On Semi Quadrangles

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Abstract

We introduce **semi quadrangles**, which are finite partial linear spaces with a constant number of points on each line, having no ordinary triangles and containing, as minimal circuits, ordinary quadrangles and pentagons, with the additional property that every two non-collinear points are collinear with at least one other point of the geometry. A semi quadrangle is called *thick* if every point is incident with at least three lines and if every line is incident with at least three points. Thick semi quadrangles generalize (thick) *partial quadrangles* (see [4]). We will emphasize the special situation of the semi quadrangles which are subgeometries of finite generalized quadrangles. Some particular geometries arise in a natural way in the theory of *symmetries* of *finite generalized quadrangles* and in the theory of *translation generalized quadrangles*, as certain subgeometries of generalized quadrangles with concurrent *axes of symmetry*; these subgeometries have interesting automorphism groups, see [17] and also [19]. Semi quadrangles axiomatize these geometries. We will present several examples of semi quadrangles, most of them arising from generalized quadrangles or partial quadrangles. We will prove an inequality for semi quadrangles which generalizes the inequality of Cameron [4] for partial quadrangles, and the inequality of Higman [7, 8] for generalized quadrangles. The proof also gives information about the equality. Some other inequalities and divisibility conditions are computed. Also, we will characterize the linear representations of the semi quadrangles, and we will have a look at the point graphs of semi quadrangles.

1 Semi quadrangles

1.1 Semi quadrangles

An **incidence structure of rank 2** consists of a set of *points* and a set of *lines*, disjoint, and with a relation — called *incidence* — between the two sets. Sometimes a line is identified with the set of points incident with it, and we will do this without further notice. The **dual** of an incidence structure is obtained by interchanging the labels 'point' and 'line' (and by interchanging the 'corresponding' parameters). Let $\mathcal{S}$ be a point-line incidence structure. A **path of length d** is a $(d+1)$-tuple of points in which consecutive elements are distinct and collinear. Distances in an incidence structure are measured in the corresponding incidence graph (where adjacency is incidence). The **diameter of an incidence structure** is the diameter of its incidence graph, and an incidence structure is **connected** if the diameter is finite. A **semi quadrangle (SQ)** is a point-line incidence structure in which any line is incident with a constant number of points, of which no two distinct points are incident with more than one line, and which contains no ordinary triangles, but contains an ordinary subquadrangle and subpentagon. Also, every two non-collinear points are always collinear with at least one common point, any line is incident with at least two points and any point is incident with at least two lines. It is clear that from this definition it does not necessarily follow that every point is incident with the same number of lines (such as in the case of thick *generalized polygons*, see [15, 21]). In order to have some more information about these structures, we introduce the **μ-parameters** and the **order** of a semi quadrangle.

Suppose that s, t_i, μ_j, where $1 \leq i \leq n$ and $1 \leq j \leq m$ for nonzero natural numbers n and m, are natural numbers satisfying $s \geq 1$ and $t_i \geq 1$. Then a semi quadrangle of **order** $(s; t_1, \cdots, t_n)$ and with **μ-parameters** $(\mu_1, \cdots, \mu_m)$ is an incidence structure with the following properties.

(SQ1) The geometry is a *partial linear space*, i.e. any two distinct points are incident with at most one line and any two distinct lines are incident with at most one point. Any point is incident with $t_1 + 1$, $t_2 + 1, \cdots$, or $t_n + 1$ lines, and every line is incident with $s + 1$ points. Also, for any $i \in \{1, 2, \cdots, n\}$ there is a point incident with $t_i + 1$ lines.

(SQ2) If two points are not collinear, then there are exactly $\mu_1, \mu_2, \cdots$, or μ_m points collinear with both, and each of the cases occurs.

(SQ3) For any two non-collinear points there is at least one point which is collinear with both (i.e. for each $i = 1, 2, \cdots, m$ there holds that $\mu_i \geq 1$).

(SQ4) The geometry contains an ordinary pentagon and an ordinary quadrangle but no ordinary triangle as subgeometry, hence there is a j for which $\mu_j \geq 2$.

REMARK 1.1 We emphasize that (SQ2) and (SQ3) should be regarded as different axioms (instead of integrating (SQ3) in (SQ2) by demanding that for every $i = 1, 2, \cdots, m$ there holds that $\mu_i \geq 1$). For instance, suppose that $\mathcal{S}$ is a GQ of order (s, t) with $s, t > 2$, and suppose $\mathcal{L}$ is an arbitrary set of k lines with $0 < k < t$. Define a geometry by taking away the lines of $\mathcal{L}$ in the GQ, with the same points as $\mathcal{S}$ and with the natural incidence. Then this geometry satisfies (SQ1), (SQ2) and (SQ4), but not (SQ3).
Other motivatons for this distinction will be clear from the following Sections, see e.g. Section 2. Also, the reason for demanding that every line has to be incident with a constant number of points is motivated by THEOREM 1.6 and the Examples (a) through (f), see below.

If a point p and a line L are incident, we simply write pIL, and if they are **not** incident, we write $p\not{I}L$. In the following we agree that $t_1 \leq t_2 \leq \cdots \leq t_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$. If there are only two possible values for the number of lines through a point, then the SQ is called **near minimal.** Since the parameters t_1, t_n, μ_1, μ_m will play an important role in the following, we call $(s; t_1, t_n)$ the **extremal order** and (μ_1, μ_m) the **extremal μ-parameters**. For a near minimal semi quadrangle, the order and the extremal order coincide. A semi quadrangle is called **thick** if every point is incident with at least three lines and if every line is incident with at least three points. A thick semi quadrangle with $t_1 = \cdots = t_n = t$ and $\mu_1 = \cdots = \mu_m = \mu$ is a thick **partial quadrangle (PQ)** — as defined by Cameron in [4] — with parameters (s, t, μ) with $\mu \neq 1$ (this notation differs somewhat from that of Cameron, but in this context it is more convenient) and a thick partial quadrangle with $\mu = t + 1$ is precisely a thick **generalized quadrangle (GQ)** with parameters (s, t) in the sense of Payne and Thas [12]. Generalized quadrangles where introduced by J. Tits in his celebrated work on triality [20]. Thick generalized quadrangles always contain quadrangles and pentagons. In the case of generalized quadrangles, the condition that the GQ must contain a pentagon is equivalent with the thickness of the GQ, see [21]. This is not the case for semi quadrangles; there are geometries with only two points per line which satisfy all the SQ-conditions. For example, define the geometry $\mathcal{S} = (P, B, I)$ as follows. The pointset P consists of six distinct 'letters' a_i, $i \in \{1, \cdots, 6\}$, lines are the sets $\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_5, a_6\}, \{a_6, a_2\}, \{a_1, a_5\}, \{a_5, a_4\}$, and incidence is the natural one. Then $\mathcal{S}$ is a semi quadrangle. Also, every ***strongly regular graph*** with parameters (v, k, λ, μ) (see e.g. Chapter 22 of [3]) and with $\mu \geq 2$ and $\lambda = 0$ is a semi quadrangle of order $(1; k-1)$ and with μ-parameters (μ) (and hence a partial quadrangle). An example is the unique strongly regular graph with parameters $(16, 5, 0, 2)$, namely the ***Clebsch graph***, see Chapter 10 (p. 440) of [3].

REMARK 1.2 A thick semi quadrangle $\mathcal{S}$ is a thick generalized quadrangle if and only if the following property is satisfied:

(GQ3) Consider a point p and a line L, p not incident with L. Then there is exactly one line which intersects L and which is incident with p.

Proof. Immediate. □

Definition and notation

A **grid** (respectively **dual grid**) is an incidence structure $\mathcal{S} = (P, B, I)$ with $P = \{x_{ij} \mid\mid i = 0, 1, \cdots, s_1 \text{ and } j = 0, 1, \cdots, s_2\}$, $s_1, s_2 > 0$ (respectively $B = \{L_{ij} \mid\mid i = 0, 1, \cdots, t_1 \text{ and } j = 0, 1, \cdots, t_2\}$, $t_1, t_2 > 0$), with $B = \{L_0, \cdots, L_{s_1}, M_0, \cdots, M_{s_2}\}$ (respectively $P = \{x_0, \cdots, x_{t_1}, y_0, \cdots, y_{t_2}\}$), $x_{ij} I L_k$ iff $i = k$ (respectively $L_{ij} I x_k$ if and only if $i = k$), and $x_{ij} I M_k$ if and only if $j = k$ (respectively $L_{ij} I y_k$ if and only if $j = k$).
A grid (respectively dual grid) with parameters s_1, s_2 (respectively with parameters t_1, t_2) is a GQ in the sense of Payne and Thas [12] if and only if $s_1 = s_2$ (respectively $t_1 = t_2$). It is clear that the grids (respectively dual grids) with $s_1 = s_2$ (respectively $t_1 = t_2$) are the GQ's with $t = 1$ (respectively $s = 1$).

Notation

Suppose that A is a set of points, respectively lines, of an SQ $\mathcal{S}$. Then $\mathrm{A}^{\perp}$ is the set of points, respectively lines, of $\mathcal{S}$ which are collinear, respectively concurrent, with every point, respectively line, of A.

The following theorem shows that a semi quadrangle is full of pentagons.

THEOREM 1.3 *Any anti-flag (p, L) (a nonincident point-line pair) of a semi quadrangle $\mathcal{S}$ which does not satisfy Property (GQ3) is always contained in a pentagon.*

Proof. Suppose (p, L) is an anti-flag which does not satisfy Property (GQ3). Suppose u is a point of L and that $x \in \{u, p\}^{\perp}$. Since p is not collinear to any point on L, there is a point v on L that is not collinear with any point on px. Let $y \in \{v,p\}^{\perp}$. Then $y \notin xp, xu$ and hence $uvypx$ is a pentagon which contains L and p. □

THEOREM 1.4 *A geometry $\mathcal{G}$ which satisfies all the SQ-conditions except that there must be a pentagon, automatically contains pentagons if and only if it is not a grid or a dual grid.*

Proof. It is clear that grids and dual grids satisfy all the SQ-conditions except the conditions that there must be a pentagon. If the geometry $\mathcal{G}$ is not a grid or a dual grid and if it satisfies (GQ3), then by REMARK 1.2, the geometry is a thick generalized quadrangle and hence there are always pentagons. If the geometry $\mathcal{G}$ does not satisfy (GQ3), then it cannot be a grid or a dual grid and applying THEOREM 1.3, the proof is complete. □

1.2 A motivation for introducing semi quadrangles

THEOREM 1.5 *Suppose that $\mathcal{S}' = (P', B', I')$ is a subgeometry*[1] *of a GQ $\mathcal{S} = (P, B, I)$ of order (s,t), with the properties that there are $s'+1$ points on a line for some s', that there is a subpentagon in $\mathcal{S}'$ and that (SQ3) is satisfied. Then $s \neq 1 \neq t$, and two points of $\mathcal{S}'$ are collinear iff they are collinear in $\mathcal{S}$. If $s' = s$, then $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order (s,t') with $t' \neq 1$.*

Proof. That $s \neq 1 \neq t$ follows from the fact that $\mathcal{S}'$ has a pentagon. Suppose p and q are collinear points of $\mathcal{S}'$. Then p and q are also collinear points in $\mathcal{S}$ trivially. Next, suppose p and q are points of $\mathcal{S}'$ which are collinear in $\mathcal{S}$ but not in $\mathcal{S}'$. Then by (SQ3) there is a point x in $\mathcal{S}'$ which is collinear with both p and q. This implies that pxq is a triangle in $\mathcal{S}$ if $x \not{I} pq$ in $\mathcal{S}$, a contradiction; hence $xIpq$ in $\mathcal{S}$. There follows that pq is a line of $\mathcal{S}'$, a contradiction. If $s = s'$, then by THEOREM 2.3.1 of [12] there follows that $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order (s,t') with $t' \neq 1$. □

In [17] we met an incidence structure which arises naturally from a set of concurrent *axes of symmetry* (see e.g. Chapter 8 of [12]) in a GQ $\mathcal{S}$, and which was proved to be a generalized quadrangle if certain additional properties are satisfied. The following theorem was proved.

THEOREM 1.6 (K.Thas [17], see also [19]) *Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle with parameters (s,t), $s \neq 1 \neq t$, and that p is a point which is incident with at least three axes of symmetry. Also,*

[1] This means that $P' \subseteq P$, $B' \subseteq B$ and that I' is the induced incidence.

suppose that G is the group generated by the symmetries about every *axis of symmetry through p. Now define the incidence structure $\mathcal{S}' = (P', B', I')$ as follows. Suppose G_* is an arbitrary G-orbit of the permutation group $(P \setminus p^\perp, G)$. The elements of P' are of three types: (1) the point p; (2) the points of G_*; (3) any point which is incident with an axis of symmetry through p. We have two types of lines: (a) the axes of symmetry through p; (b) the lines of $\mathcal{S}$ which intersect a line of the first type and contain at least one point of G_*. The incidence relation $I' \subseteq I$ is the restriction of I to $(P' \times B') \cup (B' \times P')$.*
Then we have the following properties.

1. *There are constants l and k such that any point of the first two types is incident with $l+1$ lines of $\mathcal{S}'$, and every point of the last type is incident with $k+1$ lines;*

2. *A line of $\mathcal{S}'$ contains $s+1$ points of $\mathcal{S}'$;*

3. *$|G_*| = s^2k$;*

4. *k is divisible by s, and in particular we have that $s \leq k$. Also, $l \leq k$;*

5. *The number of points of $\mathcal{S}'$ is $ks^2 + (l+1)s + 1$, and the number of lines is $(l+1)(sk+1)$.*

In [17] we called these incidence structures (l, k)-**partial quadrangles**, and in some cases they were shown to be generalized quadrangles. It is easy to see that these geometries have (1) ordinary subquadrangles and subpentagons, and that they are (2) connected. Also, they satisfy (SQ1) with $n \leq 2$, and of course (3) there are no triangles. In the theory of **translation generalized quadrangles**[2] **(TGQ's)** (see [12]), it is very important to find (and study) proper subGQ's (if at all possible), and it was one of our goals in [17, 19] to state combinatorial conditions so that the geometries $\mathcal{S}'$ would be GQ's. If $\mathcal{S}'$ also satisfies (SQ2), that is, we assume that any two non-collinear points of $\mathcal{S}'$ have a common neighbour, then it is a semi quadrangle. Also, by THEOREM 1.5 in such a case it is a GQ.

REMARK 1.7 The geometries $\mathcal{S}'$ as above are not always subGQ's. If $\mathcal{S}$ is, for example, a TGQ with $s \neq t$, then this is only the case if and only if $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits [5, 12]. For more details, see K. Thas [19].

[2]These are the GQ's with a point through which each line is an axis of symmetry.

2 Examples of semi quadrangles

We only give examples of thick semi quadrangles which are not (always) partial quadrangles, and which are near minimal. All of them are in some way related to generalized quadrangles or partial quadrangles.
We first of all emphasize again that it should be noted that (SQ3) is a very important condition. This will be clearly reflected in the following examples.

(a) Suppose that $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s,t) with $s,t \geq 3$, and suppose p is a point of $\mathcal{S}$ with the property that for every two non-collinear points q,q' of $P \setminus p^{\perp}$ there holds that

$$\text{(M)} \quad |\{p,q,q'\}^{\perp}| < t+1.$$

By Theorem 1.7.1 of [12] there follows that for every pair of non-collinear points (x,y) in $P \setminus p^{\perp}$ we have

$$\text{(Q)} \quad |\{p,x,y\}^{\perp}| < t.$$

Now define the following incidence structure $\mathcal{S}_p = (P_p, B_p, I_p)$: (a) P_p is the set of points of $P \setminus p^{\perp}$, (b) B_p is the set of all lines of $\mathcal{S}$ not incident with p, and (c) I_p is the restriction of I to $(P_p \times B_p) \cup (B_p \times P_p)$. Then $\mathcal{S}_p$ is a thick semi quadrangle with s points on every line and $t+1$ lines through every point; the condition (M) implies that (SQ3) is satisfied, (M) implies the existence of quadrangles and THEOREM 1.4 yields the existence of a pentagon.

Suppose that $\mathcal{S} = (P, B, I)$ is a GQ of order (s,t) with $s,t > 2$. Then the pair (x,y), $x \not\sim y$, is called **antiregular** provided $|z^{\perp} \cap \{x,y\}^{\perp}| \leq 2$ for all $z \in P \setminus \{x,y\}$. A point x is **antiregular** provided (x,y) is antiregular for all $y \in P \setminus x^{\perp}$, see [12]. Hence, if $\mathcal{S} = (P,B,I)$ is a GQ of order (s,t) with $s,t > 2$ and p an antiregular point, then the geometry $\mathcal{S}_p$ always satisfies condition (M) and condition (Q), thus $\mathcal{S}_p$ is a semi quadrangle, of which the μ-parameters are contained in $\{t-1, t, t+1\}$.

Now specialize, and suppose that $\mathcal{S}^{(x)}$ is a translation generalized quadrangle of order (s,t) with $s,t > 2$ and with translation point x, see [12]. If $s = t$, we furthermore suppose that s is odd. Then by Chapter 8 of [12], the conditions (M) and (Q) are satisfied, and hence $\mathcal{S}^{(x)}$ yields a thick semi quadrangle with a constant number of lines through a point. The semi quadrangles which arise from translation generalized quadrangles in the way described above all have the property that there is an elementary abelian group which acts regularly on the points of the semi quadrangle. Also, s and t are the powers of the same prime p, and there is an odd natural number a and an integer n for which $t = p^{n(a+1)}$ and $s = p^{na}$ if $s \neq t$. If $s = t$ with s odd, then by Chapter 1 of [12] the μ-parameters are given by $(s-1, s+1)$; if $s \neq t$ and if p and a are as above, then the (possible) μ-parameters are $(p^{n(a+1)} - p^n, p^{n(a+1)})$ by Chapter 8 of [12], and $\mathcal{S}_x$ is a partial quadrangle if and only if $a = 1$, and then $\mu = p^{2n} - p^n$.

Let $\mathcal{S}$ be a GQ of order (s, s^2) with $s > 2$. Then by Bose and Shrikhande [2] there follows that every triad of points has $s+1$ centers (see Section 3 for a similar result on SQ's). Now take an arbitrary point p of $\mathcal{S}$, and consider the geometry $\mathcal{S}_p$. Then $\mathcal{S}_p$ is a partial quadrangle with parameters $(s-1, s^2, s^2 - s)$.

(b) Let $\mathcal{S}$ be a GQ of order (s,t) with $s, t > 2$, and suppose that $\mathcal{S}'$ is a subGQ of order $(s, t/s)$, with the property that for every two non-collinear points x and y of $\mathcal{S} \backslash \mathcal{S}'$ there holds that $|\{x,y\}^{\perp} \cap \mathcal{S}'| < t+1$. By Chapter 2 of [12], there holds that every line of $\mathcal{S}$ intersects $\mathcal{S}'$ in 1 or $s+1$ points. Next, define a geometry $\mathcal{S}_{\mathcal{S}'} = (P_{\mathcal{S}'}, B_{\mathcal{S}'}, I_{\mathcal{S}'})$ where $B_{\mathcal{S}'}$ is the set of lines of $\mathcal{S}$ which are not contained in $\mathcal{S}'$, $P_{\mathcal{S}'}$ is the set of points of $\mathcal{S} \setminus \mathcal{S}'$, and where $I_{\mathcal{S}'}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{S}'}$ is a thick semi quadrangle of order $(s-1; t)$.
We do not know of any examples of such semi quadrangles.

(c) Suppose $\mathcal{S}$ is a GQ of order (s,t) with $s, t > 2$, and suppose that $\mathcal{O}$ is an *ovoid* (a set of $st+1$ two by two non-collinear points) with the property that for every two non-collinear points x and y of $\mathcal{S} \backslash \mathcal{O}$ there holds that $|\{x,y\}^{\perp} \cap \mathcal{O}| < t+1$. Define a geometry $\mathcal{S}_{\mathcal{O}} = (P_{\mathcal{O}}, B_{\mathcal{O}}, I_{\mathcal{O}})$ where $B_{\mathcal{O}}$ is the lineset of $\mathcal{S}$, $P_{\mathcal{O}} = \mathcal{S} \backslash \mathcal{O}$, and where $I_{\mathcal{O}}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{O}}$ is a thick semi quadrangle of order $(s-1, t)$.

Suppose $\mathcal{O}$ is an ovoid of the classical GQ $W(q)$ [12] of order q, $q > 2$. Then every point of $\mathcal{S}$ is regular, see [12]. By Theorem 1.8.4 of [12], there follows that $\mathcal{S}_{\mathcal{O}}$ is a semi quadrangle of order $(q-1; q)$ and with μ-parameters $(q-1, q+1)$.
For more on ovoids of $W(q)$, see e.g. [16].

(d) Suppose $\Gamma = (P, B, I)$ is a partial quadrangle with parameters (s, t, μ), where $s, t \geq 3$, and let $\Gamma' = (P', B', I')$ be a partial subquadrangle of

Γ with parameters (s, t', μ'). Then a simple counting argument shows that every line of Γ intersects Γ' if and only if $|P'| \times (t - t') = |B| - |B'|$, that is, if and only if

$$(t-t')(s+1)(1+(t'+1)s(1+\frac{st'}{\mu'})) = (1+(t+1)s(1+\frac{st}{\mu}))(t+1)-$$

$$(1+(t'+1)s(1+\frac{st'}{\mu'}))(t'+1). \qquad (1)$$

Note that if we interchange the words 'PQ' and 'GQ', that this condition can be simplified to $t' = t/s$, see Example (b).
Assume condition (1) is satisfied. Furthermore, we suppose that $\mathcal{S}$ has the property that (1) for every two non-collinear points q, q' of $P \setminus p^\perp$ there holds that $|\{p, q, q'\}^\perp| < \mu$, and that (2) there is a pair of non-collinear points (x, y) in $P \setminus p^\perp$ for which $|\{p, x, y\}^\perp| < \mu - 1$. Define a geometry $\Gamma_{\Gamma'} = (P_{\Gamma'}, B_{\Gamma'}, I_{\Gamma'})$ where $B_{\Gamma'}$ is the lineset of $\Gamma \setminus \Gamma'$ and $P_{\Gamma'}$ is the set of points of $\Gamma \setminus \Gamma'$, and where $I_{\Gamma'}$ is the natural incidence. Then there follows that $\Gamma_{\Gamma'}$ is a semi quadrangle of order $(s-1; t)$.
We do not know of any examples.

(e) A **partial ovoid** of a partial quadrangle is a set of mutually non-collinear points. An **ovoid** $\mathcal{O}$ of a partial quadrangle Γ with parameters (s, t, μ) is a set of non-collinear points such that every line is incident with exactly one point of the set[3].

By counting the point-line pairs (p, L) of Γ for which $p \in \mathcal{O}$, pIL with L a line of Γ, in two ways, there follows that $|\mathcal{O}| = \frac{s^2t(t+1)/\mu+(t+1)s+1}{s+1}$.
Suppose Γ is a PQ with parameters (s, t, μ) with $s, t > 2$, and suppose that $\mathcal{O}$ is an ovoid with the property that for every two non-collinear points x and y of $\Gamma \setminus \mathcal{O}$ there holds that $|\{x, y\}^\perp \cap \mathcal{O}| < \mu$. Also, we demand that there is a pair of non-collinear points (x, y) in $P \setminus p^\perp$ for which $|\{p, x, y\}^\perp| < \mu - 1$. Define a geometry $\Gamma_{\mathcal{O}} = (P_{\mathcal{O}}, B_{\mathcal{O}}, I_{\mathcal{O}})$ where $B_{\mathcal{O}}$ is the lineset of Γ, $P_{\mathcal{O}} = \Gamma \setminus \mathcal{O}$, and where $I_{\mathcal{O}}$ is the natural incidence. Then $\Gamma_{\mathcal{O}}$ is a thick semi quadrangle of order $(s-1; t)$. We do not know of any concrete examples of such semi quadrangles; we only prove the following non-existence theorem for ovoids of partial quadrangles.

[3] In the same way, one could define *(partial) ovoids* for semi quadrangles, and, dually, *(partial) spreads*.

THEOREM 2.1 *Let $\mathcal{S}$ be a GQ of order (s, s^2) with $s > 2$. Consider an arbitrary point p of $\mathcal{S}$, and consider the partial quadrangle $\mathcal{S}_p$ with parameters $(s-1, s^2, s^2-s)$ as described in Example (a). Then $\mathcal{S}_p$ cannot have ovoids.*

Proof. Suppose $\mathcal{O}$ is an ovoid of Γ. Then

$$|\mathcal{O}| = \frac{(s-1)^2 s^2 (s^2+1)/(s^2-s) + (s^2+1)(s-1)+1}{s} = s^3,$$

and hence $\mathcal{O} \cup \{p\}$ is an ovoid of $\mathcal{S}$, a contradiction by 1.8.3 of [12].□

Note. By 2.7.1 and 1.8.3 of [12] there easily follows that if $\mathcal{O}$ is a partial ovoid of the PQ $\mathcal{S}_p$, there holds that $|\mathcal{O}| \leq s^3 - s - 1$.

(f) Suppose $\mathcal{K}$ is a *complete* $(t+1)$*-cap* of $\mathbf{PG}(n-1, q)$ (see Section 5), and embed $\mathbf{PG}(n-1, q)$ in $\mathbf{PG}(n, q)$. Suppose P is the set of points of $\mathbf{PG}(n, q)$ which are not contained in $\mathbf{PG}(n-1, q)$, that B is the set of lines L of $\mathbf{PG}(n, q)$ which are not contained in $\mathbf{PG}(n-1, q)$ and for which $|\mathcal{K} \cap L| = 1$. Then the geometry $\mathcal{S} = (P, B, I)$, with I the natural incidence, is a semi quadrangle of order $(q-1; t)$. If $n = 4$ and $\mathcal{K}$ is an *ovoid* of $\mathbf{PG}(3, q)$, then $\mathcal{S}$ is a partial quadrangle, see [4]. For details and proofs, see Section 5.

REMARK 2.2 The first three constructions given above all arise by taking away a so called '*geometric hyperplane*' of a GQ. In the terminology of Pralle, see [13], this means that these geometries are all **affine generalized quadrangles**. Also, any of the Examples (a),(b),(c),(d),(e) can clearly be generalized in a natural way by considering geometric hyperplanes of semi quadrangles (instead of geometric hyperplanes of partial quadrangles).

Note. The Examples (a), (b) and (c) are the only thick semi quadrangles which are subgeometries of a GQ with the same number of points on a line as the GQ minus one.

3 Computation of some divisibility conditions, constants and inequalities

If a point p of a semi quadrangle is incident with $t_i + 1$ lines, then we denote this by $p \in \mathbf{P}_i$, and p is said to have **degree** t_i.
If we write $\lfloor \mathrm{x} \rfloor$, with $x \in \mathbb{R}$, then we mean the greatest natural number

which is at most x, and with $\lceil x \rceil$ we mean the smallest natural number which is at least x.

3.1 Some generalities

Suppose $\mathcal{S}$ is a thick semi quadrangle of order $(s;t_1,\cdots,t_n)$ and with μ-parameters $(\mu_1,\cdots,\mu_m)$ which is not a generalized quadrangle. Then by the observation we made earlier, $\mathcal{S}$ does not satisfy (GQ3), and hence there is a point-line pair (p,L) such that $p \not I L$ and for which there is no line M for which $pIM \sim L$. Suppose $p \in P_j$. By counting the points which are collinear with p and with a point of L in two ways, we obtain $(s+1)\mu_1 \leq (t_j+1)s$, from which follows that $\mu_1 + \frac{\mu_1}{s} \leq t_j + 1$.
Now suppose that $t_i = t$ **for all** i, and fix a point q. By N_k, we denote the number of points x of $P \setminus q^{\perp}$ for which there are μ_k points collinear with both x and q. Now we count the number of points in $q^{\perp} \setminus \{q\}$ in two ways, and we get that

$$\frac{N_1\mu_1 + \cdots + N_m\mu_m}{st} = (t+1)s, \tag{2}$$

and hence we have the following theorem.

THEOREM 3.1 *Suppose that $\mathcal{S}$ is a semi quadrangle with μ-parameters $(\mu_1,\cdots,\mu_m)$ and with a constant number, $t+1$, of lines through a point. Suppose the N_i are as above. Then*

$$\frac{N_1\mu_1 + \cdots + N_m\mu_m}{st} = (t+1)s. \tag{3}$$

3.2 An inequality for semi quadrangles

Definition

Suppose $\mathcal{S}$ is a semi quadrangle. A **triad** is a set of three points, respectively lines, two by two non-collinear, respectively non-concurrent. A **center** of a triad $\{\mathcal{U},\mathcal{V},\mathcal{W}\}$, where $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$ are all points or all lines, is an element of $\{\mathcal{U},\mathcal{V},\mathcal{W}\}^{\perp}$.

THEOREM 3.2 *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s;t_1,t_n)$ and with extremal μ-parameters (μ_1,μ_m). Then we have the following inequality.*

$$[(t_1-1)s\mu_1]^2 \leq \mu_m[(\mu_m-1)(\mu_m-2)+(t_n-1)s]\Big(\frac{\lfloor (t_n+1)t_ns^2 \rfloor}{\mu_1}$$
$$-s(t_1+1)+\mu_m-1\Big). \tag{4}$$

If equality holds, then there is a constant $x_0 = \frac{(t_1-1)s\mu_1}{\lfloor\frac{(t_n+1)t_n s^2}{\mu_1}\rfloor - s(t_1+1)+\mu_m-1}$ *such that each triad of points has exactly* x_0 *centers.*
Also, if each triad of points has a constant number of centers, then

$$[(t_n-1)s\mu_m]^2 \geq \mu_1[(t_1-1)s+(\mu_1-1)(\mu_1-2)](\frac{\lceil(t_1+1)t_1s^2\rceil}{\mu_m} -s(t_n+1)+\mu_1-1). \quad (5)$$

Proof. Suppose p is a point of degree t_1 of $\mathcal{S}$. We repeat our assumption that $t_1 \leq t_n$ and $\mu_1 \leq \mu_m$. There are $(t_1+1)s$ points collinear with, and different from, p and, if p' is such a point, then there are at least $t_1 s$ points collinear with, and different from, p' and not collinear with p. Since $\mu_1 \leq \mu_m$, there are at least $\frac{(t_1+1)t_1s^2}{\mu_m}$ points not collinear with p.
If q is a point not collinear with p, then there are at most $s(t_n+1)-\mu_1$ points collinear with q and not with p, but different from q. So, there are at least $a = \lceil\frac{(t_1+1)t_1s^2}{\mu_m}\rceil - s(t_n+1)+\mu_1-1$ points not collinear with p and q. Analogously there are at most $a'' = \lfloor\frac{(t_n+1)t_ns^2}{\mu_1}\rfloor - s(t_1+1)+\mu_m-1$ points not collinear with both p and q.
We suppose a' is the **precise** number of points not collinear with p and q. Now suppose $p_1, p_2, \ldots, p_{a'}$ are these a' points, and suppose x_i is the number of points collinear with p, q and p_i $(1 \leq i \leq a')$. Then we have the following inequalities.

$$b = (t_1-1)s\mu_1 \leq \sum_{i=1}^{i=a'} x_i \leq (t_n-1)s\mu_m = b' \quad (6)$$

(since for each of the at most μ_m points collinear with p and q, there are at most $(t_n-1)s$ choices for p_i and for each of the at least μ_1 points collinear with p and q there are at least $(t_1-1)s$ choices for p_i);

$$\mu_1(\mu_1-1)(\mu_1-2) \leq \sum_{i=1}^{i=a'} x_i(x_i-1) \leq \mu_m(\mu_m-1)(\mu_m-2) \quad (7)$$

(since for each pair of points collinear with p and q there are at most μ_m-2 choices for p_i and at least μ_1-2 choices for p_i).
It follows that

$$c = \mu_1(\mu_1-1)(\mu_1-2)+(t_1-1)s\mu_1 \leq \sum_{i=1}^{i=a'} x_i^2 \quad (8)$$

and

$$\sum_{i=1}^{i=a'} x_i^2 \leq \mu_m(\mu_m-1)(\mu_m-2)+(t_n-1)s\mu_m = c', \quad (9)$$

so, first of all,

$$\sum_{i=1}^{i=a'}(x_i - x)^2 = \sum_{i=1}^{i=a'} x^2 - 2\sum_{i=1}^{i=a'} xx_i + \sum_{i=1}^{i=a'} x_i^2 \leq a'x^2 - 2bx + c'. \tag{10}$$

Since the left-hand side is a positive semi-definite quadratic form, we can conclude that

$$b^2 \leq a'c' \leq a''c', \tag{11}$$

and this gives the first inequality of the theorem.
If equality holds, then with $a' = a''$ and $x_0 = \frac{b}{a'} = \frac{c'}{b} = \frac{b}{a''}$ we have that

$$\sum_{i=1}^{i=a'}(x_i - x_0)^2 = 0,$$

and so $x_i = x_0$ for $i = 1, 2, \ldots, a'$.
Now we clearly also have the following inequality:

$$ax^2 - 2b'x + c \leq \sum_{i=1}^{i=a'}(x_i - x)^2. \tag{12}$$

Since (5) is trivial if $a \leq 0$, we suppose that $a > 0$. If we suppose that $x_i = x_0$ for a certain constant x_0 and every i, then we have that

$$ax_0^2 - 2b'x_0 + c \leq 0, \tag{13}$$

and since $a > 0$, we now know that this quadratic form has at least one real root. Hence $b'^2 > ac$, which yields the complete proof of the theorem.□

Note. If equality holds in (4), we have the divisibility conditions $a''|b$ and $b|c'$, with $a'' = a'$, b and c' defined as above.

REMARK 3.3 If the dual of the SQ $\mathcal{S}$ is also a semi quadrangle, then the dual statement of THEOREM 3.2 also holds. In that case $t_1 = t_2 = \cdots = t_n = t$, and for every two non-concurrent lines there is at least one line concurrent with both[4].

REMARK 3.4 One notes that we used (SQ3) implicitely, since we divided by μ_1. Also, we did not completely use (SQ4), and hence the preceding theorem holds for some other incidence geometries (such as **partial quadrangles**).

[4]In Cameron [4] it is proved that a partial quadrangle of order (s, t, μ) has the property that the dual is also a partial quadrangle if and only if $t = s$ or $t + 1 = \mu$.

The following (well-known) corollaries of THEOREM 3.2 are easy to prove.

COROLLARY 3.5 (Cameron [4]) *Suppose $\mathcal{S}$ is a partial quadrangle with parameters (s,t,μ). Then*

$$\mu(t-1)^2s^2 \leq [s(t-1)+(\mu-1)(\mu-2)][\frac{(t+1)ts^2}{\mu}-(t+1)s+\mu-1] \quad (14)$$

Equality holds if and only if the number of points collinear with each of any three pairwise non-collinear points is a constant; if this occurs, the constant is $1+\frac{(\mu-1)(\mu-2)}{s(t-1)}$.

□

COROLLARY 3.6 (Higman [7, 8]) *Suppose $\mathcal{S}$ is a generalized quadrangle with parameters (s,t), $s \neq 1 \neq t$. Then*

$$t \leq s^2. \quad (15)$$

Equality holds if and only if the number of points collinear with every three pairwise non-collinear points is a constant, and if this occurs, the constant is $s+1$.

□

Note. Since a generalized quadrangle is a self-dual notion, COROLLARY 3.6 can be dualized.

3.3 Some inequalities

Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s;t_1,t_n)$ and with extremal μ-parameters (μ_1,μ_m), and assume that $s \leq t_1$. Suppose b is the number of lines and v is the number of points. Counting the number θ of *flags* of $\mathcal{S}$ (a flag is an incident point-line pair), we get that

$$v(t_n+1) \geq \theta = b(s+1) \geq v(t_1+1) \quad (16)$$

Note. If $t_1 = t_n = t$ in (16), then $v(t+1) = b(s+1)$. If $v = b$, then $t_n \geq s \geq t_1$.

In the case $v = b$, some refinement is possible.

THEOREM 3.7 *Suppose $\mathcal{S} = (P,B,I)$ is a semi quadrangle of order $(s;t_1,\cdots,t_n)$ and with μ-parameters $(\mu_1,\cdots,\mu_m)$. If $\mathcal{S}$ has the property that $v := |P| = |B| =: b$, then we have that either $t_1 = t_n = s$ or $t_1 < s < t_n$.*

Proof. Suppose $\mathcal{S}$ is an SQ with equally many points as lines, and suppose $t_1 \neq t_n$ (so $t_1 < t_n$). If we suppose that M_i is the number of points with degree t_i, and if we count the number θ of flags of $\mathcal{S}$, we get the following.

$$\theta = b(s+1) = v(s+1) = \sum_i M_i(t_i+1).$$

Since $\sum_i M_i = v$, the theorem easily follows. □

Notation

If two (not necessarily distinct) points p and q, respectively lines L and M, of an SQ are collinear, respectively concurrent, then we denote this by $\mathbf{p} \sim \mathbf{q}$, respectively $\mathbf{L} \sim \mathbf{M}$.

THEOREM 3.8 *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequalities*

$$s^2 t_1(t_1+1) \leq (v-(t_1+1)s-1)\mu_m, \tag{17}$$

and

$$s^2 t_n(t_n+1) \geq (v-(t_n+1)s-1)\mu_1, \tag{18}$$

and $\mathcal{S}$ is a PQ if and only if equality holds in both (17) and (18).

Proof. The inequalities are immediate by counting in two ways the ordered triples of points (p, q, r) of $\mathcal{S}$, with the property that p, q and r are not on the same line, and that $p \sim q$ and $p \sim r$. If $\mathcal{S}$ is a PQ, then $t_1 = t_n = t$, $\mu_1 = \mu_m = \mu$ and $s^2 t(t+1) = (v-(t+1)s-1)\mu$. If equality holds in (17) and (18), then from $s^2 t_1(t_1+1) \leq s^2 t_n(t_n+1)$ and $(v-(t_1+1)s-1)\mu_m \geq (v-(t_n+1)s-1)\mu_1$ follows that $s^2 t_1(t_1+1) = s^2 t_n(t_n+1) = (v-(t_n+1)s-1)\mu_1 = (v-(t_1+1)s-1)\mu_m$, and so $t_1 = t_n$ and $\mu_1 = \mu_m$, that is, $\mathcal{S}$ is a partial quadrangle. □

4 Semi quadrangles and their point graphs

A **graph** is an incidence structure in which lines are called **edges** and points are called **vertices**, and in which any edge is incident with two points and any two distinct points are incident with at most one edge. Two incident points incident with an edge are called **adjacent**, and a graph is **complete** if any two vertices are adjacent. If a vertex v is incident with t edges, then t is called the **valency** of v. The **μ-values** of a graph $\mathcal{G}$ are numbers

$\mu_1, \cdots, \mu_m$ such that any two non-adjacent vertices are both adjacent with μ_i vertices for some $1 \leq i \leq m$. The **λ-values** of a graph are numbers $\lambda_1, \cdots, \lambda_{m'}$ such that any two adjacent points are both adjacent with λ_j points for some $1 \leq j \leq m'$. An **induced subgraph** consists of a subset of points of the point set, together with all the edges joining two points in the subset, and a (maximal) **clique** is a (maximal) complete induced subgraph of a graph. The **point graph** of an incidence geometry is the graph in which two distinct points are adjacent if and only if they are collinear.

THEOREM 4.1 (P. J. Cameron [4]**)** *The point graph of a partial quadrangle is strongly regular and has no induced subgraph isomorphic to a complete graph on four points with one edge removed. Conversely, a strongly regular graph with this property is the point graph of a partial quadrangle.*

There is a similar theorem for semi quadrangles.

THEOREM 4.2 *A graph is the point graph of a semi quadrangle if and only if (a) every μ-value is strictly positive (i.e. the diameter of the graph is at most 2), (b) there is only one λ-value $s-1$, and (c) the graph contains an induced quadrangle, respectively pentagon, and it has no induced subgraph isomorphic to a complete graph on four points with one edge removed. Moreover, if the SQ has order $(s; t_1, \cdots, t_n)$ and μ-parameters $(\mu_1, \cdots, \mu_m)$, then the λ-value of the graph is $s-1$, the possible μ-values are $\mu_1, \cdots, \mu_m$, and $\{(t_1+1)s, \cdots, (t_n+1)s\}$ is the set of valencies, and, conversely, a graph which satisfies properties (a), (b) and (c), and which has these parameters is the point graph of a semi quadrangle of order $(s; t_1, \cdots, t_n)$ and with μ-parameters $(\mu_1, \cdots, \mu_m)$.*

Proof. Suppose $\mathcal{S}$ is a semi quadrangle of order $(s; t_1, \cdots, t_n)$ and with μ-parameters $(\mu_1, \cdots, \mu_m)$, where $n, m \geq 1$. There follows immediately that its point graph has valencies $(t_1+1)s, \cdots, (t_n+1)s$, that the λ-value is $s-1$, and that $\{\mu_1, \cdots, \mu_m\}$ is the set of μ-values. If there would be an induced subgraph isomorphic to a complete graph $pqp'q'$ on four points with one edge $p'q'$ removed, then p' and q' both lie on the line pq, and so they are collinear, a contradiction. The other conditions of (SQ4) are reflected in (c). Now suppose a graph $\mathcal{G}$ has one λ-value $s-1$ and μ-values $\mu_1, \cdots, \mu_m$, and suppose that it satisfies properties (a) and (c). Any edge $\{p, q\}$ is contained in a unique maximal clique $\mathcal{G}_{pq}$ whose vertex set consists of p, q and all vertices joined to both. Hence $\mathcal{G}_{pq}$ has $s+1$ vertices. If any vertex is contained in t_i+1 maximal cliques, with $i \in \{1, \cdots, m\}$, then it is clear that the vertices and maximal cliques of $\mathcal{G}$ form a semi quadrangle of order $(s; t_1, \cdots, t_n)$ and with μ-parameters $(\mu_1, \cdots, \mu_m)$. □

5 Linear representations

A **linear representation** of a semi quadrangle $\mathcal{S} = (P, B, I)$ is a monomorphism θ of $\mathcal{S}$ into the geometry of points and lines of the affine space $\mathbf{AG}(n,q)$, in such a way that P^θ is the set of all points of $\mathbf{AG}(n,q)$, that B^θ is a union of parallel classes of lines of $\mathbf{AG}(n,q)$, and that each point of L^θ is the image of some point of L for any line L in B. Usually we identify $\mathcal{S}$ with its image $\mathcal{S}^\theta$. Note that any parallel class of lines partitions the point set of $\mathbf{AG}(n,q)$. Since parallel classes of lines in an $\mathbf{AG}(n,q)$ correspond to points of $\mathbf{PG}(n-1,q)$ in a natural way, such a representation $\mathcal{S}^\theta$ defines a set of points $\mathcal{K}$ in $\mathbf{PG}(n-1,q)$. An **r-cap** in $\mathbf{PG}(n-1,q)$ (usually called **r-arc** if $n=3$) is a set of r points, no three of which are collinear. A line is **secant**, respectively **tangent**, to an r-cap according as it meets the cap in two points, respectively one point.

THEOREM 5.1 (P. J. Cameron [4]) *1. A subset $\mathcal{K}$ of the point set of $\mathbf{PG}(n-1,q)$ provides a representation of a partial quadrangle with parameters $(q-1,t,\mu)$ if and only if it is a $(t+1)$-cap with the property that any point not in $\mathcal{K}$ lies on $t-\mu+1$ tangents to $\mathcal{K}$.*

2. A subset $\mathcal{K}$ of the point set of $\mathbf{PG}(n-1,q)$ provides a representation of a generalized quadrangle $\mathcal{S}$ if and only if one of the following occurs:

(a) $n=2$ and $|\mathcal{K}|=2$;

(b) $n=3$, q is even and $\mathcal{K}$ is a hyperoval (a $(q+2)$-arc);

(c) $q=2$ and $\mathcal{K}$ is the complement of a hyperplane.

REMARK 5.2 If $n=2$ and $|\mathcal{K}|=2$, then $\mathcal{S}$ is a grid. If $q=2$ and $\mathcal{K}$ is the complement of a hyperplane, then $\mathcal{S}$ is a dual grid. If $n=3$, $q>2$, and $\mathcal{K}$ is a hyperoval, then $\mathcal{S}$ is neither a grid nor a dual grid.

Now suppose $\mathcal{S} = (P, B, I)$ is a semi quadrangle with μ-parameters $(\mu_1, \cdots, \mu_k)$, and suppose $\mathcal{S}$ has a linear representation in an $\mathbf{AG}(n,q)$. If $t+1$ is the number of parallel classes defined by this representation, then it is first of all clear that $\mathcal{S}$ is of order $(q-1,t)$ (hence every point of $\mathcal{S}$ is incident with a constant number of lines).
Suppose $\mathcal{V}$ is the set of $t+1$ points of $\mathbf{PG}(n-1,q)$ which corresponds to the semi quadrangle, and suppose that three points p, o and r of $\mathcal{V}$ are collinear. Consider an arbitrary affine point x of $\mathbf{AG}(n,q)$, and suppose $y \neq x$ is a point of $\mathbf{AG}(n,q) \cap xr$. Then the lines yp, xo and xr define a triangle which is contained in the semi quadrangle, a contradiction. Hence $\mathcal{V}$ is a $(t+1)$-cap.

Let $\mathcal{K}$ be the $(t+1)$-cap in $\mathbf{PG}(n-1,q)$ which corresponds to a semi quadrangle $\mathcal{S}$, and suppose p and o are arbitrary points of $\mathbf{AG}(n,q)$ which are non-collinear in $\mathcal{S}$. Then the line po of $\mathbf{PG}(n,q)$ intersects $\mathbf{PG}(n-1,q)$ in a point r off $\mathcal{K}$. Suppose there are μ_j points of $\mathcal{S}$ collinear (in $\mathcal{S}$) with p and o. Then this means that there are exactly $\mu_j/2$ planes through pq in the projective completion $\mathbf{PG}(n,q)$ of $\mathbf{AG}(n,q)$ which intersect $\mathcal{K}$ in exactly two points, and hence there are precisely $t-\mu_j+1$ tangents to $\mathcal{K}$ through r.
Since $|\mathbf{AG}(n,q)| = |P|$, every point of $\mathbf{PG}(n-1,q)$ off $\mathcal{K}$ is incident with μ_h tangent lines to $\mathcal{K}$ for a certain $h \in \{1,2,\cdots,k\}$. It follows that $\mu_h \equiv 0$ mod 2 for all h. Now suppose o' is such a point which is incident with $t-\mu+1$ tangents to $\mathcal{K}$ ($\mu \in \{\mu_1,\cdots,\mu_k\}$), and let L be an arbitrary line through o' and not in $\mathbf{PG}(n-1,q)$. Then every two distinct points x,y on L, $x \neq o' \neq y$, are non-collinear in $\mathcal{S}$, and $|\{x,y\}^{\perp}| = \mu$. Also, there are exactly $\frac{q(q-1)}{2}$ such (nonordered) pairs on L.

Condition (SQ3). The fact that $\mathcal{S}$ satisfies (SQ3) is clearly equivalent with the fact that for every two points p and o of $\mathbf{AG}(n,q)$ for which po does not intersect $\mathcal{K}$, there must be at least one secant to the cap. Hence there follows that the cap must be **complete.**

We now investigate how the existence of a quadrangle, respectively pentagon, is reflected on the linear respresentation.

The existence of a quadrangle. Let L be a secant to $\mathcal{K}$ and let π be a plane of $\mathbf{PG}(n,q)$ containing L, but not contained in $\mathbf{PG}(n-1,q)$. Then π contains quadrangles of $\mathcal{S}$.

The existence of a pentagon. By THEOREM 1.3 and the preceding results, the existence of a pentagon in $\mathcal{S}$ is equivalent to the condition that $\mathcal{S}$ is not a grid or a dual grid. By REMARK 5.2, this is always the case except if one of the following occurs:

1. $n=2$ and $|\mathcal{K}|=2$;
2. $q=2$ and $\mathcal{K}$ is the complement of a hyperplane.

We have proved the following theorem.

THEOREM 5.3 *1. A subset $\mathcal{K}$ of the point set of $\mathbf{PG}(n-1,q)$, $n \geq 3$, provides a linear representation of a semi quadrangle with μ-parameters $(\mu_1,\cdots,\mu_k)$ if and only if the following conditions are satisfied:*

(a) *it is a complete $(t+1)$-cap for a certain t with the property that any point off $\mathcal{K}$ in $\mathbf{PG}(n-1,q)$ lies on $t-\mu_j+1$ tangents to $\mathcal{K}$ for some $\mu_j \in \{\mu_1, \cdots, \mu_k\}$;*

(b) *If $q=2$, then $\mathcal{K}$ is not the complement of a hyperplane.*

2. *If a $(t+1)$-cap $\mathcal{K}$ of $\mathbf{PG}(n-1,q)$ provides a linear representation of the semi quadrangle $\mathcal{S}$, then every point of $\mathcal{S}$ is incident with $t+1$ lines.*

3. *Suppose $\mathcal{S}=(P,B,I)$ is an SQ with μ-parameters $(\mu_1,\cdots,\mu_k)$ which has a linear representation in $\mathbf{AG}(n,q)$, and define P_j by $P_j = \{\{x,y\} \in P \times P, x \not\sim y \parallel |\{x,y\}^\perp| = \mu_j\}$. Then for all j, $|P_j| \equiv 0$ mod $(q(q-1)/2)$.*

□

If $n=3$ and q is even, then the r-cap $\mathcal{K}$ is a hyperoval if r equals $q+2$, and a hyperoval is always complete. Suppose $q \geq 4$. If we consider the semi quadrangle $\mathcal{S}$ which corresponds with $\mathcal{K}$, then $\mathcal{S}$ is a generalized quadrangle of order $(q-1,q+1)$; this quadrangle is usually denoted by $T^*(\mathcal{K})$ and is due to R. W. Ahrens and G. Szekeres [1, 6] (see also THEOREM 5.1). If $n=4$, $q>2$, and $\mathcal{K}$ is an **ovoid** (a (complete) (q^2+1)-cap) of $\mathbf{PG}(3,q)$, then the associated semi quadrangle is a partial quadrangle with parameters $(q-1,q^2,q^2-q)$ (see e.g. [9, 14]).

5.1 Semi quadrangles and complete caps of projective spaces

The following two theorems are direct consequences of THEOREM 5.3 and the results of Section 3.

THEOREM 5.4 *Suppose $\mathcal{K}$ is a complete $(k+1)$-cap in $\mathbf{PG}(n,q)$, and let μ, respectively μ', be the integers such that every point of $\mathbf{PG}(n,q) \setminus \mathcal{K}$ is incident with at least $k+1-\mu$, respectively at most $k+1-\mu'$, tangents to $\mathcal{K}$. Then we have the following inequality.*

$$[(k-1)(q-1)\mu']^2 \leq \mu[(\mu-1)(\mu-2)+(k-1)(q-1)]\left(\frac{\lfloor (k+1)k(q-1)^2 \rfloor}{\mu'}\right.$$
$$\left. -(q-1)(k+1)+\mu-1\right).$$

If equality holds, then there is a constant $x_0 = \frac{(k-1)(q-1)\mu'}{\lfloor \frac{(k+1)k(q-1)^2}{\mu'} \rfloor -(q-1)(k+1)+\mu-1}$ such that, if we embed $\mathbf{PG}(n,q)$ as a hyperplane in $\mathbf{PG}(n+1,q)$, for every

set $\{p_1, p_2, p_3\}$ of three distinct points in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ with the property that $p_ip_j \cap \mathcal{K} = \emptyset$ for every $i \neq j$ in $\{1,2,3\}$, there holds that there are precisely x_0 points r in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ for which $|rp_i \cap \mathcal{K}| = 1$ $\forall i = 1,2,3$.
Also, if for every set $\{p_1, p_2, p_3\}$ of three distinct points in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ with the property that $p_ip_j \cap \mathcal{K} = \emptyset$ for every $i \neq j$ in $\{1,2,3\}$, it is true that there is a constant number of points r in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ for which $|rp_i \cap \mathcal{K}| = 1$ $\forall i = 1,2,3$, then we have the following.

$$[(k-1)(q-1)\mu]^2 \geq \mu'[(k-1)(q-1) + (\mu'-1)(\mu'-2)]\left(\frac{[(k+1)k(q-1)^2]}{\mu}\right.$$

$$\left. -(q-1)(k+1) + \mu' - 1\right).$$

We do not know whether THEOREM 5.4 yields new information about complete caps in projective spaces.

THEOREM 5.5 *Suppose that $\mathcal{K}$ is a complete $(k+1)$-cap in $\mathbf{PG}(n,q)$, and let μ, respectively μ', be the integers such that every point of $\mathbf{PG}(n,q) \setminus \mathcal{K}$ is incident with at least $k+1-\mu$, respectively at most $k+1-\mu'$, tangents to $\mathcal{K}$. Then we have the following inequalities*

$$(q-1)^2k(k+1) \leq (q^{n+1} - (k+1)(q-1) - 1)\mu, \tag{19}$$

and

$$(q-1)^2k(k+1) \geq (q^{n+1} - (k+1)(q-1) - 1)\mu', \tag{20}$$

and equality holds in both cases if and only if $\mu = \mu'$.

There follows that

$$k + 1 - \frac{(q-1)^2k(k+1)}{(q^{n+1} - (k+1)(q-1) - 1)} \leq k + 1 - \mu'. \tag{21}$$

From [11], we know that since $\mathcal{K}$ is complete, there holds that $k+1-\mu' < \delta(q^{n-1} + q^{n-2} + ... + 1 - k)$, where $\delta = 1$ if q is even and $\delta = 2$ otherwise. We conclude that, with $f(k,q,n) := \frac{(q-1)^2k(k+1)}{(q^{n+1}-(k+1)(q-1)-1)} - 1$, that

$$(1+\delta)k < f(k,n,q) + \delta(q^{n-1} + q^{n-2} + ... + 1). \tag{22}$$

We will come back to the connection between complete caps in $\mathbf{PG}(n,q)$ and semi quadrangles in a subsequent paper. For an excellent (updated) survey on bounds of complete caps in projective spaces and several related problems, see Hirschfeld and Storme [10].

A final remark

A semi quadrangle contains no substructure isomorphic to an ordinary 2-gon or 3-gon. With this property in mind, we could define a **semi** $2N$**-gon** of order $(s; t_1, \cdots, t_n)$ and with μ-parameters $(\mu_1, \cdots, \mu_m)$ to be an incidence structure satisfying (SQ1), and also the following conditions.

1. There is no substructure isomorphic to an ordinary M-gon, for $2 \leq M \leq 2N - 1$.
2. If two distinct points are not contained in a path of length $N - 1$ or less, then they are contained in exactly $\mu_1, \mu_2, \cdots$, or μ_m paths of length N, where $\mu_j \geq 1$ for every j. Also, each of the cases occurs.
3. There are substructures isomorphic to an ordinary $2N$-gon and an ordinary $(2N + 1)$-gon.

With this definition, a **thick partial** $2N$**-gon** [4] is just a semi $2N$-gon with $s > 1$, $t_1 = t_2 = \cdots = t_n > 1$ and $\mu_1 = \mu_2 = \cdots = \mu_m$. Also, a **generalized** $2N$**-gon** [20, 21, 15] is **precisely** a semi $2N$-gon with $t_i = \mu_j$ for arbitrary i and j.

Note. The author wants to note that this paper was partially modelled after P. J. Cameron [4].

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