] = \sum_{y\sim x} C[x,y]^{p-1}$. We say that a function u on V is *p-harmonic on a subset* X of V if $\Delta_p u = 0$ on X. If $\Delta_p u = 0$ on V we say u is *p-harmonic*. Let ν be a positive function on V. The real number λ is called an *eigenvalue* for ν and Δ_p if there exists a function $u \neq 0$ such that

$$\Delta_p u(x) = \lambda\ \nu(x)\phi_p[u(x)]$$

If so, then u is called an p-harmonic *eigenfunction* corresponding to λ and ν.

Suppose $\Gamma = (V, E)$ is a finite graph with a fixed orientation on its edges. Let $e = [x,y]$ be an edge in which x and y are endpoints. The *gradient* of a function f is defined by

$$\nabla f(e) = f(y) - f(x).$$

We now present analysis for the fundamental property of calculus and integration by parts on the graph.

Proposition 2.1. *If λ is an eigenvalue, and u its corresponding eigenfunction then*

$$\sum_{e\in E} \phi_p[\nabla u(e)]\nabla\psi(e)C(e)^{p-1} = 2\lambda \sum_{x\in V} \phi_p[u(x)]\psi(x)\nu(x)C[x]$$

whenever $\psi \in \mathbb{R}^V$.

Proof. Let $e = [x,y]$ be an edge and let $\bar{e}$ denote the edge $[y,x]$. Remark that

$$\phi_p[\nabla u(e)]\nabla\psi(e) = \phi_p[\nabla u(e)]\psi(x) + \phi_p[\nabla u(\bar{e})]\psi(y)$$

and $C(e) = C(\bar{e})$. By using the equation

$$C[x]\Delta_p\phi(x) = \sum_{y\sim x} \phi_p[\nabla u([x,y])]C[x,y]^{p-1}$$

we obtain the result. □

From the Proposition 2.1, we see that all eigenvalues are non-negative. We can easily deduce that 0 is an eigenvalue of Δ_p. Let $\mathbb{I}$ denote the constant function which assumes the value 1 on each vertex. Then $\mathbb{I}$ is an p-harmonic eigenfunction of Δ_p with eigenvalue 0.

The *first eigenvalue* λ_p of the p-Laplacian (p-harmonic) operator is defined as the smallest positive real number λ for which the equation

$$\Delta_p u(x) = \lambda\, \nu(x)\, \phi_p[u(x)]$$

has a nontrivial solution u.

The *variance* of ϕ relative to ν is denoted by $\mathrm{var}_p(\phi)$ and defined to be the quantity

$$\min_m \{\sum_{v\in V} |\phi(v) - m|^p C(v)\ \nu(v)\}.$$

The usual minimax characterization of eigenvalues will be stated as follows

Theorem 1. *let G be a connected graph. Then the first eigenvalue is the minimum of the Rayleigh quotient*

$$\lambda_p = \inf\{\frac{|\phi|_p^p}{\mathrm{var}_p(\phi)};\ \phi \textit{ is non-constant}\}$$

where $|\phi|_p$ denote the p-Dirichlet form on G.

Proof. Taking the derivative with respect to $u(x)$ of $\frac{|u|_p^p}{\mathrm{var}_p(u)}$ we have

$$\frac{p\Delta_p u(x)}{\mathrm{var}_p(u)} - p\frac{|u|_p^p \phi_p(u(x) - \bar{u})\nu(x)}{\mathrm{var}_p(u)^2} = 0.$$

After substituting λ, the above equation can be simplified to:

$$\Delta_p u(x) - \lambda\, \nu(x)\, \phi_p(u(x) - \bar{u}) = 0.$$

Therefore we obtain that $v = u - \bar{u}$ is an p-harmonic eigenfunction of Δ_p with value λ. □

3. LARGEST EIGENVALUE OF Δ_p

The material in this section is of some what different nature from the rest of this work, being primarily combinatorial. The reader should note that the only information from this section which is logically necessary in what follows consists in considering the simple case i.e. $C[x, y] = 1$ for all edge $[x, y]$. As an application consider the case $\nu(x) = 1$ for all $x \in V$. The *largest eigenvalue* γ_p of Δ_p is defined as the largest real number λ for which the equation

$$\Delta_p u(x) = \lambda\, \phi_p[u(x)]$$

has a nontrivial solution u. Let $\|\phi\|_p^p$ be the quantity $\sum_{x\in V} d_x|\phi(x)|^p$ where d_x denote the degree of the vertex x. We can characterize the eigenvalue γ_p of Δ_p in terms of the Rayleigh quotient. The largest eigenvalue satisfies:

$$\gamma_p = \sup\{\frac{|\phi|_p^p}{\|\phi\|_p^p};\ \phi \text{ nonconstant}\}.$$

Taking the derivative with respect to $\phi(x)$ of the equation of γ_p, we have

$$\frac{p\Delta_p\phi(x)}{\|\phi\|_p^p} - p\frac{|\phi|_p^p\phi_p(\phi(x))}{\|\phi\|_p^{2p}} = 0.$$

After substituting γ_p, the above expression can be simplified to:

$$\Delta_p\phi(x) - \gamma_p\phi_p[\phi(x)] = 0.$$

Therefore we obtain ϕ is an eigenfunction of Δ_p with value γ_p. □

The first comparison between largest eigenvalues of Δ_p when $p > 1$. The following is a discrete analog of Theorem 3.2 of [Lin93].

Theorem 2. *For any connected graph we have* $p(2\gamma_p)^{1/p} \le s(2\gamma_s)^{1/s}$, *if* $1 < p < s < \infty$.

Proof. To see this, choose any ϕ to be an eigenfunction corresponding to the eigenvalue γ_p; and put $\phi = |\psi|^{s/p-1}\psi$. By definition we have

$$2\gamma_p = \frac{|\phi|_p^p}{\|\phi\|_p^p} = \frac{\sum_{x\sim y}|\phi(x)-\phi(y)|^p}{\|\psi\|_s^s}.$$

Using the Rayleigh quotient for γ_p and Hölder inequality we obtain

$$(2\gamma_p)^{1/p} \le \frac{2^{1/s}|\psi|_s}{\|\psi\|_s^{s/p}}\left(\sum_{x\sim y}\frac{|\phi(x)-\phi(y)|^{p\alpha}}{|\psi(x)-\psi(y)|^{p\alpha}}\right)^{1/(\alpha p)}$$

where α is the conjugate of s/p. Then, according to Lemma 3, we have

$$(2\gamma_p)^{1/p} \le \frac{s}{p}\frac{2^{1/s}|\psi|_s}{\|\psi\|_s^{s/p}}\left(\sum_{x\sim y}\||\psi|^{(s/p)}(x) + |\psi|^{(s/p)}(y)|^p(1/2)^p\right)^{1/(\alpha p)}.$$

From convexity of t^p, we get

$$(2\gamma_p)^{1/p} \le \frac{s}{p}\frac{2^{1/s}|\psi|_s}{\|\psi\|_s^{s/p}}\left(\frac{1}{2}\sum_{x\sim y}\psi(x)^s + \psi(y)^s\right)^{1/(p\alpha)} = \frac{s}{p}\frac{2^{1/s}|\psi|_s}{\|\psi\|_s},$$

which gives

$$(2\gamma_p)^{1/p} \le \frac{s}{p}\frac{2^{1/s}|\psi|_s}{\|\psi\|_s}.$$

By taking the supremum over all ψ, we arrive at the desired inequality. □

If u and v are vertices of $\mathcal{G} = (V, E)$, then a *walk of length n from u to v* is a sequence of vertices $W = v_0, \dots, v_n$ whose initial vertex v_0 is u, whose final vertex v_n is v, and for $i = 1, \dots, n$ $[v_{i-1}, v_i]$ is an edge. If $u \ne v$ then W is called an *open walk*, if $u = v$, W is called a *closed walk*. An *open walk* is called *a path* if its vertices are distinct. A closed walk is called a *cycle* if every pair of vertices except its starting and stopping are distinct.

Proposition 3.1. *let $\mathcal{G} = (V, E)$ be a connected graph with conductance 1 and $\nu(x) = 1$ for all x. Then, every cycle is composed with an even number of vertices if and only if the largest eigenvalue of Δ_p with respect to ν is 2^{p-1}.*

A *tree* is a connected graph with no cycles. It is one of the most important kinds of graphs in both applications and theory.

Corollary 1. *If $\Gamma = (V, E)$ is a tree then $\#V = 1 + \#E$. In particular the largest eigenvalue of Δ_p with respect to $\nu = 1$ is given by 2^{p-1}.*

Following this we study several classes of examples where explicit values of the largest eigenvalue are available.

The hypercube. Let $G = \mathbb{Z}_2^n$, and for $i = 1, \ldots, n$, let e_i be the element of $\mathbb{Z}_2^n$ with all coordinates 0 except for ith, which is 1. $[x, y]$ is an edge if $x - y \in \{e_i;\ i = 1, \ldots, n\}$.

Since G satisfies the Proposition 3.1; the largest eigenvalue is 2^{p-1}.

What is important in the following proposition. is the fact that the linear case ($p = 2$) is generalized.

Proposition 3.2. *Given two graphs G_1 and G_2, the largest eigenvalue of the product graph $G_1 \times G_2$ is $\gamma_p^1 + \gamma_p^2$, where γ_p^1 and γ_p^2 are eigenvalues of G_1 and G_2 with respect to $1/d_1(x)$ and $1/d_2(x)$, respectively.*

4. Cheeger and isoperimetric inequalities

Chegeer [Che70] proved a lower bound on the next-to-smallest eigenvalue (smallest positive eigenvalue) λ of the Laplacian on a compact Riemannian manifold M. in terms of an isoperimetric constant. In the same paper Cheeger also proved a lower bound on the smallest eigenvalue λ (necessarily positive) of the Laplacian on a compact Riemannian manifold with Dirichlet boundary ∂M. More recently, Dodziuk [Dod84] proved an analogous bound for the Laplacian on a finite graph with Dirichlet boundary. In this section we prove a general version of Cheeger's inequality for the discrete p-Laplacian. We give an explanation of Cheeger's inequality of the p-Laplacian on finite graphs. Our goal is to derive the estimate concerning the best constants in the isoperimetric inequalities and the first eigenvalue of the p-Laplacian; this problem has been studied by severals authors in the linear continuous case we refer to [PS51] and [Che70]. This problem has a discrete version see [SC97] and other problems was treated by my self in [Amg97].

Throughout we assume $\mathcal{G}$ is a finite graph consisting of a set V of vertices and a set E of edges. Each edge e has an endpoint set containing two distinct elements of the vertex set V. Both the vertex set and the edge set are finite. Let X be a subset of V. We define $\partial X = \{\{x, y\} \in E;\ x \in X,\ y \notin X\}$. Write

$$\mu(\partial X) = |\partial X| = \sum_{e \in \partial X} C(e)^{p-1}.$$

Let $\mu(v)$ denote the sum of the conductances which are incident to the vertex v. In another way we have

$$\mu(v) = \sum_{x \sim v} C[v, x]^{p-1}.$$

Define the isoperimetric constant by

$$i(G) = \inf\{\frac{|\partial X|}{\mu(X)};\ X \neq \emptyset \text{ and } \mu(X) \leq 1/2\mu(V)\}.$$

Another way to write this is

$$i(G) = \inf\{\frac{|\mathbb{I}_X|_p^p}{\|\mathbb{I}_X\|_p^p};\ X \neq \emptyset \text{ and } \mu(X) \leq 1/2\mu(V)\}.$$

The analogue of Cheeger's isoperimetric constant is the rate of

$$h = \inf\{k(A) \ : \ 0 < \mu(A) \leq 1/2\mu(V)\}$$

with

$$k(A) = \mu(A)^{-1}\mu(\partial A).$$

What we actually want, is that if the operator is Δ_p, what is the bounds of λ_p? We have found quantitative answer to this question which we now present.

Theorem 3. *With the above notations, we have*

$$\frac{2^{p/q}}{p^p}h^p \leq \lambda_p \leq 2^{p-1}h.$$

As mentioned before, the estimates on λ_p that we plug into these lemmas in order to prove Theorem 3 are proved in this section.

The upper bound is obtained by the following observation. Given $S \subset V$ with $0 < \mu(S) \leq 1/2\mu(V)$, let q be the conjugate of p and set

$$u_S(x) = \mu(S)^{q-1}\mathbb{I}_{(V\setminus S)} - \mu(V \setminus S)^{q-1}\mathbb{I}_{(S)}.$$

Note that u_S have μ-mean value 0 and use Theorem 1 to conclude that

$$\lambda_p \leq \frac{|u_S|_p^p}{\text{var}_p(u_S)}.$$

By simple computation and simplification we obtain the upper bound.
For the lower bound in Theorem 3 we need some notations. For any function u use u^+ to denote $\max(0, u)$ the nonnegative part of u, and set $S(u) = \{x \in V : \ u(x) > 0\}$. Assuming that $S(u) \neq \emptyset$, first of all we have the following lemma.

Lemma 1. *For any function u, and $\lambda \in [0, +\infty[$. If $\Delta_p u \leq \lambda u^{p-1}$ on $S(u)$ then*

$$\lambda\|u^+\|_p^p \geq |u^+|_p^p.$$

Proof. Note that

$$\lambda\|u^+\|_p^p = \lambda \sum_{x\in V} u^+(x)^p\mu(x).$$

By the hypothesis we have

$$\lambda\|u^+\|_p^p \geq \sum_{x\in V} u^+(x)\Delta_p u(x)\mu(x).$$

Changing the second sum as sum on edges, shows that

$$\lambda\|u^+\|_p^p \geq \frac{1}{2}\sum_{x\sim y} c[x,y]^{p-1}(u^+(x) - u^+(y))\phi_p(u(x) - u(y)).$$

In other words we have

$$\phi_p[u^+(x) - u^+(y)](u(x) - u(y)) \geq |u^+(x) - u^+(y)|^p.$$

which gives

$$\lambda\|u^+\|_p^p \geq |u^+|_p.$$

This achieves the proof of the Lemma. □

The proof of this result appears as a consequence of the precise study of upper estimates on some constant under functional inequalities between isoperimetry and energy. Let us denote by $h(u)$ the quantity

$$\inf\{\frac{\mu(\partial S)}{\mu(S)} : \emptyset \neq S \subset S(u)\}.$$

Lemma 2. *There exists a constant α such that*

$$h(u)\|u\|_p \leq \alpha |u|_p$$

where $\alpha = p/(2^{1/q})$.

Proof. We may assume that $u \geq 0$ everywhere. Define $q(x,y)$ by $C[x,y]^{p-1} = q(x,y)$ if $x \sim y$ and $q(x,y) = 0$ otherwise. We have

$$\frac{1}{p} \sum_{u(y)>u(x)} (u(y)^p - u(x)^p) q(x,y) = \sum_{u(y)>u(x)} q(x,y) \int_{u(x)}^{u(y)} t^{(p-1)} dt.$$

Then, by Fubini Theorem, we obtain

$$\frac{1}{p} \sum_{u(y)>u(x)} (u(y)^p - u(x)^p) q(x,y) = \int_0^\infty t^{p-1} \sum_{u(y)>t\geq u(x)} q(x,y) dt.$$

Note that

$$\frac{1}{2}\sum_{x,y} |u(x)^p - u(y)^p| q(x,y) \leq |u|_p \left(\frac{1}{2}\sum_{x,y}\left(\frac{u(x)^p - u(y)^p}{u(x)-u(y)}\right)^q q(x,y)\right)^{1/q};$$

by combining this with Lemma 3, we arrive at

$$\frac{1}{2}\sum_{x,y} |u(x)^p - u(y)^p| q(x,y) \leq p|u|_p \left(\sum_{x,y}\frac{1}{4}(u(x)^p + u(y)^p) q(x,y)\right)^{1/q}$$

which gives

$$\frac{1}{2}\sum_{x,y} |u(x)^p - u(y)^p| q(x,y) \leq \frac{p}{4^{1/q}}|u|_p \left(\sum_{x,y}(u(x)^p + u(y)^p) q(x,y)\right)^{1/q}.$$

Remark that

$$p\int_0^\infty t^{p-1} \sum_{u(y)>t\geq u(x)} q(x,y) dt \geq p\int_0^\infty t^{p-1} h(u)\mu\{x : u(x) \geq t\} dt = h(u)\|u\|_p^p;$$

by combining this with previous inequality we get

$$h(u)\|u\|_p^p \leq \frac{p}{2^{1/q}} |u|_p \|u\|_p^{p/q}$$

which achieves the proof of the lemma □

Then we have

$$\lambda^{1/p} \geq h(u)\frac{2^{1/q}}{p} \quad \text{if} \quad \Delta_p u \leq \lambda u^{p-1} \quad \text{on} \quad S(u)$$

For any $\lambda \in [0, \infty[$ and any u with $S(u) \neq \emptyset$. To get the lower bound in Theorem 3 from here take $\lambda = \lambda_p$, and any u to be a normalized eigenfunction for λ_p. Because u must have μ-mean value 0, we may always arrange that $0 < \mu(S) \leq 1/2\mu(V)$ and therefore that $h(u) \geq h$. Hence the desired lower bound comes from our discussion above and this choice of λ and u. □

In the following lemma we give an elementary inequality.

Lemma 3. *If $a, b \geq 0$, and $p > 1$ is conjugate to q (i.e. $(p-1)(q-1) = 1$) then*

$$\left(\frac{1}{p}\frac{b^p - a^p}{b-a}\right)^q \leq \frac{1}{2}(a^p + b^p).$$

Proof. Note that, if, $p \geq 1$ and $x > 1$, then

$$\frac{x^{p+1}-1}{x-1} \leq \frac{p+1}{2}(x^p + 1).$$

To prove this inequality it sufficient to prove

$$\sum_{n=1}^{\infty} \frac{(p+1)^n (\log x)^n}{n!} \leq \frac{p+1}{2} \sum_{n=1}^{\infty} \frac{(p+1)^n (\log x)^n}{n!} + \frac{(\log x)^n}{n!} - \frac{(p)^n (\log x)^n}{n!}.$$

Observe that

$$1/2[(p+1)^n + 1 - p^n] - (p+1)^{n-1} = 1/2[(p+1)^{n-1}(p-1) + 1 - p^n],$$

which is nonnegative since the last part of the previous equality can be written as follows

$$1/2[(p+1)^{n-1}(p-1) + 1 - p^n] = 1/2(p-1)[(p+1)^{n-1} - \sum_{k=0}^{n-1} p^k]$$

This gives the result. Clearly we obtain the following general result

$$\frac{x^{p+1}-1}{x-1} \leq \frac{p+1}{2}(x^p + 1)$$

for all $x > 0$ and $p \geq 1$ (we may convention for $x = 1$ the right part take values $p+1$).
For $p \in [1, 2]$ and $0 < a < b$ we have

$$\left(p^{-1}\frac{b^p - a^p}{b-a}\right)^q = \left(\frac{\int_a^b t^{p-1} \mathrm{d}\, t}{b-a}\right)^q \leq \frac{\int_a^b t^p \mathrm{d}\, t}{b-a} = \frac{b^{p+1} - a^{p+1}}{b-a}\frac{1}{p+1};$$

by the previous inequality we obtain

$$\left(\frac{1}{p}\frac{b^p - a^p}{b-a}\right)^q \leq \frac{1}{2}(a^p + b^p)$$

this achieves the elementary inequality. □

Theorem 3 is unfortunately not very useful if the graph does not satisies the strong isoperimetric inequality, since the spectra can be much smaller. On the other hand Theorem 3 is very strong when case of interest is the expander graphs see [AM85] and [Lub94] for more details. It even implies results about the numerical range (and hence the spectrum) of operators. Here are some examples (which illuminate the Theorem 3).

Consider the circle graph, $\mathbb{Z}_n$. Fix $n \in \mathbb{N}$ and consider the graph Γ such that the vertices are the elements of the group of integers modulo n, denoted by $\mathbb{Z}_n$, the edge set is $E = \{[i, i+1],\ i \in \mathbb{Z}_n\}$.

Random walk on the cyclic group $\mathbb{Z}_n$. The nonzero elements of Q are

$$q_{i,i+1} = a, \quad q_{i,i-1} = 1 - a$$

Then Q is selfadjoint if and only if $a = 1/2$. The Cheeger ($a = 1/2$) constant is $k = [n/2]^{-1}$ where $[x]$ is the integer par of x.

The notation for Cayley graphs is taken to be the following. Fix G to be a finite group and S a subset of G. Then, the Cayley graph $\Gamma = X(G, S)$ with respect to $S \subset G$ is the graph which has G as the set of vertices and $E = \{(x, ax), (x, a) \in G \times S\}$ as the set of edges. Note that Γ is connected if and only if S is a generating set of G; Γ is symmetric if and only if $S = S^{-1}$; if $e \notin S$ there is no loops.

We consider S_n the group of permutation of $\{1, \ldots, n\}$. Note that this group is generated by $\tau = (1,2)$ and $\sigma = (1, 2, \ldots, n)$. Define the graph $G = (V, E)$ generated by $K = \{\tau, \sigma, \sigma^{-1}\}$ i.e $V = S_n$ and $E = \{(x, ax), x \in S_n, a \in K\}$. There exists n subset $A_1, \ldots, A_n$ of S_n such that $\sigma A_i = A_{i+1}$ for $i = 1, \ldots, n-1$ and $\sigma A_n = A_1$; $\tau A_i = A_i$ for $i = 3, \ldots, n$. For example take $A_i = \{x \in S_n : x(n) = i\}$. By choosing the function u_n defined by $u_n = \sum_{k=1}^{n} (k/n) \mathbb{I}_{A_k}$ on S_n we obtain

$$\lim_{n \longrightarrow \infty} \lambda_p(n) = 0$$

then the strong isoperimetric inequality is not satisfied.

Other works on the first eigenvalues of the Laplacian includes [DSC96a], [DS91], [Dod84] and [LS88] [Lin93], [Chu97], [CRS97], [DK88] and [Gro93]. In particular, [Lin93] study the p-Laplacian in the continous case which is the analogues of our contribution in the discrete case. p-harmonic functions are only interesting in infinite graphs; since if the graph is finite and connected, then the constants are p-harmonic but no other function are.

5. Isoperimetry and functional inequalities

In this section we will give some functional estimates for the upper bound and the lower bound for λ_p. The development thus far in this section has excluded the case $p = 1$, a situation which almost always requires special treatment. As for the log Sobolev inequality studies [DSC96b] and [Dav89], we give some analog results for the p-Laplacian.

We now formally introduce d-dimensional that play the most important role in the estimates of the smallest eigenvalues of some subgraphs of the lattice $\mathbb{Z}^d$.

Definition 1. The d-dimensional isoperimetric constant of a graph G is defined by

$$I_d(G) = \min_{A \subset V,\ \mu(A) \leq 1/2\mu(V)} \frac{\mu(\partial A)}{\mu(A)^{1-1/d}}.$$

where μ is defined in section 4 with $\mu(V) = 1$.

A similar argument leads to the following result. We formulate two facts along this section. Since the analysis will depend upon estimates involving, $\|\nabla u\|_p$, it is not surprising that the co-area formula will play a crucial role.

Proposition 5.1. *Let μ and ν be measures in V. Asume that for all sets A such $\mu(A) \leq 1/2\mu(V)$ we have*

$$\mu(A)^{1/q} \leq \alpha\sigma(\partial A)^{\delta} \nu(A)^{(1-\delta)/r}$$

where $\alpha > 0$, $\delta \in [0,1]$, $r, q > 0$, $1/\gamma = \delta + (1-\delta)/r \geq 1/q$. *Then*

$$\|u\|_{q,\mu} \leq C|\nabla u|_{1,\sigma}^{\delta}\|u\|_{r,\nu}^{1-\delta}$$

for all u *such that* $\mu(u \neq 0) \leq 1/2\mu(V)$, *where* $C = \gamma^{1/\gamma}\alpha$.

Proof. By

$$\|u\|_{q,\mu} = [\int_0^\infty \mu(u > t)\mathrm{d}t^q]^{1/q} = [\int_0^\infty \frac{q}{\gamma}[t^\gamma \mu(u > t)^{\gamma/q}]^{q/\gamma - 1}\mu(u > t)^{\gamma/q}\mathrm{d}t^\gamma]^{1/q}$$

Since $\mu(u > t)$ is a nonincreasing function, then,

$$\|u\|_{q,\mu} \leq [\int_0^\infty \frac{q}{\gamma}[\int_0^{t^\gamma} \mu(u > x)^{\gamma/q}\mathrm{d}x]^{q/\gamma - 1}\mu(u > t)^{\gamma/q}\mathrm{d}t^\gamma]^{1/q}$$

by applying simple integration we obtain

$$\|u\|_{q,\mu} \leq [\int_0^\infty \mu(u > t)^{\gamma/q}\mathrm{d}t^\gamma]^{1/\gamma}.$$

Using the fact that $\mu(u > t) \leq 1/2\mu(V)$ we obtain

$$\|u\|_{q,\mu} \leq \gamma^{1/\gamma}\alpha[\int_0^\infty t^{\gamma-1}\mathrm{d}t\sigma(\partial A_t)^{\delta\gamma}\nu(A_t)^{(1-\delta)\gamma/r}]^{1/\gamma}.$$

Since $\gamma\delta + \gamma(1-\delta)/r = 1$, we get by Hölder's inequality

$$\|u\|_{q,\mu} \leq \gamma^{1/\gamma}\alpha[\int_0^\infty \sigma(\partial A_t)\mathrm{d}t]^{\delta}[\int_0^\infty \nu(A_t)t^{r-1}\mathrm{d}t]^{(1-\delta)/r}$$

which is equivalent to (1). □

The following theorem determined the constant on the functional inequality.

Theorem 4. *Let* $1 \leq p < n$ *If the graph* G *satisfies the* n*-dimensional isoperimetric inequality with constant* α. *Then for all function* u *which satisfies* $\mu\{u \neq 0\} \leq 1/2$, *we have*

$$\|u\|_{np/(n-p)} \leq C|u|_p$$

where $C = \alpha(1/2)^{(p-1)/p}\frac{p(n-1)}{(n-p)}$.

Proof. From the classical isoperimetric inequality

$$|A|^{(n-1)/n} \leq \alpha|\partial A|,$$

it follows that for all u with finite support

$$\|u\|_{n/(n-1)} \leq \alpha|\nabla u|_1$$

with the best constant. In the case $n > p \geq 1$ we replace u by $|u|^{p(n-1)/(n-p)}$ in the previous inequality we get

$$\||u|^{p(n-1)/(n-p)}\|_{n/(n-1)} = \|u\|_{np/(n-p)}^{p(n-1)/(n-p)} \leq \alpha|\nabla|u|^{p(n-1)/(n-p)}|$$

and then estimate the right hand side by Hölder's inequality and Lemma 3. We have

$$\sum_{x\sim y} c[x,y]^{p-1}||u|^{p(n-1)/(n-p)}(x) - |u|^{p(n-1)/(n-p)}(y)| \leq A^{(p-1)/p}|u|_p$$

where $A^{(p-1)/p}|u|_p$ denotes the quantity

$$|u|_p[\sum_{x\sim y} c[x,y]^{p-1}\frac{||u|^{p(n-1)/(n-p)}(x) - |u|^{p(n-1)/(n-p)}(y)|}{|u(x) - u(y)|}^{p/(p-1)}]^{(p-1)/p}.$$

Applying Lemma 3, we obtain

$$A \leq \xi \sum_{x\sim y} c[x,y]^{p-1}[|u(x)|^{p(n-1)/(n-p)} + |u(y)|^{p(n-1)/(n-p)}]^{n/(n-1)},$$

where $\xi = [1/2(\frac{p(n-1)}{(n-p)})^{p(n-1)/(pn-n)}]^{n/(n-1)}$, which gives

$$A \leq \eta \sum_{x\sim y} c[x,y]^{p-1}[|u(x)|^{np/(n-p)} + |u(y)|^{np/(n-p)}],$$

where $\eta = 1/2(\frac{p(n-1)}{(n-p)})^{p/(p-1)}$. Then, we have

$$\|u\|^{p(n-1)/(n-p)}_{np/(n-p)} \leq \eta^{(p-1)/p}|u|_p\|u\|^{n(p-1)/(n-p)}_{np/(n-p)}.$$

Consequently,

$$\|u\|_{np/(n-p)} \leq (1/2)^{(p-1)/p}\frac{p(n-1)}{(n-p)}\alpha|u|_p.$$

This terminate the proof. □

Again, for the sake of convenience, we shall give a direct proof instead of applying the Proposition 6.1.

Proposition 5.2. *If for all sets $A \subset V$*

$$\mu(A)^{1/q} \leq \alpha\mu(\partial A) + \beta\nu(A)$$

where α, β are constants $q \geq 1$ then

$$\|u\|_{q,\mu} \leq \alpha|\nabla u|_{1,\mu} + \beta\|u\|_{1,\nu}$$

Proof. By definition,

$$\|u\|_{q,\mu} = [\int_0^\infty \mu(u > t)dt^q]^{1/q} = [\int_0^\infty q[t\mu(u > t)^{1/q}]^{q-1}\mu(u > t)^{1/q}]dt]^{1/q}.$$

Since $\mu(u > t)$ is a nonincreasing function,

$$\|u\|_{q,\mu} \leq [\int_0^\infty q[\int_0^t \mu(u > x)^{1/q}dx]^{q-1}\mu(u > t)^{1/q}dt]^{1/q}.$$

Applying simple integration, we obtain

$$\|u\|_{q,\mu} \leq \int_0^\infty \mu(u > t)^{1/q}dt.$$

Using the fact that

$$\|u\|_{q,\mu} \leq [\int_0^\infty (\alpha\mu(\partial A_t) + \beta\nu(A_t))dt],$$

which gives

$$\|u\|_{q,\mu} \leq \alpha|\nabla u|_\mu + \beta\|u\|_{1,\nu}$$

which is the desired result follows. □

6. LOGARITHMIC SOBOLEV INEQUALITIES

In this section we shall study the connection between a functional property of the graph and spectral property of this graphs in more detail. We shall do this first for the case of a finite graphs; we plan to investigate this and its connection with the Wiener test for the infinite graph in a forthcoming paper. We consider a measure μ on V as in section 4. For any real valued function u on V and $p \in]1, \infty[$ we write

$$\mathcal{L}_p(f) = \sum_{x \in V} |f(x)|^p \log(|f(x)|^p)\mu(x).$$

A Logarithmic Sobolev inequality (or Log-Sobolev) inequality is an inequality of type

$$\mathcal{L}_p(f) \leq c|f|_p^p$$

holding for all function f such that $\|f\|_p^p = \mu(V)$. The smallest constant

$$\alpha = \inf\left\{\frac{|f|_p^p}{\mathcal{L}_p(f)} : \|f\|_p^p = \mu(V),\ \mathcal{L}_p(f) \neq 0\right\}$$

such that this inequality holds is called the Log-Sobolev constant for G.

Lemma 4. *For a graph G, we suppose $f \longrightarrow \mathbb{R}$ achieves the log-Sobolev constant and $\sum_{x \in V} |f(x)|^p \mu(x) = \mu(V)$. Then f satisfies, for any vertex x.*

$$\Delta_p f(x) = \alpha\phi_p(f(x)) \log |f(x)|^p.$$

Proof. We use Lagrange's multiplicator, by taking the derivative with respect to $f(x)$:

$$\frac{\Delta_p f(x)}{\sum_{x \in V} |f(x)|^p \log |f(x)|^p \mu(x)} - \frac{|f|_p^p}{\mathcal{L}_p(f)^2}[\phi_p(f) \log |f(x)|^p + \phi_p(f)] + c\phi_p(f) = 0$$

for some constant c. After substituting for α, the above expression can be simplified:

$$\Delta_p f(x) - \alpha[\phi_p(f) \log |f(x)|^p + \phi_p(f)] + c\phi_p(f)\mathcal{L}_p(f) = 0 \quad (eq.x)$$

After multiplying by $\mu(x)f(x)$ and summing over all x in V, we have

$$\frac{1}{2}\sum_{x \sim y} |f(x) - f(y)|^p - \alpha(\mathcal{L}_p(f) + \|f\|_p^p) + c\mathcal{L}_p(f)\|f\|_p^p = 0$$

This implies $c\mathcal{L}_p(f) = \alpha$. Therefor we obtain from $(eq.x)$ that

$$\Delta_p f(x) = \alpha\phi_p(f) \log |f(x)|^p$$

This completes the proof. □

The first eigenvalue is the minimum of the Rayleigh quotient

$$\lambda_p = \inf\{\frac{|\phi|_p^p}{\mathrm{var}_p(\phi)};\ \phi \text{ is non-constant}\}.$$

Here $1 < p < \infty$, and in the linear case $p = 2$, one gets the principal frequency of the vibrating membrane. Let us finally mention that the best constant in the Poincaré-Friedrichs inequality

$$\sum_{x \in V} \phi(x)^p \mu(x) \leq c \sum_{e \in E} \nabla\phi(e)^p$$

is the reciprocal of the principal frequency $c = \frac{1}{\lambda_p}$.

The eigenvalue are positive and the least of them, say λ_p, is obtained as the minimum of the Rayleigh quotient.
To any graph there is a eigenfunction u_p corresponding to the least eigenvalue λ_p. The existence is a standard calculus of variation, see Theorem 3 and the strict positivity follows from connction applied to the nonnegative minimizing function $|u|_p$. The first eigenfunctions are essentially not unique, they are merely constant multiples of each other see section 9. The first eigenfunctions are the only eigenfunctions "not changing very much the sign".

Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \{r\}$ (root) consider the graph $G = G_1 \uplus G_2$ which is obtained by attaching G_1 and G_2 at their roots. in other word the vertex set of $G_1 \uplus G_2$ is $V(G_1) \cup V(G_2)$ and the edge set is $E(G_1) \cup E(G_2)$.

Proposition 6.1. *Let G be a rooted graph at r and connected. Consider two copies G_1, G_2 of G with the same root r. Then the first eigenfunction of $G_1 \uplus G_2$ not changing the sign at each part. And $f(1,x) = -f(2,x)$ for all $x \in V(G)\backslash\{r\}$, where (x,i) are vertecies of G_i.*

Proof. Let f be the first eigenfunction of $G_1 \uplus G_2$. By Theorem 1 we have

$$\lambda_p = \frac{|f|_p^p}{\|f\|_p^p} \geq \frac{|f|_p^p}{\|f - f(r)\|_p^p}.$$

Let f_i be the restruction of $f - f(r)$ on G_i for $i = 1, 2$. Without loss of generality we can assum that $|f_1|_p/\|f_1\|_p \leq |f_2|_p/\|f_2\|_p$. Then define a function g on $G_1 \uplus G_2$ by $g = |f_1|+$ on G_1, $g = -|f_1|$ on G_2 this gives $\mathrm{var}_p(g) = \|g\|_p^p = \|f - f(r)\|$, $g(r) = 0$ and $|g| = g$ on G_i. Then

$$\lambda_p \geq \frac{|g|_p^p}{\mathrm{var}_p(g)}$$

by applying Theorem 1 we get the result. □

Theorem 5. *For any connected graph G. If $1 < p < s < \infty$ then*

$$p\lambda_p^{1/p}(G \uplus G) \leq s\lambda_s^{1/s}(G \uplus G).$$

Proof. To see this, choose any ψ to be an eigenfunction corresponding to the eigenvalue λ_s; and put $\phi = |\psi|^{s/p-1}\psi$. By proposition 7.1 we have $\mathrm{var}_p(\phi) = \|\phi\|_p^p$. By definition we get

$$2\lambda_p \leq \frac{|\phi|_p^p}{\|\phi\|_p^p} = \frac{\sum_{x\sim y} |\phi(x) - \phi(y)|^p}{\|\psi\|_s^s}.$$

Using the Rayleigh quotient for λ_p and Hölder inequality we obtain

$$(2\lambda_p)^{1/p} \leq \frac{2^{1/s}|\psi|_s}{\|\psi\|_s^{s/p}} \left(\sum_{x\sim y} \frac{|\phi(x) - \phi(y)|^{p\alpha}}{|\psi(x) - \psi(y)|^{p\alpha}} \right)^{1/(\alpha p)}$$

where α is the conjugate of s/p. Then, according to Lemma 3, we have

$$(2\lambda_p)^{1/p} \leq \frac{s}{p} \frac{2^{1/s}|\psi|_s}{\|\psi\|_s^{s/p}} \left(\sum_{x\sim y} \||\psi|^{(s/p)}(x) + |\psi|^{(s/p)}(y)|^p (1/2)^p \right)^{1/(\alpha p)}.$$

From convexity of t^p, we get

$$(2\lambda_p)^{1/p} \le \frac{s}{p}\frac{2^{1/s}|\psi|_s}{\|\psi\|_s^{s/p}}\left(\frac{1}{2}\sum_{x\sim y}\psi(x)^s + \psi(y)^s\right)^{1/(ps)} = \frac{s}{p}\frac{2^{1/s}|\psi|_s}{\|\psi\|_s}.$$

which gives

$$(2\lambda_p)^{1/p} \le \frac{s}{p}\frac{2^{1/s}|\psi|_s}{\|\psi\|_s} = \frac{s}{p}(2\lambda_s)^{1/s}.$$

Which achieves the proof of the Theorem 6. □

Remark 1. The hypothesis on the graph is used to satisfy the equation $\mathrm{var}_p(\phi) = \|\phi\|_p^p$ when ψ is the first eigenfunction of Δ_s.

Remark 2. If the graph is not connected then

$$\inf\{\frac{|f|_p^p}{\mathrm{var}_p(f)}, f \text{ nonconstant}\} = 0.$$

7. Finite tree and λ_p

A question which was often asked during early stages of development is whether the minimax property can be used towards establishing some explicit bounds for λ_p. Below we give two examples. There are of course many ways of combining graphs to form new ones, and these may be used to establish structural properties of the class of some graphs. On the other hand, we have already seen that λ_p must be a solution of the variational problem. Moreover, it is a consequence of another result that the values taking of eigenfunction associated to the largest eigenvalue is $+1, -1$ for some graph. In line with these interests, this section reports results on investigations on eigenvalues. Apart from its importance and independent interest, and connection to other areas. The case $p = 2$ has a long history and an extensive literature and has been the subject of considerable recent research of algebra graph theory [CRS97] and [Chu97]; the least named work contains a substantial bibliography of earlier writing in this topic.

In the following lemma we show how the method developed in [AM85] to study the first eigenvalue for the discrete Laplacian can be adapted to prove analogous results for eigenvalues based on the p-Laplacian operator. We first begin by some notations. Let A and B be two disjoint subsets of V. Let E_A (E_B) be the set of edges of G with both endpoints in A (in B). Put $a = |A|/|V|$, $b = |B|/|V|$. For many applications of general interest, the quantities a and b will be given. The following result is obtained.

Lemma 5. *Let A and B be a subset of V. Then*

$$\lambda_p \le [\frac{1}{a^{q-1}} + \frac{1}{b^{q-1}}]^{p-1}[|E| - |E_A| - |E_B|].$$

This result on the linear discrete case, for example is applied for the first time from the work of [AM85] on the linear case.

Proof. Define a function g by

$$g(v) = \frac{1}{a^{q-1}}\mathbb{I}_A - \frac{1}{b^{q-1}}\mathbb{I}_B.$$

If $v \in A$ then $g(v) = \frac{1}{a^{q-1}}$ and if $v \in B$ then $g(v) = \frac{1}{b^{q-1}}$. For the quantity $|g|_p^p$ we use the following majoration

$$|g|_p^p \leq \left(\frac{1}{a^{q-1}} + \frac{1}{b^{q-1}}\right)^p [|E| - |E_A| - |E_B|]$$

which may be written in the form of Lemma 6. By using the variational principle we obtain the result. □

The lemma has established the relation between the first eigenvalue of Δ_p and certain edge connectivity property of the graph. We can get some bound on Δ_p, just from graph theory bounds. It follows that for a connected graph on n vertices we have

$$\lambda_p \geq \frac{1}{n^p}$$

where Δ_p is the p-Laplacian for a simple graph.

When the graph Γ is a tree then we can choose A and B such that $|E| - |E_A| - |E_B| = 1$, which gives an upper bound of λ_p.

8. Examples

In this section we discuss the usual method for finding the largest eigenvalue. We will give an analysis of certain aspects of it which we will need. Product graphs are of interest both in their own right and as a basis for comparison in analyzing more complex graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with conductances functions $q_i = 1$, let $d^i(x)$ denote the degree of the vertex x for the graph G_i. We define a new graph $G = (V, E)$ where $V = V_1 \times V_2$ and $\{(x_1, x_2), (y_1, y_2)\}$ is an edge if and only if $x_1 = y_1$ and $x_2 \sim y_2$, or $x_2 = y_2$ and $x_1 \sim y_1$; in this case we define $q((x_1, x_2), (y_1, y_2)) = q(x_1, y_1) + q(x_2, y_2) = 1$. Let $d(x)$ denote the degree of the vertex x for the graph G We have the following proposition.

Proposition 8.1. *If λ_i is an eigenvalue with respect to $1/d_i(x)$ for G_i for $i \in \{0, 1\}$ and with the associated eigenfunction f_i , then $\lambda_1 + \lambda_2$ is an eigenvalue with respect to $1/d(x)$ and with the associated eigenfunction $f(x, y) = f_1(x) f_2(y)$.*

Proof. Remark that

$$d(x, y)\Delta_p^G f(x, y) = d^1(x)\Delta_p^{G_1} f(., y)(x) + d^2(y)\Delta_p^{G_2} f(x, .)(y)$$

which gives

$$d(x, y)\Delta_p^G f(x, y) = d^1(x) f_2(y)^{p-1}\Delta_p^{G_1} f_1(x) + d^2(y) f_1(x)^{p-1}\Delta_p^{G_2} f_2(y).$$

Then the result is a simple consequence. □

From this, we deduce a bound on the order of the smallest eigenvalue of the product graphs that we found surprisingly difficult.

Let G be a connected general graph. The length of the shortest walk between two vertices a and b is the *distance* $d(a, b)$ between a and b in G. The maximum value of the distance function over all pairs of vertices is called the diameter of G.

Proposition 8.2. *Let G be a connected general graph of diameter d. Then G has at least $d + 1$ distinct eigenvalues of its spectrum.*

Actually, this fact has led to the following **question 1.** : what is the optimal number of eigenvalues of a graph of diameter d?

9. Eigenfunctions and eigenvalues for complete graphs

One original motivation for this section was to compute the number of eigenvalues of the p-Laplacian. For example, we do not know the number of eigenvalues for the p-Laplacian for general graphs. The purpose of this section is not only to describe our results, but also to give a brief connection of the theory of eigenvalues of the complete graphs and the standard eigenvalue of the Laplacian in $\mathbb{R}^d$. If $e = [a, b]$ is an edge, we call a and b the endpoints of e. Two vertices on some edge or two distinct edges with common vertex are adjacent. A *complete* graph is a graph in which all possible pairs of vertices are edges. Two graphs G_1 and G_2 are isomorphic provided there exists a 1-1 correspondence between their vertex set that preserves adjacency. Let $G = (V, E)$ be a graph the number n of elements in the finite set V is called the *order* of the graph G. Two complete graphs with the same order are isomorphic, and we denote a complete graph of order n by K_n, where the vertices set of K_n is $\{1, \ldots, n\}$. The aim in the following theorem is to describe, both theoretically and explicitely, eigenfunctions and eigenvalues for K_n.

Theorem 6. *Let be $n \geq 3$ and $p \neq 2$ conjugate to q. The positive eigenvalues of Δ_p are*

$$\frac{1}{n-1}\left(n - (h+k) + (h^{q-1} + k^{q-1})^{p-1}\right)$$

where h, k are positives integers with $h + k \leq n$.

Proof. Let us remark that, if A and B are disjoint subsets of V, the function

$$a\mathbb{1}_A - b\mathbb{1}_B$$

is an eigenfunction of Δ_p. The second step consists in showing that every eigenfunction has the above form. For this, let consider a positive eigenvalue λ and f the eigenfunction corresponding to λ. Suppose that $f(1) \leq f(2) \leq \ldots \leq f(n)$. This does not lead to any loss of generality in the problems. Since $\lambda > 0$ we have $f(1) < 0$ and $f(n) > 0$. Let α be the largest k such that $f(k) < 0$; β the smallest k such that $f(k) > 0$. From the equation of eigenvalue we obtain

$$(n-1)\lambda = -\sum_{k=1}^{i}\left(-1 + \frac{f(k)}{f(i)}\right)^{p-1} + \sum_{k=i}^{n}\left(1 - \frac{f(k)}{f(i)}\right)^{p-1}$$

for $i = 1, \ldots, \alpha$. For $i = \beta, \ldots, n$ we have

$$(n-1)\lambda = \sum_{k=1}^{i}\left(1 - \frac{f(k)}{f(i)}\right)^{p-1} - \sum_{k=i}^{n}\left(-1 + \frac{f(k)}{f(i)}\right)^{p-1}.$$

In particular, we have

$$\sum_{k=1}^{\alpha}\left(-1 + \frac{f(k)}{f(\alpha)}\right)^{p-1} + \sum_{k=1}^{n}\left(1 - \frac{f(k)}{f(n)}\right)^{p-1} = \sum_{k=\alpha}^{n}\left(1 - \frac{f(k)}{f(\alpha)}\right)^{p-1}.$$

If $\alpha \geq 2$ and $p > 2$, then

$$\sum_{k=1}^{\alpha}\left(-\frac{f(k)}{f(n)} + \frac{f(k)}{f(\alpha)}\right)^{p-1} + \sum_{k=\alpha+1}^{n}\left(1 - \frac{f(k)}{f(n)}\right)^{p-1} \leq \sum_{k=\alpha}^{n}\left(1 - \frac{f(k)}{f(\alpha)}\right)^{p-1}.$$

Since $\sum_{k=1}^{\alpha}(-f(k))^{p-1} = \sum_{k=\alpha+1}^{n} f(k)^{p-1}$, we have

$$\sum_{k=\alpha+1}^{n} \left(\frac{f(k)}{f(n)} - \frac{f(k)}{f(\alpha)}\right)^{p-1} + \left(1 - \frac{f(k)}{f(n)}\right)^{p-1} \leq \sum_{k=\alpha}^{n} \left(1 - \frac{f(k)}{f(\alpha)}\right)^{p-1}.$$

which gives

$$\sum_{k=\alpha+1}^{n} \left(1 - \frac{f(k)}{f(\alpha)}\right)^{p-1} \leq \sum_{k=\alpha}^{n} \left(1 - \frac{f(k)}{f(\alpha)}\right)^{p-1}.$$

Thus f is constant in $\{1, \ldots, \alpha\}$ (if not the above inequalities are strict) The same method is applicable if $p \in]1, 2[$. If we change f to $-f$ we obtain f that is constant in $\{\beta, \ldots, n\}$; this achieve the first part.

Consider again the above equation

$$(n-1)\lambda = \beta + 1 - \alpha + (n+1-\beta)\left(1 - \frac{f(n)}{f(1)}\right)^{p-1} = \beta - 1 - \alpha + \alpha\left(1 - \frac{f(1)}{f(n)}\right)^{p-1}.$$

This gives

$$(n-1)\lambda = \beta - 1 - \alpha + \left((n+1-\beta)^{q-1} + \alpha^{q-1}\right)^{p-1}.$$

where q is the conjugate of p. All the eigenfunctions of Δ_p have now been described, which achieves the proof. □

Remark 3. If $p = 2$, then $q = 2$ and the unique positive eigenvalue is $n/(n-1)$. In particular; this is satisfied in Theorem 6. but the space of eigenfunctions corresponding to λ_p in the case $p = q = 2$ is an $(n-1)$-dimensional vector space.

As a consequence, we obtain the following corollary which is the best constant in the Poincaré-Friedrichs inequality for K_n.

Corollary 2. *If $1 < p \leq 2$, then $(2n-1)\lambda_p(K_{2n}) = 2^{p-1}n$ and $\gamma_p =$ for K_n. If $p > 2$ then $(n-1)\lambda_p(K_n) = [n - 2 + 2^{p-1}]$ and $\gamma_p =$ for K_n.*

Proof. To establish this result, note that by considering Theorem 6 we may prove □

The Theorem 6 gives a direct proof of Theorem 2, Theorem 5 in particular casa and leads us to the follwing question

Question 2. Is there a version of theorem 2 for other eigenvalues for general graph ?

It would be desirable to find sufficiently general criteria for the graph in which the p-Laplacian has optimal number of eigenvalues.

Question 3. Is there a graph with n vertices of which the number of positive eigenvalues is larger than the number of positive eigenvalues of K_n?

Using Corollary 2, we have thus derived the following result which is a discrete version of Poincaré-Friedrichs inequality.

Corollary 3. *If $a_1, \ldots, a_n$ is a sequence of real numbers such that*

$$\sum_{i=1}^{n} |a_i|^{p-2} a_i = 0$$

then

$$\lambda_p \sum_{i,j=1}^{n} |a_i - a_j|^p \geq \sum_{i=1}^{n} |a_i|^p.$$

with equality if and only if a is the first eigenfunction of Δ_p.

However the whole motivation of Corollary 3 was to show that this does, in fact hold by minimal constant (explicite values of the optimal constant).

Acknowlegdement. The author thanks C. DELLACHERIE who turned his attention to several references and who carefully read the first draft and made many useful comments.

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