SELF-DUAL CONFINED CONFIGURATIONS WITH TEN POINTS

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1. Introduction. Let Σ be a non-empty set of points. Let σ be a non-empty set of lines. Let $\mathcal{I}$ be a **symmetric** relation, a subset of $\Sigma \cup \sigma \times \Sigma \cup \sigma$ where $(\alpha, \beta) \in \mathcal{I}$ implies either $\alpha \in \Sigma$ & $\beta \in \sigma$, or $\alpha \in \sigma$ & $\beta \in \Sigma$. Informally, we say point P *is on* line ℓ for $(P, \ell) \in \mathcal{I}$, line ℓ *goes through* point P for $(\ell, P) \in \mathcal{I}$, and point P *is off* line ℓ for $(P, \ell) \notin \mathcal{I}$. Moreover, if P and Q$\neq$P are on ℓ, we say that ℓ *joins* P and Q. Likewise, for P on k and on $\ell \neq$k , we say that k and ℓ *intersect in* P. Furthermore, the *closure of join* condition means that every pair of distinct points are joined by some line; *closure of intersection* is the dual condition. The triple $(\Sigma, \sigma, \mathcal{I})$ is called a partial plane exactly when *two distinct lines never join two distinct points.*

A *configuration* is a partial plane with finite sets Σ and σ. A partial plane is *confined* means that each line has at least 3 distinct points on it and each point has at least 3 distinct lines through it, (see Hall [10]).

Partial plane $(\Sigma, \sigma, \mathcal{I})$ is *isomorphic to* partial plane $(\Pi, \pi, \mathcal{J})$ if and only if (1) there is a bijection $\mathcal{B}$ from Σ to Π, (2) there is a bijection β from σ to π, and (3) for every P in Σ and ℓ in σ, $(P, \ell) \in \mathcal{I}$ if and only if $(\mathcal{B}(P), \beta(\ell)) \in \mathcal{J}$.

A partial plane $(\Sigma, \sigma, \mathcal{I})$ is *self-dual* if and only if it is isomorphic to its dual $(\sigma, \Sigma, \mathcal{I})$. We use the phrase *our configuration* for any self-dual confined configuration having exactly 10 points. By duality, we have exactly 10 lines. Thus, one cannot have a line with 5 points, since each of these points would lie on two more lines yielding more than 10 lines.

A line with 4 points on it is called *a long line*; a point with 4 lines through it is called *a strong point.* A long line with a strong point on it is called *a strong line*; otherwise the long line is *weak.*

We find all the self-dual confined configurations with exactly 10 points in Σ, their collineation groups, their invariants, and we show in which structures some of them lie. No configuration has 5 long lines, only one has 4 long lines, four have 3 long lines, twelve have 2 long lines, eighteen have one long line, and ten have no long lines. The last ten are the known Martinetti configurations [7], including the well-known Desargues configuration.

2. Projective planes. A projective plane may be defined as a confined partial plane closed under join and intersection (lemmas below, [11],p.346).

Lemma 1: *Every confined partial plane has a quadrangle (four, no three collinear, points).*

Proof. Let P have k, m, n through it. Each line has an additional point so chosen that the three are not collinear.

Lemma 2: *Any partial plane closed under join and intersection having a quadrangle is a confined partial plane.*

Proof. Let the quadrangle be A, B, C, D. By the closures, the *sides* AB, AC, AD, BC, BD, CD are present (yielding a *complete quadrangle* [3],p.17) as well as the *diagonal points*, E=AB∩CD, F=AC∩BD, G=AD∩BC. Let P be any point. If P is off AB, then P joins A, B, E. If P is off AC, then P joins A, C, F. If P=A, then A joins B, C, G. Any line k has some point P off it, and the 3 lines through P intersect k in three points.

Let Π_1 be any partial plane containing a quadrangle. Then let Π_{2n} be the partial plane Π_{2n-1} and any new lines which join any pairs of distinct points. Also let Π_{2n+1} be the partial plane Π_{2n} and any new points of intersections of pairs of distinct lines. Then

$$\bigcup_{i=1}^{\infty} \Pi_i$$

is Hall's free projective plane generated by Π_1. Lemma 2 essentially verifies that this is a projective plane.

The fundamental theorem of free projective planes: *Any confined configuration lies in the generating partial plane* Π_1.

Each of our configurations lies in some free plane. This follows from lemma 1. **There is a free plane in which none of our configurations lie.** This follows from the fundamental theorem where the generating Π_1 is a quadrangle with one of its sides (so that σ is non-empty).

A *standard construction of Pappian projective planes*: For any three-dimensional vector space over a field, the one-dimensional subspaces will represent points while the two-dimensional subspaces will represent lines. Any non-zero vector (a,b,c) generates a one-dimensional subspace by taking all scalar multiples of that vector. The associated point has homogeneous coordinates $(\lambda a, \lambda b, \lambda c)$, $\lambda \neq 0$. The two-dimensional subspaces can be described by equations such as Ax+By+Cz=0, where at least one of A, B, C is non-zero. Then the associated line has homogeneous coordinates $[\mu A, \mu B, \mu C]$, $\mu \neq 0$. For convenience we often use (a,b,c) for $(\lambda a, \lambda b, \lambda c)$ and use [A,B,C] likewise. A point is on a line if and only if the associated one-dimensional subspace is a subspace of the associated two-dimensional subspace. Moreover, (a,b,c) is on [A,B,C] if and only if Aa+Bb+Cc=0.

Quadrangle theorem: *Given any two quadrangles, there is a collineation sending one to the other.*

Proof. Any linear map will induce a bijection preserving collinear sets of the associated projective plane, i.e. a collineation. Any quadrangle is created from 4 vectors where any three span the space. We shall find explicitly the map which takes quadrangle (1,0,0), (0,1,0), (0,0,1), (1,1,1) to quadrangle (a_1,a_2,a_3), (b_1,b_2,b_3), (c_1,c_2,c_3), (d_1,d_2,d_3). Then there is an inverse map taking the second quadrangle to the first. Then the old trick of taking any given quadrangle to our special quadrangle followed by taking our special quadrangle to any other given quadrangle accomplishes the mission.

$$\begin{pmatrix} \lambda a_1 & \mu b_1 & \nu c_1 \\ \lambda a_2 & \mu b_2 & \nu c_2 \\ \lambda a_3 & \mu b_3 & \nu c_3 \end{pmatrix}$$

This matrix fulfills the first three requirements (but if all the scalars are 1, the fourth may not be fulfilled). If we can solve the following linear system for non-zero scalars, then the fourth condition also holds. But Cramer's rule does that since the associated determinants are all non-zero.

$$\lambda a_1 + \mu b_1 + \nu c_1 = d_1$$
$$\lambda a_2 + \mu b_2 + \nu c_2 = d_2$$
$$\lambda a_3 + \mu b_3 + \nu c_3 = d_3$$

Note: every configuration residing in this projective plane using the real field for scalars also resides in E_2, the Euclidean plane. From this result, any quadrangle of a configuration can be placed arbitrarily in E_2. But some line might be lost.

Desargues theorem: *Let O, P, Q, R and O, P′, Q′, R′ be two quadrangles sharing exactly one vertex and the three sides OPP′, OQQ′, ORR′, then the points K=PQ∩P′Q′, L=PR∩P′R′, M=QR∩Q′R′ are collinear.*

Proof. (This is motivated by the statement of Wagner's theorem [20].) Let O:(0,0,1), P:(1,0,0), Q:(0,1,0), R:(1,1,1); then the shared sides have coordinates [0,1,0], [1,0,0], [1,–1,0] and P′:(1,0,a), Q′:(0,1,b), R′:(1,1,c) where a+b≠c. Note: this hypothesis fails for the field of two elements. We introduce a collineation.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c-a-b \end{pmatrix}$$

Any line through O, say [p,q,0], has other points (–q,p,t). The collineation fixes O, sends P to P′, Q to Q′, R to R′, and fixes the lines through O. Once (u,v,w) is a fixed point then (u,v,au+bv+cw–aw–bw) = $(\lambda u, \lambda v, \lambda w)$. Hence $\lambda = 1$ or u=v=0. Other fixed points than O satisfy $\lambda = 1$ and (c–a–b–1)w=–au–bv. Then [a,b,c–a–b–1] is the line containing these fixed points including O if and only if c–a–b–1=0. Moreover, each point on this line is fixed.

Now PQ intersects line OK in K. Then PQ goes to P′Q′, and OK is fixed. Hence K goes to the intersection of P′Q′ and OK which is K.

Therefore K is a fixed point. Similarly L and M are fixed, none is O, so they lie on the line of fixed points. **Hence both the Desarguesian configuration and the little Desarguesian configuration lie in E_2.**

m=OPP′	q=PRL	M=opp′	Q=prℓ
ℓ=OQQ′	q′=P′R′L	L=oqq′	Q′=p′r′ℓ
k=ORR′	p=QRM	K=orr′	P=qrm
r=PQK	p′=Q′R′M	R=pqk	P′=q′r′m
r′=P′Q′K	o=OKLM	R′=p′q′k	O=okℓm

This display serves several purposes. First, we can see that this is one of our configurations by counting the number of points and lines. Also confinement is noted. To check the self-dual condition merely take each point in the the second half of the display and make sure that its lines do go through the point in the first half of the display. For example, we see M is on lines o, p, p′ from the first line third column. Then in the second column we observe in the last three rows that indeed M is on each of those lines. In later displays only the first half will be given; the second half can be constructed by dualizing the first half.

By comparison with the proof just given, we find that this configuration is the little Desargues configuration. We can obtain a display for the Desargues configuration from this one by changing o=OKLM to o=KLM and O=okℓm to O=kℓm. This illustrates a process that we call *diminishing*. Given any one of our configurations with the isomorphism showing duality, if a strong line corresponds in the isomorphism to one of its strong points, then we can remove two ordered pairs from $\mathcal{I}$. In this example, they are (o,O) and (O,o). Sometimes strong point A is on long line b, and strong point B is on long line a; then we can *double diminish* by removing (A,b), (B,a), (a,B), (b,A) from $\mathcal{I}$. Warning! There is often more than just one isomorphism to show duality. For example, with the Desargues configuration, we can change 6 labels as noted: r′=PQK, r=P′Q′K, q′=PRL, q=P′R′L, p′=QRM, p=Q′R′M.

Any point of the standard Desargues configuration can be taken as center and the three points not joining it form the axis; thus we can find in this way 20 complete quadrangles. But each quadrangle has been counted 4 times; hence, there are 5 different complete quadrangles: OPQR, OP′Q′R′, PP′KL, QQ′KM, RR′LM. There is a collineation of this configuration interchanging any two of the quadrangles; for example, O, K, L, M fixed, (PP′), (QQ′), (RR′). These involutions can be used as generators for the collineation group. Just as ten such "abstract" involutions produce S_5, these ten produce **the collineation group which is isomorphic to S_5**. (See [11],pp.54,405).

Our invariant counts the number of quadrangles with two sides, three sides, four sides, five sides, six sides, and the number of quadrangles with no, one, two, or three diagonal points. **The invariant for the standard Desargues configuration is (0,60,75,0,5,125,0,15,0)**; in words, 60 quadrangles with three pairs of vertices joined and no other sides, 75 with four sides, the 5 complete quadrangles already noted, and 125 of these 140 quadrangles have no diagonal points (the sides do not intersect in the configuration) while 15 do support two diagonal points.

Quadrangle PQLM has four sides and diagonal points K and R; quadrangle PQRP′ has four sides and no diagonal points; quadrangle PQP′R′ has three sides and no diagonal points.

Every quadrangle of every configuration of ours has at least two sides; therefore, we start our invariant with the number of quadrangles with two sides. This is almost a complete set of invariants with just one pair of configurations sharing an invariant without being isomorphic. We show directly that these two are not isomorphic while the invariant insures all other pairs we introduce as different are different. We place all the invariants in section 10. The invariants and the collineation groups are found by computer.

3. Finite Projective Planes. First we use GF(2) ([5],p.165) and quadrangle completion. Let A:(1,0,0), B:(0,1,0), C:(0,0,1), D:(1,1,1). Then AB:[0,0,1], AC:[0,1,0], AD:[0,1,1], BC:[1,0,0], BD:[1,0,1], CD:[1,1,0] as well as E:(1,1,0), F:(1,0,1), G:(0,1,1). Finally EFG:[1,1,1]. This is also called the *Fano configuration*. Thus the smallest projective plane is Pappian.

The counting theorem(e.g. [11],p.348): *If one line of a projective plane has exactly n+1 points on it, then every line has exactly n+1 points on it, every point has exactly n+1 lines through it, there are exactly n^2+n+1 points and n^2+n+1 lines.* We say this plane has **order n**.

The Fano configuration is a plane of order 2 with 3 points per line, 3 lines through each point, 7 points and 7 lines.

Corollary: *None of our configurations is a projective plane; no plane except the smallest is contained in any of our configurations.*

Consider any field of characteristic 2 with 4 distinct elements 0, 1, α, β. We extend the Fano configuration just given by X:$(\alpha,\beta,0)$, Y:$(\alpha,0,\beta)$, Z:$(\alpha+\beta,\beta,\beta)$. Then XYG:$[\beta,\alpha,\alpha]$, XZF:$[\beta,\alpha,\beta]$, YZE:$[\beta,\beta,\alpha]$. Hence one of our configurations is: e=ABEX, f=ACFY, g=ADGZ, d=BCG, c=BDF, b=CDE, a=EFG, z=XYG, y=XZF, x=YZE. We call this *the general extended Fano configuration*. We later show that this is not in any ordered plane; note that E_2 is an ordered plane. For the collineation group use the dihedral group ([11],p.19) with generators (BYDXCZ)(EFG) and (BC)(XY)(EF).

Consider any field of characteristic 2 with $\iota^2+\iota+1=0$ (e.g. GF(4^η)). Let X:$(1,\iota,0)$, Y:$(1+\iota,0,\iota)$, Z:$(1,\iota,1+\iota)$; then XYD:$[\iota,1,1+\iota]$, XZC:$[\iota,1,0]$, YZB:$[\iota, 0, 1+\iota]$. Thus another of our configurations is: c=ABEX, b=ACFY, g=ADG, a=BCG, f=BDF, e=CDE, d=EFGZ, z=XYD, x=XZC, y=YZB. We call this *the special extended Fano configuration.* We later show that this is not in any ordered plane. Its collineation group of order 2 has the identity element and involution (BC)(EF)(XY).

Let us perform some quadrangle completions for fields which do not have characteristic 2; A:(1,0,0), B:(0,1,0), C:(0,0,1), D:(1,1,1), as usual, then AB:[0,0,1], AC:[0,1,0] AD:[0,1,–1], BC:[1,0,0], BD:[1,0,–1], CD:[1,–1,0], E:(1,1,0), F:(1,0,1), G:(0,1,1), but no longer collinear as previously. Therefore EF:[–1,1,1], EG:[1,–1,1], FG:[1,1,–1], are *the sides of the diagonal triangle* [3]. The harmonic points arise as the intersection of a side of the quadrangle and the appropriate side of the diagonal triangle; AB,FG yield H:(1,–1,0), AC,EG yield I:(1,0,–1), AD,EF yield J:(2,1,1), BC,EF yield K:(0,1,–1), BD,EG yield L:(1,2,1), CD,FG yield M:(1,1,2). More is said about harmonic sets later. This completion is equivalent to Π_5 of the free plane generated from $\Pi_1 = \{A, B, C, D, AB\}$.

For fields which do not have characteristic 3, in the next extension we have 12 lines joining vertices of the quadrangle and harmonic points and KLM:[–3,1,1], IJM:[1,–3,1], HJL:[1,1,–3], HIK[1,1,1]. Ignoring the 12 lines noted and removing harmonic points H, I, J and the three new lines involving them, we have a Desargues configuration center A, axis KLM, with exactly 3 long lines, namely, BCGK, BDFL, CDEM.

Furthermore, if we keep A, B, C, D, E, F, K, ABE, ACF, AD, BCK, BDF, CDE, EFK instead and add X:[4,1,1] on AD, yielding EX:[–1,1,3] and FX:[–1,3,1], and add Y:[1,–2,1] on EX and BDF, and Z:[1,1,–2] on FX and CDE, yielding K on YZ:[1,1,1]. Thus we have another Desargues configuration center A, axis KYZ, with exactly two long lines, namely BDFY, and CDEZ.

If the field does have characteristic 3, then we have lines AKLM:[0,1,1], BIJM:[1,0,1], CHJL:[1,1,0], DHIK:[1,1,1]. But if a projective plane of order 3 has a subplane of order 2, then each new point on each line of the subplane is different from the others yielding 14 points contrary to the counting theorem. Thus we have shown uniqueness of this plane by quadrangle completion as we essentially did above for order 2. Every extension was free except the last one; it is forced by the counting theorem that we can only add 4 more lines. Without the guidance of the algebra, no other completion is viable.

Since this projective plane of order three is represented by subspaces, we can use the theorem concerning collineations and find that the permutations of A, B, C, D can be extended to collineations.

If we delete the diagonal triangle and its vertices from this plane of order 3, we get the following configuration: a=AKLM, b=BIJM, c=CHJL, d=DHIK, h=CDM, i=BDL, j=BCK, k=ADJ, ℓ=ACI, m=ABH. It's one of ours. We later show that it is not in any ordered plane. The collineation group is S_4 acting on A, B, C, D. The "short" lines (those with three points) are thus permuted and the rest of the collineation can be determined by that. Moreover, this shows that one may diminish by any of A, B, C, D.

We diminish by D. This yields: a=AKLM, b=BIJM, c=CHJL, d=HIK, h=CDM, i=BDL, j=BCK, k=ADJ, ℓ=ACI, m=ABH. But this is isomorphic to the Desargues configuration just noted with 3 long lines. The isomorphism from that to this diminished configuration is: A→D, B→J, C→L, D→M, E→A, F→B, G→C, K→H, L→I, M→K. Since the little Desargues configuration has one long line, and this has three, we call this *the littlest Desargues configuration*. Its collineation group is S_3 acting on A, B, C.

Then *the littler Desargues configuration* has two long lines obtained from the littlest from diminishing by C. But this is isomorphic to the earlier Desargues configuration with 2 long lines. The isomorphism from that to this diminished configuration is: A→D, B→J, C→L, D→M, E→A, F→B, K→H, X→C, Y→I, Z→K. Its collineation group is Klein's 4-group with non-identity elements (AB)(IK)(JL), (CD)(IJ)(KL), (AB)(CD)(IL)(JK).

Diminishing by B yields the little Desargues configuration, best rendered with A as center and on axis KLM; but in terms of the others one could also use B as center, quadrangles BHLK, ABCD, axis IJM. Its collineation group, also dihedral, has generators (BIDHCJ)(KML) and (BC)(HI)(LM). Finally, diminishing by A yields the standard Desargues configuration as noted already.

All of these Desargues configurations lie in E_2; however, once we establish a configuration in E_2, we know that in some sense all configurations obtained from this one by any diminishing operation will also lie in E_2.

The general extended Fano configuration also can be diminished using any one of its strong points. If we diminish by G, then the new configuration maps onto the littler Desargues configuration thus: A→M, B→L, C→J, D→D, E→A, F→B, G→H, X→K, Y→I, Z→C.

The special extended Fano configuration can be double diminished by B and C; the collineation group is the same as that of the "parent" configuration. For completeness in this case, this diminished configuration lies in E_2. We shall show an embedding by giving the coordinates of the points: A:(0,0), B:(31/23,24/23), C:(5/13,8/13), D:(1,0), E:(0,1), F:(2,3), G:(−1,0), X:(0,6), Y:(4/5,6/5), Z:(1/3,4/3).

In the vector space, the one-dimensional subspaces (a,b,c) not contained in the two-dimensional subspace [0,0,1] are characterized by c≠0. In turn, by dividing by c, or λc for $(\lambda a, \lambda b, \lambda c)$, we have (u,v,1). Thus these special vectors with tails at the origin have their heads lie on the plane z=1.

The points of a *field plane* are all ordered pairs of field elements. Thus these associated projective points correspond to points of a field plane. The lines of a field plane are defined by equations Ax+By+C=0, where A≠0 or B≠0. The other two-dimensional subspaces [A,B,C] are also characterized by A≠0 or B≠0. Furthermore, they intersect z=1 yielding the equations for field planes. Now two-dimensional subspaces intersecting in a one-dimensional subspace (–B,A,0) in [0,0,1] correspond to parallel lines in the field plane, namely, Ax+By+C=0 is parallel to Dx+Ey+F=0 if and only if AE=BD. Thus we see how to go from a Pappian projective plane to a field plane, and conversely, to go from a field plane to a Pappian projective plane add *ideal points* for each parallel class and the *ideal line* consists of the ideal points.

4. Affine planes. An affine plane is either a complete quadrangle or a confined partial plane closed under join for which Playfair's axiom holds. Two lines are *parallel* (denoted by ||) if and only if they share no points or they share all points.

Playfair's axiom: *For any point P and any line k there exists a unique line m, P on m, and k||m.*

It is routine to show that every field plane is an affine plane. Likewise we can obtain a projective plane from any affine plane, but the projective plane may not be Pappian, e.g. the free projective plane which is explicitly non-Desarguesian.

A *near-field* ([11],p.364) is a structure sharing all but the commutative law of multiplication and the left distributive law with a field. The near-field of order nine can be obtained from GF(9) by the same addition and the new multiplication o defined in terms of GF(9) by $a \circ b = 2ab + 2a^3b + 2ab^5 + a^3b^5$. The non-zero elements form a copy of the quaternion group ([11],p.23) under o. Just as one views the complex numbers as an extension of the reals, we view GF(9) as the extension of GF(3) obtained by adjoining i, a solution to the equation $x^2 + 1 = 0$. For convenience, the additional field elements i, i+1, i+2, 2i, 2i+1, 2i+2 are denoted 3, 4, 5, 6, 7, 8, respectively. The lines are defined by equations x=k, y=k, y=xom+b. In general, any **finite** near-field produces an affine plane in this manner. When we extend such non-field planes, the projective plane is not Pappian, so we replace homogeneous coordinates by Hall coordinates. The ideal line is labelled L_∞. The ideal point for the lines with slope m has coordinate (m), including (0) for y=k type lines. For x=k type lines the ideal point has coordinate (∞).

In this near-field projective plane, the extension of the affine plane, we do some quadrangle completions as follows: A:(0,0), B(0,1), C:(1,3), D:(1,7); AB:x=0, AC:y=xo3, AD:y=xo7, BC:y=xo5+1, BD:y=xo6+1, CD:x=1. Then E:(∞), F:(3,2), G:(7,2), EF:x=3, EG:x=7, FG:y=2. Next

H:(0,2), I:(7,4), J:(3,8), K:(3,5), L:(7,6), M:(1,2). Of special interest are these: HIJ:y=xo2+2, HKL:y=x+2, IKM:y=xo4+7, JLM:y=xo8+3. This configuration is called a *Fano cluster* in that it contains 4 Fano configurations. This projective plane is not self-dual; the dual plane also has a Fano cluster [13]. The remaining non-Desarguesian projective plane of order 9, generalized by Hughes, also has a Fano cluster [12], [6]p.377.

The Fano cluster can be displayed: ABEH, ACFI, ADGJ, BCGK, BDFL, CDEM, EFJK, EGIL, FGHM, HIJ, HKL, IKM, JLM. Any one of the 6 permutations on $\{B, C, D\}$ can be extended to a collineation of the Fano cluster. We discard the points K, L, M and any line with two of them, and we have: ABEH, ACFI, ADGJ, BCG, BDF, CDE, EFJ, EGI, FGH, HIJ. Then we have a potential configuration of ours. The first three lines are the long lines and they are concurrent in A. The points E, F, G are strong but they are not collinear. Hence this fails to be self-dual. Changing ADGJ to ADJ yields another candidate, which we see later and called *the two strong line* configuration. When this is diminished by F, we have the k-configuration [5]pp.294-295. When this is twice diminished we obtain the *anti-Desargues* configuration (named by private communication), another Martinetti configuration. We can also get this anti-Desargues configuration from the Fano cluster by discarding the diagonal triangle and any other occurrences of the diagonal points. In the latter case, we have h=ABH, i=ACI, j=ADJ, k=BCK, ℓ=BDL, m=CDM, a=HIJ, b=HKL, c=IKM, d=JLM. In [15] we showed that this configuration does not lie in E_2; it follows from this that neither the k-configuration nor the two strong line configuration lies in E_2. Nevertheless we showed in [16] that it lies in a Moulton plane, which is a non-Desarguesian affine complete ordered plane. We introduce the last structure for embedding, ordered planes, in the next section; we could also have investigated embedding the Martinetti configurations in Steiner-triple systems which would overlap with [7].

One more use of the near-field plane is to place a copy of the asymmetric disjoint configuration there. The points are: A:(1,0), B:(0,0), C:(2,0), D:(5,0), E:(2,1), F:(6,1), G:(3,1), H:(5,1), I:(5,4), J:(7,5). The lines are: ABCD:y=0, EFGH:y=1, AEI:y=x+2, AFJ:y=xo4+8, BEJ:y=xo2, BGI: y=xo6, CFI:y=xo5+5, CHJ:y=xo6+6, DGJ:y=x+7, DHI:x=5.

5. Ordered planes. We have a non-empty set of points and a ternary relation expressed by ωPQR. In ordered field planes this relation can be expressed: if P:(p_1, p_2), R:(r_1, r_2), and ωPQR, then there is a field element λ, where $0 < \lambda < 1$ and Q:$(\lambda p_1 + (1-\lambda)r_1, \lambda p_2 + (1-\lambda)r_2)$. We have three models for five of the axioms of an ordered plane. In the field plane using GF(5), ωABC will mean that B is the midpoint of A and C, where midpoint is computed in the usual way. In the integer plane, the points are ordered pairs of integers and the lines are defined by linear Diophantine equations; here ωABC is consistent to that used for ordered field planes. In E_3, analytic ωABC involves also $\lambda p_3 + (1-\lambda)r_3$. Thus five axioms allow a finite plane, a plane with gaps on the lines (some pairs of distinct points without midpoints), a 3-dimensional space, and any ordered field plane.

Five axioms of an ordered partial plane.

1.$(\forall A)(\forall B)(\forall C)(\omega ABC \Rightarrow A \neq B \neq C \neq A)$
2.$(\forall A)(\forall B)(\forall C)(\omega ABC \Rightarrow$ not $\omega BCA)$
3.$(\forall A)(\forall B)(A \neq B \Rightarrow (\exists C)(\omega ABC))$
Line AB for A≠B is $\{P | P = A$ or $P = B$ or ωPAB or ωAPB or $\omega ABP\}$
4. Two distinct lines never join two distinct points.
5. There are three non-collinear points.

An *ordered plane* is an ordered partial plane with an additional axiom, called the Pasch axiom. In [8] one finds that Pasch forces an infinite number of points on a line, there are no gaps, and from any triangle (using all points on each side) by joining and intersecting, we can reach every point, i.e. we have a plane.

6.(Pasch): *For all A, B, C, D, d, if A, B, C are non-collinear points off line d, ωADB, and D is on d, then there is a point E such that E is on d and either ωAEC or ωBEC.*

In ordered field planes, the axiom of Pasch is a corollary to **Hilbert's Separation Theorem**: *For any line k, there are sets $\mathcal{A}$ and $\mathcal{B}$ where these three are disjoint, their union is the whole plane, all are convex, and $P \in \mathcal{A}$, $R \in \mathcal{B}$ yield point Q on k and ωPQR.* The trick is to define k: Ax+By+C=0, $\mathcal{A}$: Ax+By+C>0, $\mathcal{B}$: Ax+By+C<0. For convexity of $\mathcal{A}$ simply add $\lambda(Ap_1 + Bp_2 + C) > 0$ to $(1-\lambda)(Ar_1 + Br_2 + C) > 0$. The last is resolved by examining three cases.

Geometric Dedekind cut axiom: *For any line k, there are disjoint sets $\mathcal{M}$ and $\mathcal{N}$, each having at least two points, S in $\mathcal{M}$ implies S is on k, likewise for T in $\mathcal{N}$, any point V on k is in one of these two sets, if P,R in $\mathcal{M}$ then any Q, ωPQR is in $\mathcal{M}$, if X,Z in $\mathcal{N}$ then any Y, ωXYZ is in $\mathcal{N}$, then there is a point C such that for all M≠C in $\mathcal{M}$, and all N≠ C in $\mathcal{N}$, ωMCN.* Note: the two points in each set have a midpoint in the GF(5) plane yielding at least 6 distinct points on the line is part of the hypothesis. Moreover, these axioms, with Pasch out, are irrefutably consistent.

A *complete ordered plane* is an ordered plane with the Dedekind cut axiom; i.e. there are no holes.

An *affine ordered plane*, or, equivalently, an *ordered affine plane* is an ordered plane with the Playfair's axiom holding.

A *Pappian affine ordered plane* is an affine ordered plane with a special Pappus configuration (see [1], p.103) universal. It is a field plane where the field is ordered. Conversely, every field plane has this Pappus configuration universal.

The 6 axioms of an ordered plane with Playfair, Dedekind, and Pappus define E_2. Again, leaving Pasch out, the GF(5) plane satisfies all the others.

These axioms, with Playfair out, are satisfied by the classical hyperbolic plane. These axioms, with Dedekind out, are satisfied by the rational plane. These axioms, with Pappus out, are satisfied by Moulton planes we now introduce.

Any Moulton plane overlaps E_2 by sharing all points and all lines except those with positive slope. For each positive r$\neq$ 1, we define a particular Moulton plane by having as each of its lines of positive slope, in terms of E_2, two rays with common endpoint on the x-axis, one in the upper half plane, the other in the lower half plane whose slope is r times that of the slope of the other ray.

In the Euclidean plane let H:(2,–3), G:(0,0), F:(–10/7,15/7) be on line y=(–3/2)x; let H, I:(0,3), J:(–4,15) be on line y=–3x+3; let H, C:(3,0), D:(16/5,3/5) be on line y=3x–9. We claim that CGHI is a quadrangle; lines CH, GH, HI are already different, and CG:y=0, GI:x=0. Also DFHJ is a quadrangle; DH, FH, HJ are already different, and FJ:y=–5x–5, DJ:y=–2x+7. Then E=GI∩FJ:(0,–5); B=CG∩DF:(5,0), since DF:x+3y=5; A:DJ∩CI:(4,–1), since CI:x+y=3. But ABE:y=x–5 producing a Desargues configuration. However, H is also on ABE yielding a little Desargues configuration.

If we change the setting of this to a Moulton plane, the 10 points remain unchanged, and likewise for 8 of the 10 lines. Even though ABEH has new points in the upper half-plane, since A, B, E, H are in the lower half-plane, those points are still collinear. But C, D, H are no longer collinear. For r=5/9, the ray CD is still y=3x–9 while the ray EC is y=(5/3)x–5 giving us the k-configuration from Moulton line CDE.

Transversal Corollary (of Pasch): *For non-collinear points A, B, C, if* ωBCD *and* ωAEC *then* ωAFB *and* ωDEF.

Crossing Corollary (of Pasch): *Given* ωABC, ωADE, *and* $AB\neq AD$, *then* ωBFE *and* ωCFD.

Triple order theorem: $\omega ABC \Rightarrow \omega CBA$, *and none of* ωACB, ωCAB, ωBAC, ωBCA *hold.* The proof is routine.

The Sylvester-Gallai Theorem: *In any ordered plane there is no confined configuration closed under join.*

Proof: (Essentially that of [2],pp.181-182.) From lemma 1 of section 2, there is a quadrangle A, B, C, D in the configuration. Hence point A is off line BC. From the infinitude of points on BC, and the finite number of points and lines of the configuration, there is a point E on BC, not of the configuration and line AE is not of the configuration.

Let $\mathcal{T} \equiv \{E\} \cup \{P|\omega EPA\}$. Let $\mathcal{S}$ be all points of $\mathcal{T}$ which lie on some line of the configuration. Note that $\mathcal{S}$ is non-empty since E is in $\mathcal{S}$. Let Z of $\mathcal{S}$ be E or for every point P≠Z of $\mathcal{S}$, ωAZP. In other words, there is a line k of the configuration intersecting AE in Z, and no point between A and Z is the intersection of a line of configuration and line AE.

On k there are three points U,V,W of the configuration. We can label them such that ωZUV and either ωWZU, case 1, or ωUVW, case 2. In either event, on line AV, a line of the configuration from closure of join, there is a third point of the configurature, be it P where ωPAV, or be it Q where ωAQV or be it R where ωAVR. In both cases, if PU were a line of the configuration, by the crossing corollary there would be an X on that line in the forbidden region; likewise if RU were a line of the configuration, by the transversal corollary there would be a Y on the line in the forbidden region. Thus Q must be the third point. In case 1, by the crossing corollary there is a point X′ on line WQ of the configuration in the forbidden region. In case 2, by the transversal corollary, there is a point Y′ on line WQ in the forbidden region.

The Fano configuration is thus not allowed in any ordered plane; hence the two configurations of ours containing the Fano configuration are not in any ordered plane. In particular, neither are in E_2. But each of them occur in many field planes where the field has characteristic 2; thus we search no further for embeddings for them.

The four long line configuration we found is unfortunately not closed under join; for example, H and M are not joined. But we now show that this configuration cannot be in any ordered plane also.

First, we make the assumption that none of ωAHB, ωAIC, ωAJD, ωBKC, ωBLD, ωCMD occurs. Then without loss of generality (from the collineation group) ωABH. Also ωACI or ωCAI. In the former case the crossing corollary yields ωBJI and ωCJH. Using triangle BHI and the transversal corollary, we find ωHDI and ωAJD. Since the latter is forbidden, we now assert ωCAI. We now apply this very argument several times. Next ωCAI and ωCDM yield ωCKB from which we assert ωDCM. Then ωDCM and ωDBL yield ωDJA from which we assert ωBDL. Also ωBDL and ωBCK yield ωBHA from which we have ωCBK. But then established ωCAI and ωCBK yield ωCMD which thus shows that the six denials assumed lead to a contradiction.

Without loss of generality we thus have ωAHB. Then for triangle ABC and collinear points H, I, K, we cannot have ωBKC and ωAIC from a theorem in Forder [8], p,49. But denying either of these forces the other from our Pasch corollaries. By the collineation group we can choose ωBKC. The crossing corollary then yields ωALK and ωCLH. Applying it again yields ωBDL and ωHDK. Once again we use it to find ωCMD and ωKML. Note that ωALK & $\omega LMK \Rightarrow \omega AMK$ from a theorem in Forder [8] on p.52. Now using triangle ACK, the transversal corollary yields ωBMI and ωAIC. The last ordered triple produces the contradiction which shows that indeed this configuration does not lie in an ordered plane. Nevertheless, it occurs in all the planes of order 9.

We now purportedly embed the asymmetric disjoint configuration in the real projective plane. We use some of the axioms for the real projective plane given by Coxeter [3]. Since we do not use the Desargues axiom, our approach may be valid in other projective planes such as the projective extension of a Moulton plane. However, we confine our discussion to the real projective plane. For a more general approach, see [18] pp. 174-175.

There are four useful functions available in projective planes but not in affine planes in this generality. Since we are using functions, and "range" is part of their vocabulary, we change Coxeter's "range" to *queue*. In other words, a queue is the set of all points on a line. The dual is *pencil*; i.e. a pencil is the set of all lines through a point. Let P be off k, then the lines of the pencil through P intersect in the points of the queue on k. This illustrates the function called a *perspectivity* from the pencil to the queue. It also illustrates another perspectivity, a function from the queue to the pencil. If P is off lines k,m≠k, by composition of such perspectivities we introduce a perspectivity from the queue of k to the queue of m via P and the inverse function as well. We also have a dual perspectivity from a pencil of P to a pencil of Q≠P via k where P and Q are off k. These are the four functions. On page 22 [3], it is shown how to perform some permutations on four points via three perspectivities.

We replace the ternary order relation of ordered planes by a *separation* relation on four points for some projective planes. (Unfortunately, there are topological projective planes [19] and "separation" has two meanings. In earlier editions of this work, we had axioms of separation; in this edition these are axioms of order which we now give [pages 25, 26].) The notation AB//CD is read: A and B separate C and D.

1. *There is a line containing four distinct points.*
2. *If AB//CD, then A, B, C, D are four distinct collinear points.*
3. $AB//CD \Rightarrow AB//DC$
4. *If A, B, C, D are distinct collinear points, then at least one of the three relations BC//AD, CA//BD, AB//CD must hold.*
5. $AB//CD$ & $AC//BE \Rightarrow AB//DE$

The remaining axiom states that separation is invariant under perspectivities. This allows one to prove the corollary that the duals of these six axioms are theorems for ab//cd. Thus the principle of duality is maintained.

Using this invariant, the three perspectivities to do certain permutations noted above, and axiom 3 yield the following are all equivalent: AB//CD, AB//DC, BA//CD, BA//DC, CD//AB, CD//BA, DC//AB, DC//BA. Using axioms 2 and 5, AB//CD $\Rightarrow$ not AC//BD. Further work shows that no additional permutation of A, B, C, D produces a separation equivalent to the 8 shown.

In the asymmetric disjoint configuration, one can estabish a perspectivity using I yielding A$\rightarrow$E, B$\rightarrow$G, C$\rightarrow$F, D$\rightarrow$H, and another using J yielding A$\rightarrow$F, B$\rightarrow$E, C$\rightarrow$H, D$\rightarrow$G. Then trying AB//CD, AC//BD, AD//BC, only the last applied to both perspectivities noted for the asymmetric disjoint configuration can occur. In particular the first perspectivity yields EH//GF while the second yields FG//EH.

When placed in the ordered affine plane, and letting ωWXYZ mean ωWXY, ωWXZ, ωWYZ and ωXYZ as well as ωZYXW, then AD//BC becomes one of ωABDC, ωBDCA, ωDCAB, or ωCABD. Similarly, FG//EH is changed to one of such four point orders. The collineation group is also a dihedral group with generators (AFCHDGBE)(IJ) and (AE)(BF)(CG)(DH). Two "starts" need to be considered: ωABDC, ωEFHG; ωABDC, ωEGHF; the collineations show this. A computer program using these starts and all 6561 possible orders for the points on the other 8 lines and getting contradictions using Hilbert's separation in various ways resulted in 8 candidates. These 8 were easily found wanting.

6. Partitioning by quadrangles. A well-known conjecture [4] p.145 states: *every finite Fano-free projective plane is Desarguesian.* The free projective plane generated by a complete quadrangle is Fano-free and non-Desarguesian but it is not finite. Gleason's theorem [9] states: *if every quadrangle of a finite projective plane completes to a Fano configuration then it is Desarguesian.* Every Desarguesian plane of even order has this property while every Desarguesian plane of odd order is Fano-free. A less ambitious conjecture to settle is this: *every finite Fano-free projective plane has odd order.* It could be true with the original one false. In [14] by introducing the Coxeter configuration we found that *every finite Fano-free Coxeter-free projective plane has odd order.* In this section we show that every finite Fano-free projective plane with the *little hexagonal configuration* (defined below) universal has odd order.

In a finite projective plane we pick a line k and two points P and Q off k. Let B=PQ$\cap$k, and A be another point on k. We partition the remaining points on k. Let C be any third point on k, and we construct a quadrangle in this way: R=CQ$\cap$AP, S=AQ$\cap$BR. Then D=PS$\cap$AB. This is the classical way to get harmonic set A, B, C, D in Fano-free projective planes. If the quadrangle P, Q, R, S generates a Fano configuration, then C=D, and we have a singleton point of our partition. Thus a quadrangle yields a harmonic set of three points, the diagonal points, when it generates a Fano configuration. Otherwise, the harmonic sets are obtained by intersecting one side of the diagonal triangle by all sides of the quadrangle (e.g. [3] pp.18-19). Thus, in section 3, we named these non-diagonal points as harmonic points.

If we apply this algorithm to D, i.e. start with D joins Q NOT P, and the resulting harmonic set is the same, then we have constructed a *little hexagonal configuration* (for the name, consult [4] p.163). Thus we created two quadrangles sharing one side and two diagonal points. In any event let $R_1=DQ\cap AP$, $S_1=AQ\cap BR_1$, $E=PS_1\cap AB$. Then C=E for the little hexagonal configuration. But if C≠E, then we can apply the algorithm again introducing R_2, S_2, F. If F=C then we have a triple for our partition, etc. **When the little hexagonal configuration is universal, the partition is composed of pairs of points. Hence the plane has odd order.** In the self-dual projective plane of order 9 generalized by Hughes, by selecting all starts on a computer we find all partitions except these six: 21^6, 31^5, 41^4, 341, 23^2, 4^2. The nearfield plane of order 9, as well as its dual, share partitions 2^4, 1^8, 51^3, 17, 421^2, and have just one not shared, namely 21^6.

The little hexagonal configuration is also called [17] the two-fold degenerate Pappus configuration. When it is universal every quadrangle generates a Pappian subplane coordinatized by a field having no proper subfields [4] p.163. It is one of ours: a=BPQ, b=$APRR_1$, c=DPS, d=CPS_1, p=ABCD, q=$AQSS_1$, r=BRS, s=CQR, $r_1=BR_1S_1$, $s_1=DQR_1$. Moreover, it resides in the real projective plane; e.g. P(0,0,1), Q:(0,1,0), R(1,0,0), S:(1,1,,1), A:(1,0,1), B:(0,1,1), C:(1,–1,0), D:(1,1,2), R_1:(1,0,2), S_1:(1,–1,1). Its collineation group, again dihedral, has generators $(CRSDR_1S_1)(BPQ)$ and $(CS)(DS_1)(BQ)$.

We have another configuration of ours from diminishing by Q; this we call *the diminished little hexagonal configuration*. Its collineation group is Klein's 4-group with generators $(CD)(RR_1)(SS_1)$ and $(BP)(CR_1)(DR)$.

Likewise we can get *the double diminished little hexagonal configuration*. Its collineation group is the same Klein four group.

Furthermore, we can do both yielding *the combination diminished little hexagonal configuration* with collineation group the same dihedral group. For convenience of describing the collineation group, we included that here. Normally, when diminishing processes yields a configuration without a long line, we merely indicate which Martinetti configuration it is and postpone the collineation group for section 9 below. In this case we have MVIII. Here we begin the policy of usually not embedding diminished configurations.

7. Two or more long lines. We start with two disjoint long lines ABCD and EFGH. If we have a third long line AEIJ, then a third line with A cannot have two more points. Thus, we can have lines AEI, AFJ, and no fourth line through A. Also we can have BEJ, but we could have then BFI or BGI. In the first case, without loss of generality, we finish with CGI, CHJ, DGJ, DHI. In the second case, without loss of generality, CFI, CHJ, DGJ, DHI. Both are our configurations. The symmetric one: i=ABCD, j=EFGH, a=AEI, e=AFJ, f=BEJ, b=BFI, c=CGI, g=CHJ, h=DGJ, d=DHI. The asymmetric one: i=ABCD, j=EFGH, a=AEI, e=AFJ, f=BEJ, c=BGI, b=CFI, g=CHJ, h=DGJ, d=DHI.

The symmetric one occurs in classical geometry as two isosceles triangles ABI and CDI and A, B, C, D collinear, of course, and sharing centroid J. Then E, F, G, H are midpoints on the sides of the triangles. This configuration has a collineation group which is the semi-direct product ([11],p.88) of the quaternion group ([11],p.23), with generators (AEBF)(CGDH)(IJ) and (AGBH)(CFDE), by Klein's 4-group with generators (AB)(EF) and (EF)(GH)(IJ). (Actually J can be placed anywhere on the common altitude except at I and its foot; then E, F, G, H also move to a new parallel.)

The collineation group of the asymmetric one occurs in section 5. We settled the embedding question in sections 4 and 5.

The other extreme would be a strong point supporting long lines. In the previous section we find a point supporting three long lines. But we see there that all points are accounted for on those three lines, so no other line goes through the common point. We begin with ABCD, AEFG, AHIJ, and from duality we need exactly three strong points. Since the three long lines are concurrent, from duality, the three strong points are collinear. If B is strong, then C and D cannot be. Similarly choose E instead of F or G. Let H be the third one yielding line BEH. Without loss of generality, we also have BFI, and BGJ. Then we can continue with CEX, CFH, DEY, DGH with two cases: either both X=I and Y=J or both X=J and Y=I. The former is the general extended Fano configuration with Fano configuration ACB, AFE, AIH, CFH, CIE, FIB, BEH. The latter is the little hexagonal configuration. The isomorphism from its earlier display to this one is: each of A, B. C, D goes to itself and P→E, Q→H, R→F, S→I, R_1 →G, S_1 →J.

For 4 or more long lines we need no three concurrent and no two disjoint. Let us build four; if more, such can be added. For example, AKLM, BIJM, CHJL, DHIK. We have ten points. A fifth long line could have either A or K on it. But K can only join B, C, J, which yields line BCK. Using this pattern, we also find BDL, CDM, ACI, ADJ, ABH, which yields ten lines. Thus no configuration of ours has five long lines, and only one has four long lines. In section 3 we find this as one of ours, its collineation group, and its embedding in Pappian projective planes, where the field has characteristic 3. In section 5, we see it cannot be placed in an ordered plane.

For 3 long lines not concurrent, and no two disjoint, we have ABDE, ACFG, BCHI, with J special. Without loss of generality we can continue with AHJ, BFJ, CDJ. Since A has joined 8 others, A cannot be a strong point. Likewise for B and C. From C already located, E cannot join it; hence E does not have enough to join to be a strong point. Similar arguments hold for G and I. If J is not a strong point, then we have the littlest Desargues configuration with center J and axis EGI. Since the six lines already noted have S_3 as a collineation group, we add EGJ, and join E and I. If we assume G does not join I, then we cannot complete; otherwise, by joining G and I, we can complete uniquely and have the special extended Fano configuration with Fano configuration ADB, AFC, AJH, DFH, DJC, FJB, BCH. Furthermore, if one of our configurations includes a Fano configuration, then three strong points are forced. Hence we have found the two such configurations.

We return to considering configurations with two long lines. We had two disjoint ones earlier. So we start with ABCD, AEFG, AHI. Now we select various pairs of strong points which are joined. We choose B and E for the two strong line case. Then we have BEH, case 1, or BEJ, case 2. In case 1 we can continue with BFI and BGJ, then specify C, so to speak, rather than D to join G, forcing CGH, CEJ, DFJ, DEI. It's ours: b=ABCD, e=AEFG, h=AHI, a=BEH, c=BFI, d=BGJ, i=CGH, f=CEJ, g=DEI, j=DFJ. Its collineation group has only one non-identity element, namely, (BE)(CG)(DF). This is the one called *the two strong line configuration*. To go from the earlier version to this one, fix A and E, interchange B and F, $C \to D \to I \to C$, $G \to J \to H \to G$. Hence, (section 4) it is in every non-Desarguesian projective plane of order 9 and cannot be placed in E_2.

In case 2 for two strong lines, we continue with BFH, BGI, CEH, DEI, yielding two subcases: 2.1 CFJ, DGJ; 2.2 CGJ, DFJ. Then 2.1 is the littler Desargues configuration with center H and axis DGJ which can be placed in E_2 "as is". But 2.2, the diminished little hexagonal configuration, when placed in E_2, becomes its parent. We are not into denial, so we accept that the configuration may not always be found in E_2. We find the "x's" of the Pappus configuration, namely, BHF,CHE; BIG,DIE; CJG,DJF; from collinear triple BCD to collinear triple EFG, with centers H, I, J. But unlike the Pappus configuration, they are not collinear when we deny J to be on AHI. Then it cannot be embedded in E_2 which has this Pappus configuration. On the other hand, sometimes a configuration not embeddable in E_2 has a diminished configuration which is embeddable, and, when we can show that, we report that in one way or another.

b=ABCD	b=ABCD	b=ABCD	b=ABCD	h=ABCD	i=ABCD	h=ABCD
h=AEFG	h=AEFG	h=AEFG	h=AEFG	j=AEFG	h=AEFG	i=AEFG
e=AHI	e=AHI	e=AHI	e=AHI	b=AHI	a=AHI	a=AHI
a=BEH	a=BEH	a=BEH	a=BEH	a=BHJ	f=BHJ	d=BHJ
d=BFI	c=BGI	d=BFI	d=BGI	f=EIJ	d=EIJ	g=EIJ
c=BGJ	d=BFJ	c=BGJ	c=BFJ	i=BFI	b=BFI	f=BFI
g=CHJ	g=CHJ	g=CFH	f=CFH	d=CEH	c=CGI	e=CGI
j=CGI	i=CEI	j=CIJ	j=CIJ	g=CGJ	g=CEH	c=CEH
f=DFH	f=DFH	f=DGH	g=DGH	e=DFJ	j=DFJ	b=DFH
i=DEJ	j=DGJ	i=DEJ	i=DEJ	c=DGH	e=DGH	j=DGJ

We claim that we display the remaining two long line configurations. All but the last one have a collineation group with only the identity element. We first consider the mixed ones; the first long line is strong and the second is weak. We still begin ABCD, AEFG, AHI, pick B to be strong, and then the other lines start with B, C, or D; thus J is not strong, and we pick H to be strong. Then we "define" E, so to speak, by using line BEH. Then the first four columns and first four rows make sense including the line names b, h, e, a. At this point case 1, called the extra strong from extra point J joins H; case 2, called the extra weak from J does not join H. In case 1 we have CHJ and we can have DFH, leaving case 1.1 with E joining D and case 1.2 with E joining C. Both then complete uniquely. Case 1.1 has an embedding in E_2: A:(0,0), B:(12,0), C:$(-12\sqrt{5}/5,0)$, D:(4,0), E:(0,4), F:(0,12), G:$(0,-24+12\sqrt{5})$, H:(3,3), I:(6,6), J:$(6-2\sqrt{5},-2+2\sqrt{5})$. Case 1.2 has this embedding in E_2: A:(0,0), B:(6,0), C:(3,0), D:$(6+3\sqrt{2},0)$, E:(0,6), F:$(0,3\sqrt{2})$, G:(0,3), H:(3,3), I:(2,2), J:$(3,3\sqrt{2}/2)$.

Case 2 continues with CFH, DGH, then either the pair CEJ, DIJ or the pair CIJ, DEJ. The first pair does complete to two "new" configurations, but they are isomorphic to the displayed ones via C↔D, F↔G. Then case 2.1 has BFI and BGJ with embedding A:(0,0), B:(12,0), C:(4,0), D:(12/7,0), E:(0,4), F:(0,12), G:(0,–4), H:(3,3), I:(6,6), J:(3,–3). Hence case 2.2 has BFJ and BGI, but one can show by algebra that there is no embedding in E_2. In the future we simply say "no embedding in E_2" whereas in this case we illustrate the process (also shown in [15] for the anti-Desargues configuration). We can assign any three non-collinear points coordinates (0,0), (0,1), (1,0); we could risk problems of parallels and assign a fourth point of this quadrangle. Instead we assign (α,β), and (γ,δ) to two more points and do the related algebra.

Let A:(0,0), B:(0,1), E:(1,0) so ABCD:x=0, AEFG:y=0, BEH:x+y=1. Now we can let I:(α,β) so AHI:$y=\beta x/\alpha$ and H:$(\alpha/(\alpha+\beta),\beta/(\alpha+\beta))$ as well as BGI:$y=(\beta-1)x/\alpha+1$ yielding G:$(\alpha/(1-\beta),0)$ which then leads to DGH:$y=\beta(\alpha+(\beta-1)x)/(\alpha+2\beta-1)$ yielding D:$(0,\beta/(\alpha+2\beta-1))$.

Now DEJ:$y=(\beta x-\beta)/(1-\alpha-2\beta)$ and $\delta=\beta(\gamma-1)/(1-\alpha-2\beta)$

since J:(γ, δ). Next BFJ: y=$(\delta - 1)x/\gamma + 1$ yielding F:$(\gamma/(1-\delta)$,0). Finally CFH:y=$\beta(\gamma+(\delta-1)/(\alpha\gamma+\alpha\delta+\beta\gamma-\alpha)$ yielding C:(0,$\beta\gamma/(\alpha\gamma+\alpha\delta+\beta\gamma-\alpha)$. Thus for C, I, J to be collinear, the determinant of the following matrix must be zero.

$$\begin{pmatrix} 0 & \beta\gamma/(\alpha\gamma + \alpha\delta + \beta\gamma - \alpha) & 1 \\ \alpha & \beta & 1 \\ \gamma & \delta & 1 \end{pmatrix}$$

At this point we use Mathematica to substitute $\beta(\gamma - 1)/(1 - \alpha - 2\beta)$ for δ in the elements of the matrix. Then the determinant is computed yielding factors β and $\alpha + \beta + 1$ and a complicated one in the numerator. In some other examples, such as the one shown in [15], each factor directly yields a contradiction when set to 0.

If β=0, then lines AHI and ABCD coincide. If $\alpha + \beta = 1$, then points H and I coincide. The remaining factor over the denominator is set to zero and Mathematica gives the following solution for γ:

$$(\alpha^2 + (\alpha^2 - \alpha + 2\alpha\beta)i)/(1 - 2\alpha + 2\alpha^2 - 4\beta + 4\alpha\beta + 4\beta^2)$$

and its conjugate. The denominator can be written as $\alpha^2 + (\alpha + \eta)^2$ where $\eta = 2\beta - 1$. For γ to be real, α or $\alpha + 2/\beta - 1$ must be zero. If $\alpha = 0$, then lines AHI and AEFG coincide. If $\alpha + \beta = 1 - \beta$, then G and H have the same first coordinate making DGH parallel to ABCD; hence, lines DGH and ABCD coincide.

For two weak lines we let H be a strong point, and either I or J will be the other. We consider J strong first. Then we can have B on HJ. We can place e, f, g arbitrarily on lines with J. Thus h=ABCD, j=AEFG, b=AHI, a=BHJ, i=BFI (from f=IJ below), g=CvJ, CwH, e=DxJ, DyH, f=zIJ. Hence, except for c an d, we have the same labels as in the display above. We first try c and d opposite to the display getting c=CwH, d=DyH. Then D=deh, y=E. Also C=cgh, w=G. Then v is E or F, x is F or G, z is E or G. The choice of v determines the others, and in both cases the duality test fails. So we go back and have d=CwH, c=DyH. Then D=ceh, w=E. Also C=dgh, y=G. Then v=F or G, x=E or F, z=E or G. The choice for v determines the other choices, so we first choose v=F, contrary to the display above. The "new" configuration is isomorphic to the displayed one via $C \leftrightarrow D$, $E \leftrightarrow G$. We name this the *strong extra weak lines configuration.*

In this case we will start with a quadrangle because we can get away with it. Let A:(1,0), E:(2,3), H:(0,0), J:(0,1). We find five of the possible six sides of our quadrangle, namely, AEFG:y=3x–3, AHI:y=0, CEH:y=3x/2, EIJ:y=x+1, BHJ:x=0. The only diagonal point in this configuration is I:(–1,0). We let B:(0,β) yielding ABCD:y=$-\beta x + \beta$ and BFI:y=$\beta x + \beta$. Now C:$(2\beta/(3 + 2\beta), 3\beta/(3 + 2\beta))$ yielding CGJ:y=$(\beta - 3)x/2\beta$+1. Hence

G:$(8\beta/(3+5\beta), 9(\beta-1)/(3+5\beta))$. Also F:$((\beta+3)/(3-\beta), 6\beta/(3-\beta))$ yielding DFJ:$y=(7\beta-3)x/(\beta+3)+1$, D:$(\eta/(\beta^2+10\beta-3), 8\beta^2/(\beta^2+10\beta-3))$ where $\eta=(\beta+3)(\beta-1)=\beta^2+2\beta-3$. Furthermore GH:$y=9(\beta-1)x/8\beta$. Finally, plugging in the coordinates for D into the equation for GH yields $55\beta^3-9\beta^2+45\beta-27=0$. We then solve for β with Mathematica. The only real root (approximately .49838) is exactly:

$$3/55+\sqrt[3]{(37152/166375+96\sqrt{69}/3025}-816/(55\sqrt[3]{37152+5280\sqrt{69}}$$

Clearly h=ABCD, i=AEFG, a=AHI in these, the last cases. Then we can have BH, BI, CH, CI, DH but not DI since the last line would only involve E, F, G, J. We next place J with BHJ, DJ, IJ. We can place E with EIJ, CHE. We can place F on BI yielding G on CI. So far all of this agrees with both last columns of our display. The two "holes" in DH and DJ will be filled with F and G in two ways. We chose to display the configuration with the identity only collineation first. Thus we have *two weak lines 1 configuration* and *two weak lines 2 configuration* with collineation (BE)(CF)(DG)(HI) and only the identity. The former has no embedding in E_2. The latter is embedded: A:(0,0), B:(6,0), C:(9-3$\sqrt{5}$,0), D:(3,0), E:(0,9+3$\sqrt{5}$), F:(0,6), G:(0,-3-3$\sqrt{5}$), H:(2,2), I:(3,3), and J:((24+6$\sqrt{5}$)/11,(21-3$\sqrt{5}$/11).

8. One long line. These were found by computer; we still organize them as if they had been done by hand. Often we do not verify these results by arguments as given earlier. We left the embedding question open for the weak line configurations; one of the new strong line configurations lies in E_2. These new strong line ones will be named by XMn (extended Mn) where upon diminishing we have Martinetti configuration Mn. The weak line ones not already found shall be named LLm. We begin the strong line ones with ABCD, AEH, AFI, AGJ, and two more B-lines, C-lines, and D-lines. We next see how E, F, G either are joined or not. When G is not joined to either E or F, then it is joined to H and I. Isomorphism interchanging E and H as well as F and I yields G is connected to both E and F. Then interchanging AEH with AGJ yields E is connected to F and G. In particular, we can have BEF, CEG, DFG as a start. Then H, I, J are joined; one example is BHI, CHJ, DIJ, which is the little Desargues configuration with axis ABCD and quadrangles AEFG, and AHIJ. A minor change is this: CIJ and DHJ, the double diminished little hexagon configuration; we map the earlier version to this one via $A \leftrightarrow B$, $B \leftrightarrow J$, $C \leftrightarrow F$, $D \leftrightarrow E$, $P \leftrightarrow G$, $Q \leftrightarrow A$, $R \leftrightarrow I$, $R_1 \leftrightarrow H$, $S \leftrightarrow C$, $S_1 \leftrightarrow D$. Finally we can instead change to BHJ, CIJ, DHI which we call XMIX. Note: a=ABCD, b=AEH, c=AFI, d=AGJ, e=BEF, h=BHJ, f=CEG, i=CIJ, g=DFG, j=DHI. Its collineation group is S_3 with generators (BC)(EJ)(FI)(GH) and (CD)(EH)(FJ)(GI). It is not embedded in E_2.

We now break the FG connection of our first one and one result is CFJ, DGH, and DIJ, the k-configuration with collineation group Klein 4 having generators (CD)(EH)(FI) and (EF)(GJ)(HI). We map the earlier version to this one via $A \leftrightarrow D$, $B \leftrightarrow C$, $C \leftrightarrow G$, $D \leftrightarrow J$, $E \leftrightarrow A$, $G \leftrightarrow E$, $H \leftrightarrow B$, $I \leftrightarrow H$, $J \leftrightarrow I$, with F "fixed". A minor change is this: DGI and DHJ which we reject since it is not self-dual. Finally we can instead change to BIJ, CEG, CFH, DGI, DHJ which we call XMI. Note: a=ABCD, c=AEH, b=AFI, d=AGJ, f=BEF, i=BIJ, h=CEG, e=CFH, j=DGI, g=DHJ. Its collineation group has only one non-identity element, namely, (CD)(EJ)(FI)(GH). It is the one of these three which is embedded: A:(1,0), B:(0,1), C:(39/88,49/88), D:(6/55,49/55), E:(0,0), F:(0,3/10), G:(195/97,245/97), H:(−117/226,0), I:(−3/7,3/7), J:(3,5).

Another way to break the FG connection of our first one yields CHJ, DFJ, DGI, called XMVIII. Note: a=ABCD, b=AEH, c=AFI, d=AGJ, e=BEF, h=BHI, f=CEG, i=CHJ, g=DFJ, j=DGI. Its collineation group is dihedral with generators (EGIHJF)(BCD) and (CD)(EF)(GJ)(HI). A minor change is this: CIJ, DFJ, DGH which we reject since it is not self-dual. All others which can be found are isomorphic to these.

We begin the one weak line configurations by noting two possibilities. Either the strong point joins all the points of the long line, or else we have for a start: ABCD, AEF, AGH, BI, CI, DI, I, BE, CF, DJ. This second group has case 1: BEJ, CFG, DHJ; case 2: BEJ, CFH, DGJ; case 3: BEG, BFJ, DHJ; case 4: BEH, BFJ, DGJ. Every configuration of this type is a subcase of case 1; i.e. we do not have to consider the other three cases. First we have LL1: i=ABCD, j=AEF, g=AGH, d=BIF, c=CIH, b=DIE, a=GIJ, e=BEJ, h=CFG, f=DHJ with only one collineation. Next we have i=ABCD, e=AEF, h=AGH, b=BIF, c=CIJ, d=DIG, a=EIH, f=BEJ, j=CFG, g=DHJ, which is the double diminished special extended Fano configuration whose collineation group was given earlier. The isomorphism from the past to the present is: $A \to J$, $B \to G$, $C \to F$, $D \to I$, $E \to B$, $F \to D$, $G \to C$, $X \to E$, $Y \to H$, $Z \to A$.

We continue with LL2: i=ABCD, f=AEF, j=AGH, b=BIG, d=CIH, c=DIE, a=FIJ, g=BEJ, e=CFG, h=DHJ with only one collineation. Next we have LL3: i=ABCD, h=AEF, f=AGH, c=BIG, b=CIJ, d=DIE, a=FHI, j=BEJ, g=CFG, e=DHJ with only one collineation. Now LL2 and LL3 are the two configurations with the same invariant. Suppose they were isomorphic. Then I, A, J are "fixed" in that there is only one strong point, one long line, A does not join I, and J is the only other one that does not join A. Then FIJ would go to CIJ, but F is not on the long line while C is on it. Hence there is no such isomorphism.

Finally we finish this type with LL4: i=ABCD, e=AEF, h=AGH, d=BIG, c=CIJ, b=DIF, a=EIH, f=BEJ, j=CFG, g=DHJ with the only non-identity collineation (BD)(EH)(FG).

e=ABCD	e=ABCD	e=ABCD	e=ABCD	e=ABCD	e=ABCD	e=ABCD
b=AEF	c=AEF	c=AEF	a=AEF	a=AEF	d=AEF	c=AEF
g=AGH	h=AGH	h=AGH	f=AGH	f=AGH	i=AGH	h=AGH
a=BEG	d=BEG	b=BEG	b=BEG	b=BEG	b=BEG	d=BEG
d=CEH	a=CEH	a=CEH	c=CEH	d=CEH	c=CEH	a=CEH
c=DEI	b=DEI	d=DEI	d=DEI	c=DEI	a=DEI	b=DEI
f=BFJ	i=BFJ	g=BHJ	g=BFI	g=BFI	g=BFI	i=BFI
i=CIJ	f=CIJ	f=CIJ	h=CFJ	i=CGJ	h=CIJ	f=CIJ
h=DGJ	g=DHJ	i=DFJ	i=DGJ	h=DFJ	f=DGJ	g=DHJ
j=FHI	j=FGI	j=FGI	j=HIJ	j=HIJ	j=FHJ	j=FGJ

The last seven long line configurations are displayed above. We have a weak line and the strong point joins all the points of the weak line. Thus we start with ABCD, AEF, AGH, BEG, and either CEH, DEI or CEI, DEJ. The latter yields configurations isomorphic to the former. Then J joins B, C, D or joins two of them.

We first consider all three are so joined. Now both F and I join J and each needs another line; so they join each other on the tenth line. So we have either BFJ or BHJ without loss of generality. In either case, we have CIJ. Thus we have LL5, LL6, LL7, the first, second, third columns, respectively, each having only the identity collineation.

Let J not join B and again F and I join J and each other yielding BFI. Then LL8, LL9, LL10, LL11 are in the fourth, fifth, sixth, seventh columns, respectively. In the first two H joins I and in the latter two H does not join I with CIJ in common. The collineation (AB)(CD)(FG)(HI) is a collineation of LL8, LL9, and LL11. The only other collineation for any of these four is the identity.

9. The Martinetti configurations. We obtained them from [7].

h=ABC	d=ABC	e=ABC	f=ABC	g=ABC	a=ABC	h=ABC
i=ADE	b=ADE	a=ADE	g=ADE	f=ADE	c=ADE	i=ADE
j=AFG	f=AFG	d=AFG	a=AFG	a=AFG	b=AFG	j=AFG
a=HIJ	e=HIB	i=HIB	h=HIJ	h=HIB	g=HIB	a=HIJ
b=HBD	c=HJD	g=HJD	i=HBD	i=HJC	e=HJC	b=HBD
c=HCF	g=IJF	h=IJG	j=HCE	j=IJD	j=IJE	c=HCF
d=IBE	j=HCG	j=HCG	b=IBF	b=HEG	h=HEG	d=IBG
e=IDG	i=IEG	b=IEF	d=IDG	c=ICG	d=ICF	e=IEF
f=JCG	h=JCE	c=JCE	e=JCG	d=JEF	i=JDG	f=JCE
g=JEF	a=BDF	f=BDF	c=JEF	e=BDF	f=BDF	g=JDG

Here MII, MVIII, MX are not shown in the display below. MII, the anti-Desargues configuration, is displayed in section 4; in terms of that display, its collineation group is simply S_4 acting on A, B, C, D which determines the mapping of the lines and the other six poiints. Moreover, it is not embeddable in E_2. MVIII, the combination diminished little hexagonal configuration is in section 6 along with its collineation group. MX, the standard Deargues configuration, is discussed in section 2.

MI displayed in the first column has only non-identity collineation (BC)(DF) (EG)(IJ); recall that XMI is embedded in E_2. MIII displayed in the second column has collineations: identity, (BDF)(CEG)(HJI), and its inverse, and is embedded thus: A:(0,0), B:(–4,0), C:(1,0), D:(0,–6), E:(0,1), F:(–2,–3), G:(2,3), H:(8,21), I:(–8,–7), J:(8/5,–3/5). MIV displayed in the third column has cyclic collineation group of order 4 with generator (AHFJ)(BIEC)(DG) and is embedded thus: A:(0,1), B:(0,2), C:(0,–2), D:(2,0), E:$(8 - 2\sqrt{13}, -3 + \sqrt{13})$, F:(2/3,4/3), G:(2,2), H:(1,0), I:$((12 + 2\sqrt{13})/23, (22 - 4\sqrt{13})/23)$, J:$(-3 + \sqrt{13}, 0)$.

MV displayed in the fourth column has Klein 4 for its collineation group with generators (BD)(CE)(FG); (BE)(CD)(IJ) and is embedded as follows: A:(0,0), B:(1,0), C(–1, 0), D:(0,–3), E:(0,1), F:(–3,–5), G:(333/22,555/22); H:(2,3), I:(37/13,30/13), J:(40/31,111/31). MVI displayed in the fifth column has collineations: identity, (ADE)(BJG)(CIH), and its inverse, and by letting δ=481, we have the embedding: A:$((33 + \sqrt{\delta})/76, (-23 - 3\sqrt{\delta})/76)$, B:$((71 - \sqrt{\delta})/76, (81 + \sqrt{\delta})/76)$, C:(2/3,2/3), D:$((-3 + \sqrt{\delta})/118, 0)$, and E:(0,1/3), F:(1/4,1/4), G:(0,1), H:(0,2), I:(2,0), J:(1,0).

MVII displayed in the sixth column has a cyclic collineation group with generator (ACJDBHGFIE) which is embedded thus: A:(0,1), B:(0,2), C:(0,–4), D:(1,0), and E:(7/4,–3/4), F:(2/3,2/3), G:(2,0), H:(4,6), I:(1,3), J:(8/5,0). MIX displayed in the last column has S_3 as collineation group with generators (BC)(DF)(EG)(IJ); (BE)(CD)(FG)(HJ) and is embedded thus: A:(104/69,157/69), B:(0,1), C:(–13/19,8/19), D:(2,3), E:(1/9,2/9), F:(2,4), G:(0,–3), H:(–1,0), I:(0,0), J:(1,0).

10. The invariants. We give two dictionaries. The first places the invariants in lexicographical order followed by the name of the associated configuration (or in one case, two configurations); thus the displayed invariants are all different as promised earlier. The second dictionary goes by the number of long lines, the configurations therein, and the invariants for them.

(0, 0, 3, 36, 29, 1, 12, 39, 16)	four long lines
(0, 2, 27, 43, 14, 18, 40, 21, 7)	special extended Fano
(0, 6, 27, 36, 17, 18, 45, 9, 14)	general extended Fano
(0, 6, 27, 36, 17, 25, 24, 30, 7)	littlest Desargues
(0, 8, 44, 52, 0, 36, 68, 0, 0)	asymmetric disjoint
(0, 9, 50, 41, 4, 43, 51, 9, 1)	two weak lines 1
(0, 11, 45, 45, 3, 41, 54, 9, 0)	strong extra weak line
(0, 14, 9, 48, 15, 20, 33, 27, 6)	little hexagonal

(0, 14, 47, 37, 6, 47, 41, 15, 1)	extra strong mixed 1
(0, 18, 47, 30, 9, 53, 28, 21, 2)	littler Desargues
(0, 19, 44, 33, 8, 50, 34, 18, 2)	two strong lines
(0, 33, 60, 28, 1, 76, 43, 3, 0)	double dim. special ext. Fano
(0, 36, 63, 18, 5, 86, 21, 15, 0)	little Desargues
(0, 38, 57, 24, 3, 80, 33, 9, 0)	k-configuration
(0, 60, 75, 0, 5, 125, 0, 15, 0)	(standard) Desargues, MX
(0, 64, 63, 12, 1, 113, 24, 3, 0)	anti-Desargues, MII
(1, 9, 47, 43, 4, 44, 48, 12, 0)	two weak lines 2
(1, 9, 55, 32, 7, 49, 38, 15, 2)	extra weak mixed 1
(1, 11, 50, 36, 6, 47, 41, 15, 1)	extra strong mixed 2
(1, 13, 46, 38, 6, 47, 41, 15, 1)	extra weak mixed 2
(1, 27, 71, 20, 3, 82, 31, 9, 0)	LL11
(1, 29, 65, 26, 1, 76, 43, 3, 0)	LL7
(1, 30, 63, 27, 1, 76, 43, 3, 0)	LL1
(2, 0, 56, 44, 2, 42, 56, 6, 0)	symmetric disjoint
(2, 20, 35, 40, 7, 49, 34, 21, 0)	dim. little hexagonal
(2, 25, 71, 22, 2, 79, 37, 6, 0)	LL9
(2, 28, 65, 25, 2, 79, 37, 6, 0)	LL10
(2, 29, 62, 28, 1, 76, 43, 3, 0)	LL5
(3, 25, 68, 24, 2, 79, 37, 6, 0)	LL6
(3, 27, 63, 28, 1, 76, 43, 3, 0)	LL2 as well as LL3
(4, 25, 65, 26, 2, 79, 37, 6, 0)	LL8
(4, 27, 60, 30, 1, 76, 43, 3, 0)	LL4
(4, 30, 60, 26, 2, 77, 39, 6, 0)	XMI
(4, 32, 57, 26, 3, 80, 33, 9, 0)	double dim. little hexagonal
(5, 50, 75, 10, 0, 110, 30, 0, 0)	MVII
(5, 53, 69, 13, 0, 110, 30, 0, 0)	MI
(6, 26, 63, 24, 3, 80, 33, 9, 0)	XMVIII
(6, 30, 54, 30, 2, 77, 39, 6, 0)	XMIX
(6, 50, 72, 12, 0, 110, 30, 0, 0)	MVI
(6, 50, 75, 6, 3, 119, 12, 9, 0)	comb. little hexagonal, MVIII
(6, 52, 69, 12, 1, 113, 24, 3, 0)	MV
(7, 50, 69, 14, 0, 110, 30, 0, 0)	MIV
(9, 45, 75, 9, 2, 116, 18, 6, 0)	MIX
(9, 46, 72, 12, 1, 113, 24, 3, 0)	MIII

The (only) four long lines type has invariant (0,0,3,30,29,1,12,39,16).

The three long lines type:
the littlest Desargues has invariant (0,6,27,36,17,25,24,30,7)
the little hexagonal has invariant (0,14,9,48,15,20,33,27,6)
the general extended Fano has invariant (0,6,27,36,17,18,45,9,14)
the special extended Fano has invariant (0,2,27,43,14,18,40,21,7)

The two long lines type:
the symmetric disjoint has invariant (2,0,56,44,2,42,56,6,0)
the asymmetric disjoint has invariant (0,8,44,52,0,36,68,0,0)
the littler Desargues has invariant (0,18,47,30,9,53,28,21,2)
the diminished little hexagonal has invariant (2,20,35,40,7,49,34,21,0)
the two strong lines has invariant (0,19,44,33,8,50,34,18,2)
the extra strong mixed 1 has invariant (0,14,47,37,6,47,41,15,1)
the extra strong mixed 2 has invariant (1,11,50,36,6,47,41,15,1)
the extra weak mixed 1 has invariant (1,9,55,32,7,49,38,15,2)
the extra weak mixed 2 has invariant (1,13,46,38,6,47,41,15,1)
the strong extra weak lines has invariant (0,11,45,45,3,41,54,9,0)
the two weak lines 1 has invariant (0,9,50,41,4,43,51,9,1)
the two weak lines 2 has invariant (1,9,47,43,4,44,48,12,0)

The one long line type:
the little Desargues has invariant (0,36,63,18,5,86,21,15,0)
the double diminished hexagonal has invariant (4,32,57,26,3,80,33,9,0)
the k-configuration has invariant (0,38,57,24,3,80,33,9,0)
XMI has invariant (4,30,60,26,2,77,39,6,0)
XMVIII has invariant (6,26,63,24,3,80,33,9,0)
XMIX has invariant (6,30,54,30,2,77,39,6,0)
the double dim. special extended Fano has invariant (0,33,60,28,1,76,43,3,0)
LL1 has invariant (1,30,63,27,1,76,43,3,0)
LL2 as well as LL3 has invariant (3,27,63,28,1,76,43,3,0)
LL4 has invariant (4,27,60,30,1,76,43,3,0)
LL5 has invariant (2,29,62,28,1,76,43,3,0)
LL6 has invariant (3,25,68,24,2,79,37,6,0)
LL7 has invariant (1,29,65,26,1,76,43,3,0)
LL8 has invariant (4,25,65,26,2,79,37,6,0)
LL9 has invariant (2,25,71,22,2,79,37,6,0)
LL10 has invariant (2,28,65,25,2,79,37,6,0)
LL11 has invariant (1,27,71,20,3,82,31,9,0)

The Martinetti configurations with no long line:
MI has invariant (5,53,69,13,0,110,30,0,0,)
MII, the anti-Desargues, has invariant (0,64,63,12,1,113,24,3,0)
MIII has invariant (9,46,72,12,1,113,24,3,0)
MIV has invariant (7,50,69,14,0,110,30,0,0)
MV has invariant (6,52,69,12,1,113,24,3,0)
MVI has invariant (6,50,72,12,0,110,30,0,0)
MVII has invariant (5,50,75,10,0,110,30,0,0)
MVIII, the combination (diminished and double diminished) little hexagonal, has invariant (6,50,75,6,3,119,12,9,0)
MIX has invariant (9,45,75,9,2,116,18,6,0)
MX, the standard Desargues, has invariant (0,60,75,0,5,125,0,15,0)

BIBLIOGRAPHY

1. L. M. Blumenthal, **A Modern View of Geometry**, W. H. Freeman, San Francisco, 1961.

2. H. S. M. Coxeter, **introduction to GEOMETRY**, John Wiley, New York, 1961.

3. H. S. M. Coxeter, **THE REAL PROJECTIVE PLANE**, Third Edition, Springer-Verlag, New York, 1992.

4. P. Dembowski, **Finite Geometries** (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 44), Springer-Verlag, New York, 1968.

5. J. Denes and A. D. Keedwell, **Latin Squares and their Applications**, Academic Press, New York, 1974.

6. J. Denes and A. D. Keedwell, **Latin Squares, New Developments in the Theory and Applications** (Annals of discrete mathematics 46), North-Holland, New York, 1991.

7. J. W. Di Paola and H. Gropp, *Hyperbolic Graphs from Hyperbolic Planes*, **Congressus Numer.**, 68(1989), 23-44.

8. H. G. Forder, **The Foundations of Euclidean Geometry**, Dover, New York, 1958.

9. A. M. Gleason, *Finite Fano planes*, **Amer. J. Math.**, 78(1956)797-807.

10. M. Hall, *Projective planes*, **Trans. Amer. Math. Soc.**, 54(1943)229-277.

11. M. Hall, **The theory of groups**, Macmillan, New York, 1959.

12. R. B. Killgrove and D. I. Kiel, *Completion of quadrangles, revisited*, **Congressus Numer.**, 26(1980)187-198.

13. R. B. Killgrove and M. Cates, *Completion of quadrangles, revisited II*, **Congressus Numer.**, 36(1982)261-277.

14. R. B. Killgrove, D. I. Kiel, and R. W. Sternfeld, *Local Motions*, **Congressus Numer.**, 44(1984)185-196.

15. R. Killgrove, C. Roberts, R. Sternfeld, R. Tamez, R. Derby, and D. Kiel, *Latin squares and other configurations*, **Congressus Numer.**, 117(1996)161-174.

16. R. Killgrove, R. Sternfeld, and R. Tamez, *Quadrangle completions and the anti-Desargues configuration*, **Congressus Numer.**, 127(1997)57-66.

17. G. Pickert, **Projektive Ebenen**, Springer-Verlag, Berlin, 1955.

18. S. Priess-Crampe, **Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen**, Springer-Verlag, New York, 1983.

19. H. R. Salzmann, *Topological Planes* (Advances in Mathematics, vol. 2), Academic Press, 1967, pp. 1-60.

20. A. Wagner, *On projective planes transitive on quadrangles*, **J. London Math. Soc.**, 33(1958)25-33.