New Families of Graceful Graphs

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Abstract

In this paper, we show that if G is an "α-labeled" graph and if H is a "pseudograceful" graph, then $G \cup H$ can be graceful or "pseudograceful" under some conditions on the α-labeling function of G. This generalizes Theorem 2.1 of [7]. We also show that if G is a Skolem-graceful, then $G + \overline{K}_n$ is graceful for all $n \geq 1$. We also give a partial answer to the question in [1] about the gracefulness of $\overline{K}_n + mK_2$ for $m \geq 3$. Finally, we complete the characterization of graceful graphs in the family $C_m \cup S_n$.

All graphs in this paper are finite, simple and undirected. We follow the basic notations and terminology of graph theory as in [2]. Recall that the union $G \cup H$ of two disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ while the join G+H of G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

A graph G with vertex set V(G) and edge set E(G) with $q = |E(G)|$ is said to be graceful if there exists an injective function f, called a graceful labeling of G,

$$f : V(G) \to \{0, 1, \ldots, q\}$$

such that the induced function

$$f^* : E(G) \to \{1, 2, \ldots, q\}$$

defined by

$$f^*(xy) = |f(x) - f(y)|, \text{ for all } xy \in E(G)$$

is an injection. The image of $f\,(= I_m(f))$ is called the corresponding set of vertex labels. The concept of graceful graphs was first introduced by Rosa [5].

A survey of the present status of graceful graphs can be found in [4].

In this paper we obtain some new families of graceful graphs. The paper is divided into two sections. In Section 1, we obtain our main result, Theorem 1.1, where we show that if G is an "α-labeled" graph and if H is a "pseudograceful" graph, then $G \cup H$ is graceful or "pseudograceful" under some conditions on the α-labeling function of G. This generalizes Theorem 2.1 of [7]. In Theorem 2.2 of Section 2, we show that if G is Skolem-graceful, then $G + \overline{K}_n$ is graceful for all $n \geq 1$. We also give in Corollary 2.3 a partial answer to the question in [1] about the gracefulness of $\overline{K}_n + mK_2$ for $m \geq 3$. Finally, in Theorem 2.4, we complete the characterization of graceful graphs in the family $C_m \cup S_n$ given in [3] and [7].

1. Main Theorem.

An "α-labeled" graph, is a graph G with a graceful labeling f such that there exists an integer k_f so that for each edge xy of G either $f(x) \leq k_f < f(y)$ or $f(y) \leq k_f < f(x)$. It follows that such a k_f must be the smaller of the two vertex labels that yield the edge label 1. Also, a graph with an α-labeling is necessarily bipartite with vertex partition $\{x \in V(G) : f(x) \leq k_f\}$ and $\{x \in V(G) : f(x) > k_f\}$.

A graph G with vertex set V(G) and edge set E(G) with $q = |E(G)|$ is pseudograceful if there exists an injective function f, called a pseudograceful labeling of G,

$$f : V(G) \rightarrow \{0, 1, \ldots, q-1, q+1\}$$

such that the induced function f^* on E(G) defined by

$$f^*(xy) = |f(x) - f(y)|, \text{ for all } xy \in E(G)$$

is also an injection : from E(G) to $\{1, 2, \ldots, q\}$.

A survey of the known results on α-labeled graphs and pseudograceful graphs can be found in [4].

Our main theorem can now be stated.

Theorem 1.1

Let G be a graph having an α-labeling f, and let H be a pseudograceful graph. Then

(i) If $k_f + 2$ or $k_f - 1 \notin I_m(f)$, then $G \cup H$ is graceful, and

(ii) If $\{k_f + 2, |E(G)| - 1\} \cap I_m(f) = \Phi$ and $k_f + 2 \neq |E(G)| - 1$

or $\{1, k_f - 1\} \cap I_m(f) = \Phi$ and $1 \neq k_f - 1$, then $G \cup H$ is pseudograceful.

Proof.

As f is an α-labeling of G, $V(G)$ can be partitioned into two independent sets V_1 and V_2, $V(G) = V_1 \cup V_2$, where $V_1 = \{u : f(u) \le k_f\}$ and $V_2 = \{v : f(v) > k_f\}$. Suppose that h is a pseudograceful labeling function of H.

(i) If $k_f + 2 \notin I_m(f)$. Define a labeling g,

$$g : V(G \cup H) \to \{0, 1, \ldots, |E(G \cup H)|\}$$

as follows :

$$g|_{V_1} = f|_{V_1}$$

$$g|_{V_2} = f|_{V_2} + |E(H)|$$

$$g|_{V(H)} = k_f + 1 + h.$$

It is easy to check that g is a graceful labeling of $G \cup H$.

If $k_f - 1 \notin I_m(f)$, then one can easily show that $\bar{f} = |E(G)| - f$ is an α-labeling of G with $k_{\bar{f}} = |E(G)| - k_f - 1$ so that $k_{\bar{f}} + 2 \notin I_m(\bar{f})$ and $G \cup H$ is graceful by the above argument.

(ii) Let $\{k_f + 2, |E(G)| - 1\} \cap I_m(f) = \Phi$ and $k_f + 2 \neq |E(G)| - 1$. Define

$$g : V(G \cup H) \to \{0, 1, \ldots, |E(G \cup H)| + 1\}$$

as

$$g|_{V_1} = f|_{V_1} + 1$$

$$g|_{V_2} = f|_{V_2} + |E(H)| + 1$$

$$g|_{V(H)} = k_f + 1 + h + 1$$

Note that g and g^* are both injective and $|E(G \cup H)| + 1 \notin I_m(g^*)$ as in (i) since $k_f + 2 \notin I_m(f)$. Observe also that $|E(G \cup H)| \notin g(V(G))$ since $|E(G)| - 1 \notin I_m(f)$ and $|E(G \cup H)| \notin g(V(H))$ since $|E(G)| - 1 \neq k_f + 2$ and $k_f \neq |E(G)| - 1$, hence $|E(G \cup H)| \notin I_m(g)$ and g is a pseudograceful labeling of $G \cup H$ as desired.

If $\{1, k_f - 1\} \cap I_m(f) = \Phi$ and $1 \neq k_f - 1$, then as in (i) $\bar{f} = |E(G)| - f$ is an α-labeling of G and $k_{\bar{f}} = |E(G)| - k_f - 1$ so that

$\{k_{\bar{f}} + 2, |E(G)| - 1\} \cap I_m(\bar{f}) = \Phi$ and $k_{\bar{f}} + 2 \neq |E(G)| - 1$ and $G \cup H$ is pseudograceful by the above argument. □

Corollary 1.2 [7, Theorem 2.1]

Let H be a pseudograceful graph, then

$K_{m,n} \cup H$ is graceful for $m, n \geq 2$ and is pseudograceful for $m, n \geq 2$ and $(m, n) \neq (2, 2)$.

Proof.

If $V(K_{m,n}) = V_1 \cup V_2$ where V_i, $i = 1, 2$ are independent sets of vertices and $|V_1| = m \leq n = |V_2|$, then the function $f : V(K_{m,n}) \rightarrow \{0, 1, \ldots, mn\}$ defined by

$$f(V_1) = \{ in : 1 \leq i \leq m \}$$

$$f(V_2) = \{ i : 0 \leq i \leq n-1 \}$$

is an α-labeling of $K_{m,n}$ with $k_f = n - 1 \geq 1$ and $k_f + 2 = n + 1 \notin I_m(f)$ (since $n \geq 2$). Therefore, by Theorem 1.1 (i), $K_{m,n} \cup H$ is graceful. Further, if $m,n \geq 2$ and $(m,n) \neq (2,2)$, $n+1 = k_f+2 \neq |E(K_{m,n})| - 1 = mn - 1 \notin I_m(f)$, and so by Theorem 1.1 (ii), $K_{m,n} \cup H$ is pseudograceful. □

2. Some Other Families of Graceful Graphs

In this section we show in Theorem 2.1 that if G is a (p, q) graceful graph with $p = q + 1$, then $G + S_n$ is graceful for all $n \geq 1$. We also obtain in Corollary 2.3 a partial answer to the question of [1] on the gracefulness of the graphs $mK_2 + \overline{K}_n$, $m \geq 3$. Finally, in Theorem 2.4 we complete the characterization of graceful graphs in the family $C_m \cup S_n$ given in [3] and [7].

Theorem 2.1

If G is a (p, q) graceful graph with $p = q + 1$, then $G + S_n$ is graceful for all $n \geq 1$.

Proof.

Let $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ where v_0 is the center vertex of S_n and let f be a graceful labeling of G. Define a labeling function

$$g : V(G + S_n) \rightarrow \{0, 1, \ldots, 2q + 1 + n(q+2)\}$$

by

$$g|_{V(G)} = f$$

$$g(v_0) = 2q + 1 + n(q+2)$$

$$g(v_i) = q + i\,(q+2) \qquad , \qquad 1 \le i \le n\,.$$

Then $\{g^*(v_0v_i) : 1 \le i \le n\} = \{q + 1 + i\,(q+2) : 0 \le i \le n-1\}$. Then g and g^* are injective and g is a graceful labeling as desired. □

Recall that a (p, q) graph G is said to be Skolem-graceful if there exists an injective function

$$f : V(G) \to \{1, 2, \ldots, p\}$$

such that the induced function

$$f^* : E(G) \to \{1, 2, \ldots, q\}$$

defined by

$$f^*(xy) = |f(x) - f(y)|\,, \text{ for all } xy \in E(G)$$

is also injective.

Theorem 2.2

If G is Skolem-graceful, then $G + \overline{K}_n$ is graceful for all $n \ge 1$.

Proof.

Let G be of order p and size q and let f be a Skolem-graceful labeling of G. Define a labeling function

$$g : V(G + \overline{K}_n) \to \{0, 1, \ldots, np + q\}$$

as

$$g|_{V(G)} = f - 1$$

$$g\,(V(\overline{K}_n)) = \{ip + q : 1 \le i \le n\}$$

then g and g^* are injective as desired. □

The converse of Theorem 2.2 is not true. For $m \ge 1$ the graph $K_{1,1,m} + \overline{K}_n = K_{1,1,m,n}$ is greaceful for all $n \ge 1$ by Theorem 4 of [6], but the graph $K_{1,1,m}$ is not Skolem-graceful, for all $m \ge 1$ since $|E(K_{1,1,m})| = 2m+1 > m+1 = |V(K_{1,1,m})|-1$.

The following corollary gives a partial answer to the question in [1] about the gracefulness of $mK_2 + \overline{K}_n$, $m \geq 3$.

Corollary 2.3

For all $n \geq 1$, $mK_2 + \overline{K}_n$ is graceful if $m \equiv 0$ or $1 \pmod 4$ and it is not graceful if n is odd and $m \equiv 2$ or $3 \pmod 4$.

Proof.

Note that mK_2 is Skolem-graceful if and only if $m \equiv 0$ or $1 \pmod 4$ [8], hence the first statement follows from Theorem 2.2. The second statement follows by applying the graceful parity condition [5]. □

Finally, we complete the characterization of the graceful graphs in the family $C_m \cup S_n$. We have $C_m \cup S_n$ is graceful for $m \geq 7$ and $n \geq 1$ by [3], and $C_3 \cup S_n$ is not graceful for all $n \geq 1$ and $C_4 \cup S_n$ is graceful if and only if $n = 2$ by [7].

Theorem 2.4

(1) $C_5 \cup S_n$ is graceful if and only if $n = 1$ or 2

(2) $C_6 \cup S_n$ is graceful if and only if n is odd or $n = 2$ or 4.

Proof.

(1) $\Leftarrow$ A graceful labeling of $C_5 \cup S_1$ (resp. $C_5 \cup S_2$) is indicated in Figure 1.

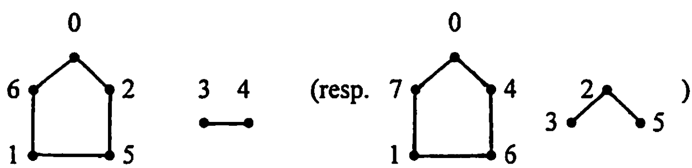
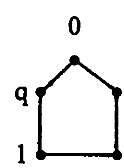


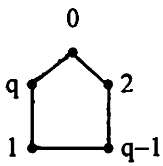
Figure 1.

$\Rightarrow$ Suppose that f is a graceful labeling of $C_5 \cup S_n$ for some $n \geq 3$. Then $q = |E(C_5 \cup S_n)| = n + 5 \geq 8$ and let $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ where v_0 is the center vertex of S_n. We have $0 \in f(V(C_5))$ by Lemma 2.3 (b) of [7] and hence we have one of two cases :

Case 1 : The vertex labels of C_5 have the pattern

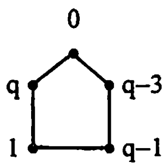


In this case, this labeling must further extend to



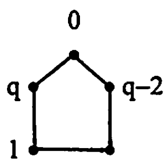
which gives $q-4 \notin I_m(f^*)$, which is absurd,

or



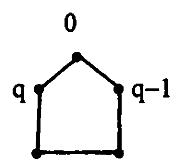
which gives $2 \in f^*(E(C_5)) \cap f^*(E(S_n))$, which is absurd,

or



hence $q-3 \notin f^*(E(C_5))$ and once again we get a contradiction.

Case 2 : The vertex labels of C_5 have the pattern



This pattern can be represented by the ordered 5-tuple (0, q, *, *, q–1). (The order, clockwise or anticlockwise, in which C_5 is traced is immaterial). In this case, there are three ways in which the labeling can be extended.

(i) (0, q, q–2, 1, q–1), which gives $q-4 \notin I_m(f^*)$, a contradiction.

(ii) (0, q, 3, 1, q–1), in this subcase $q-4 \in f^*(E(S_n))$ and we may assume that $\{f(v_0), f(v_1)\} = \{2, q-2\}$. This gives $2 \in f^*(E(C_5)) \cap f^*(E(S_n))$, an impossibility.

(iii) (0, q, 2, *, q–1), hence $q-3 \notin f^*(E(C_5))$ and we may assume that $\{f(v_0), f(v_1)\} = \{1, q-2\}$ which gives 1 or $q-4 \in f^*(E(C_5)) \cap f^*(E(S_n))$, which is absurd.

(2) Let $q = |E(C_6 \cup S_n)| = n + 6$ and let $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ where v_0 is the center vertex of S_n.

$\Leftarrow$ Suppose that n is odd and $i = (q-1)/2$, then a graceful labeling of $C_6 \cup S_n$ is indicated in Figure 2.

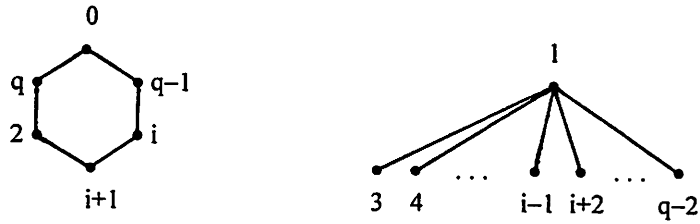


Figure 2.

A graceful labeling of $C_6 \cup S_2$ (resp. $C_6 \cup S_4$) is shown in Figure 3.

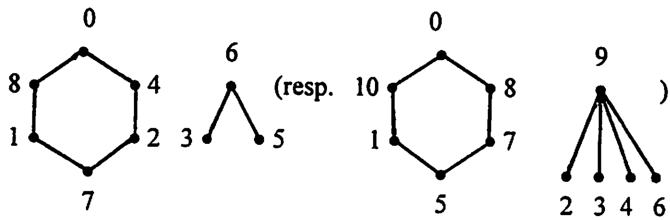


Figure 3.

$\Rightarrow$ Suppose that f is a graceful labeling of $C_6 \cup S_n$ for some n even ≥ 6. We have $0 \in f(V(C_6))$ by Lemma 2.3 (b) of [7], and hence we must have one of the following two cases :

Case 1 : The vertex labels of C_6 have the pattern $(0, q, *, *, *, q-1)$.

In this case, the labeling must further extend to one of the following three subcases :

(i) $(0, q\ \ , 4, q-2, 1, q-1)$, which gives $q-6 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.

(ii) $(0, q, 3, i, 1, q-1)$, with $4 \leq i \leq q-4$ and we may assume that $\{f(v_0), f(v_1)\} = \{2, q-2\}$ which gives $i-1$ or $i-3 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.

(iii) $(0, q, 2, *, *, q-1)$, hence $q-3 \notin f^*(E(C_6))$ and we may assume that $\{f(v_0), f(v_1)\} = \{1, q-2\}$. If $f(v_0) = q-2$, then the labeling of C_6 must have the pattern $(0, q, 2, q-3, 3, q-1)$, and $q-6 \in f^*(E(C_6)) \cap f^*(E(S_n))$, a contradiction. If $f(v_0) = 1$, then, since q is even, the labeling of C_6 must be one of the following two ways :

(a) $(0, q, 2, q-i, i, q-1)$, with $4 \leq i \leq q-4$, $i \neq q-4$, hence $q-i-2 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.

(b) $(0, q, 2, *, q/2, q-1)$, and since $1 \in f^*(E(C_6))$, we must have 3 or $q/2 - 1 \in f(E(C_6))$ and as $n \geq 6$, $q/2 - 2 > 3$, and this gives that $q/2 - 3 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.

Case 2 : The vertex labels of C_6 have the pattern $(0, q, 1, *, *, *)$.

In this case, the labeling must further extend to one of the following three subcases :

(i) $(0, q, 1, q-1, 2, q-4)$, and we may assume that $\{f(v_0), f(v_1)\} = \{3, q-2\}$ which gives $q-6 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.

(ii) $(0, q, 1, q-1, i, q-3)$, and we may assume that $\{f(v_0), f(v_1)\} = \{2, q-2\}$ and $4 \leq i \leq q-5$ which gives $q-i-3 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.

(iii) $(0, q, 1, *, *, q-2)$, and we may assume that $\{f(v_0), f(v_1)\} = \{2, q-1\}$. If $f(v_0) = 2$, then the labeling of C_6 must be one of the following two ways:

(a) $(0, q, 1, q-3, i, q-2)$, with $4 \leq i \leq q-4$, $i \neq q-i$, hence $q-i-2 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.

(b) $(0, q, 1, q-3, q/2, q-2)$, and since $q/2 - 1 > 3$, we have $q/2 - 3 \in f^*(E(C_6)) \cap f^*(E(S_n))$, which is absurd.
If $f(v_0) = q-1$, then the labeling of C_6 must have the pattern $(0, q, 1, j, i, q-2)$, with $4 \leq i, j \leq q-3$, and since $1 \in f^*(E(C_6))$ we must have either $q-i-2$ or $|i-j| = 1$. If $i = q-3$, then $j = q-5$ and we have $q-6 \in f^*(E(C_6)) \cap f^*(E(S_n))$, since $5 < q-5$, which is absurd. If $|i-j| = 1$, then $q-i-2 \in f^*(E(C_6)) \cap f^*(E(S_n))$, a contradiction. □

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