Isomorphic Path Decompositions of Crowns

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ABSTRACT. For positive integers $k \leq n$, the crown $C_{n,k}$ is the graph with vertex set $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ and edge set $\{a_i b_j\colon\ 1 \leq i \leq n,\ j = i, i+1, \ldots, i+k-1 \pmod{n}\}$. In this paper we give a necessary and sufficient condition for the existence of P_l decomposition of $C_{n,k}$.

1 Introduction

For positive integers $k \leq n$, the *crown* $C_{n,k}$ is the graph with vertex set $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ and edge set $\{a_i b_j\colon\ 1 \leq i \leq n,\ j = i, i+1, \ldots, i+k-1 \pmod{n}\}$. Let K_n be a complete graph on n vertices and C_k, a cycle with k vertices. A path of length l is denoted by P_l. If the edges of a graph G can be decomposed into subgraphs isomorphic to a graph H, then we say that G has an *H-decomposition*.

Decomposition of graphs continues to be a popular topic of research (see Bosák's book on the topic[1] for an overview). In particular K_k decompositions and C_k decompositions of K_n have attracted considerable attention. Decompositions of various graphs into paths have also attracted

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a fair share of interest. In 1977, S. H. Y. Hung and N. S. Mendelsohn [2] found a necessary and sufficient condition of balanced path decomposition of complete multigraphs. In 1983, M. Tarsi [6] gave a necessary and sufficient condition for the existence of path decomposition (nonbalance) of complete multigraphs. In 1985, M. Truszczyński [7] presented a necessary and sufficient condition for the existence of path decomposition of complete bipartite symmetric multidigraphs and some necessary and/or sufficient conditions of path decomposition of complete bipartite multigraphs. Edge decompositions of crowns have been investigated for isomorphic stars[4], isomorphic cycles[5] and isomorphic complete bipartite graphs[3].

In this paper we deal with the isomorphic path decomposition of crowns, and obtain the following.

Theorem. The crown $C_{n,k}$ has a P_l decomposition if and only if the following conditions are satisfied:

$A.\ \ nk \equiv 0 \pmod{l}.$

$$B.\ \ l \leq \begin{cases} 2 & \text{if } k = n = 2, \\ 2n-3 & \text{if k is even and n} \geq 3, \\ k & \text{if k is odd.} \end{cases}$$

2 Main Result

We need the following lemmas for our discussion.

Lemma 1 *Suppose that a graph G can be decomposed into t nontrivial paths and that G contains v vertices with odd degrees. Then $v \leq 2t$.*

Proof. Suppose that G is decomposed into nontrivial paths $Q_1, Q_2, \ldots, Q_t$. Since each vertex of G with odd degree must be an end vertex of at least one Q_i $(1 \leq i \leq t)$, the result follows. □

For our discussion we need the following definitions and notations. A $v_0 - v_r$ *walk* of length r in a graph G is a sequence of vertices of the form $v_0, v_1, \ldots, v_r$ where $v_{i-1}v_i \in E(G)$ for $i = 1, 2, \ldots, r$; this walk is denoted by $v_0v_1 \ldots v_r$. A *trail* is a walk in which all edges are distinct. Suppose Q: $x_1x_2x_3 \ldots x_n$ is a trail in a graph. Then the trail T: $x_kx_{k+1}x_{k+2} \ldots x_s$ $(1 \leq k \leq s \leq n)$ is called a *subtrail* of Q. A trail is *closed* if its starting vertex and ending vertex are the same. An *Eulerian trail* of a graph G is a closed trail containing all edges of G. For vertices $x_1, x_2, \ldots, x_{l+1}$, we use $< x_ix_{i+1} >_{i=1}^{l}$ to mean the walk $x_1x_2 \ldots x_{l+1}$, and $< x_1 >$ the trivial walk x_1. Suppose W_1: $x_1x_2 \ldots x_t$ and W_2: $y_1y_2 \ldots y_s$ are walks. If x_ty_1 is an edge, we use W_1W_2 to denote the walk $x_1x_2 \ldots x_ty_1y_2 \ldots y_s$. If

$x_t = y_1$, we use $W_1 + W_2$ to denote the walk $x_1x_2 \dots x_ty_2y_3 \dots y_s$. For walks $W_1, W_2, \dots, W_v$ in a graph, $W_1 + W_2 + \cdots + W_v$ is similarly defined. For an integer t and a walk W: $x_1x_2 \dots x_s$ in the crown $C_{n,k}$, we use $W + t$ to denote the walk $x_{1+t}x_{2+t} \dots x_{s+t}$; here and in the sequel, the subscripts of vertices of $C_{n,k}$ are taken modulo n. Suppose G is a subgraph of $C_{n,k}$ and H is a subgraph of G such that the edges of G can be decomposed into subgraphs $H, H+1, H+2, \dots, H+j$ for some integer j. Then H is called a *base graph* of this decomposition. The *girth* $g(T)$ of a trail T in a graph G is the least number of edges between two appearances of the same vertex along T. For example, let T be the trail $a_1b_3a_5b_4a_1b_6a_5b_2a_6b_3$ in $C_{6,5}$. Then $g(T) = 4$.

Lemma 2 *Let $4 \le k \le n$, and k is an even integer. Then $C_{n,k}$ contains an Eulerian trail with girth $2n-4$.*

Proof. For $i = 2, 4, \dots, k$, let C_i denote the Hamiltonian cycle $< a_j b_{i+(j-1)} >_{j=1}^{n} < a_1 >$ of $C_{n,k}$, i.e., C_i is the trail $a_1b_ia_2b_{i+1}a_3b_{i+2} \dots a_n b_{i+(n-1)}a_1$. Then $C_2+C_4+\cdots+C_k$ is an Eulerian trail of $C_{n,k}$. We see that $C_i + C_{i+2}$ is the trail $< a_jb_{i+(j-1)} >_{j=1}^{n} < a_jb_{i+2+(j-1)} >_{j=1}^{n} < a_1 >$, i.e., $C_i + C_{i+2}$ is the trail $a_1b_ia_2b_{i+1}a_3b_{i+2} \dots a_nb_{i+(n-1)}a_1b_{i+2}a_2b_{i+3}a_3b_{i+4} \dots a_nb_{i+2+(n-1)}a_1$. There are $2n$ edges between two appearances of a_j ($j = 1, 2, \dots, n$), $2n-4$ edges between two appearances of b_j for $j = i+2, i+3, i+4, \dots, i+n-1$, and $4n-4$ edges between two appearances of b_j for $j = i, i+1$. Thus $g(C_i + C_{i+2}) = 2n-4$. □

The edges of $C_{n,k}$ are labeled as follows. Each edge can be assumed to be a_ib_j with $1 \le i \le n$, $i \le j \le i+k-1$. We refer to this edge as an s-edge where $s = j - i$.

Lemma 3 *Let l, k, n be positive integers such that n, l are even, k is odd, and $4 \le l+3 \le k < n$. Let G be a spanning subgraph of $C_{n,k}$ with $E(G) = \{a_ib_j\colon\ i \text{ is odd},\ j = i,\ i+2 \pmod n\} \cup \{a_ib_j\colon\ i = 1, 2, \dots, n,\ j = i+l+1, i+l+2, \dots, i+k-1 \pmod n\}$. Then G contains an Eulerian trail with girth $k-1$.*

Proof. Let D be the trail $< b_{2i-1}a_{2i-1} >_{i=1}^{\frac{n}{2}} < b_1 >$, i.e., D is the trail $b_1a_1b_3a_3 \dots b_{n-1}a_{n-1}b_1$. For $i = l+3, l+5, l+7, \dots, k$, let D_i be the trail $< b_ja_{n-i+(j+2)} >_{j=1}^{n} < b_1 >$, i.e., D_i is the path $b_1a_{n-i+3}b_2a_{n-i+4}b_3a_{n-i+5} \dots b_na_{n-i+2}b_1$. Since each D_i contains all edges with labels $i-1$ and $i-2$, $D + D_k + D_{k-2} + \cdots + D_{l+3}$ is an Eulerian trail of G. To determine its girth, we first determine that of $D + D_k$. It is easy to see that $D + D_k$ is the trail $< b_{2i-1}a_{2i-1} >_{i=1}^{\frac{n}{2}} < b_ja_{n-k+(j+2)} >_{j=1}^{n} < b_1 >$, i.e., $D + D_k$ is the trail $b_1a_1b_3a_3 \dots b_{n-1}a_{n-1}b_1a_{n-k+3}b_2a_{n-k+4}b_3a_{n-k+5} \dots b_na_{n-k+2}b_1$. There are n edges between two appearances of b_1, and more than n edges between two appearances of b_i for $i = 3, 5, \ldots, n-1$. There are $k-1$ edges between two appearances of a_{n-k+4}, and more than $k-1$ edges between two appearances of a_i for $i \in \{1, 3, \ldots, n-1\} - \{n-k+4\}$. Thus $g(D + D_k) = k-1$. By a similar argument in the proof of Lemma 2, we have $g(D_k + D_{k-2}) = g(D_{k-2} + D_{k-4}) = \cdots = g(D_{l+1} + D_{l+3}) = 2n-4$. Since $k - 1 \le 2n - 4$, we have $g(D + D_k + D_{k-2} + \cdots + D_{l+3}) = k-1$. □

Now we prove the main theorem.

Proof of Theorem.(Necessity) Condition A is trivial. Now we prove Condition B. The case $k = n = 2$ is trivial. Consider the case that k is even and $n \ge 3$. Since P_l has $l+1$ vertices, and $C_{n,k}$ has 2n vertices, we have $l + 1 \le 2n$, i.e., $l \le 2n-1$. Suppose $l = 2n-1$. Then, by Condition A, $nk \equiv 0 \pmod{2n-1}$, which implies $k \equiv 0 \pmod{2n-1}$ since $gcd(2n-1, n) = 1$. This is impossible since $k \le n$. This contradiction excludes $l = 2n-1$. Suppose $l = 2n-2$. Then $nk \equiv 0 \pmod{2n-2}$, which implies $k \equiv 0 \pmod{n-1}$ since $gcd(n-1, n) = 1$. Then $k = n-1$ since $k \le n$. But then, $n(n-1) \equiv 0 \pmod{2n-2}$, which implies $n \equiv 0 \pmod 2$. Then $k = n-1$ is odd. This contradiction asserts that $l \le 2n-3$. Now consider the case that k is odd. Since all the $2n$ vertices in $C_{n,k}$ has odd degree, and there are $\frac{nk}{l}$ paths in the decomposition, we have, by Lemma 1, $2n \le 2\frac{nk}{l}$. Thus $l \le k$.
(Sufficiency) The case $n \le 3$ is trivial. Suppose $n \ge 4$.

Case 1. k is even.

For $k = 2, C_{n,k}$ is in fact a cycle; the result is obvious. Now $k \ge 4$. By Lemma 2, $C_{n,k}$ contains an Eulerian trail E with girth $2n-4$. We consider the following three cases.

Case 1.1. $l \le 2n-5$.

From the starting vertex we cut the Eulerian trail E into subtrails with l edges. These subtrails are all paths since $l < g(E)$. Thus $C_{n,k}$ have a P_l decomposition.

Case 1.2. $l = 2n-4$.

As in Case 1.1, from the starting vertex we cut the Eulerian trail E into subtrails with l edges. Again these subtrails are paths. This follows from the following facts: $g(E) = 2n-4$, and the only vertices with the number of edges between two appearances of the same vertex along E equal to $2n-4$ are in $\{b_1, b_2, \ldots, b_n\}$, and each subtrail which we have has end vertices in $\{a_1, a_2, \ldots, a_n\}$. Thus $C_{n,k}$ have a P_l decomposition.

Case 1.3. $l = 2n-3$.

Since $gcd(2n-3, n) = 1$ or 3, we consider two cases as follows.

Case 1.3.1. $gcd(2n-3, n) = 1$.

Since $nk \equiv 0 \pmod{l}$, $l = 2n-3$, we have $nk \equiv 0 \pmod{2n-3}$. Thus $k \equiv 0 \pmod{2n-3}$; this is impossible since $n \geq k$, 4.

Case 1.3.2. $gcd(2n-3, n) = 3$.

Since $n \equiv 0 \pmod{3}$, $n \geq 4$, we write $n = 3t$ where t is an integer ≥ 2. From $nk \equiv 0 \pmod{l}$, we obtain $k \equiv 0 \pmod{2t-1}$, which implies $2(2t-1) \leq k$ since k is even. Thus $2(2t-1) \leq k \leq n = 3t$, which implies $t \leq 2$. Hence $t = 2$, which implies $k = n = 6$, $l = 9$. Now we exhibit a P_9 − decomposition of $C_{6,6}$. The decomposition consists of the following paths: $a_1b_2a_2b_3a_3b_4a_4b_5a_5b_6$, $a_1b_1a_2b_4a_5b_2a_3b_5a_6b_6$, $a_2b_6a_3b_1a_4b_2a_6b_3a_1b_4$, $a_2b_5a_1b_6a_4b_3a_5b_1a_6b_4$.

Case 2. k is odd.

Case 2.1. l is odd.

Let G_1 which consists of all the edges with labels $0, 1, \ldots, l-1$ be the spanning subgraph of $C_{n,k}$. And let $G_2 = C_{n,k} - E(G_1)$. From the facts that G_2 is isomorphic to $C_{n,k-l}$, $n(k-l) \equiv 0 \pmod{l}$, $k-l$ is even, and $l \leq k \leq n \leq 2n-3$, we see, by Case 1, that G_2 has a P_l decomposition. For the decomposition of G_1, let Q be the base path $< a_ib_{l+1-i} >_{i=1}^{\lceil \frac{l}{2} \rceil}$, i.e., Q is the path $a_1b_la_2b_{l-1}a_3b_{l-2}\ldots a_{\lceil \frac{l}{2} \rceil}b_{\lceil \frac{l}{2} \rceil}$. We can see that Q is a path of length l, and consists of edges with labels in order of $l-1, l-2, \ldots, 1, 0$. It is easy to see that G_1 can be decomposed into paths $Q, Q+1, Q+2, \ldots, Q+(n-1)$. Thus G_1 has a P_l decomposition. From the P_l decomposition of G_1 and that of G_2, we obtain the P_l decomposition of $C_{n,k}$.

Case 2.2. l is even.

Let H be a spanning subgraph of $C_{n,k}$ with $E(H) = \{a_ib_j\colon 1 \leq i \leq n,$ i is odd, $j = i+1, i+3, i+4, i+5, \ldots, i+l \pmod{n}\} \cup \{a_ib_j\colon 1 \leq i \leq n,$ i is even, $j = i, i+1, i+2, \ldots, i+l \pmod{n}\}$. We first give a P_l − decomposition of H. we consider two cases as follows.

Case 2.2.1. $l \equiv 0 \pmod{4}$.

Let Q be the path $< a_ib_{l+2-i} >_{i=1}^{\frac{l}{2}} < a_{\frac{l}{2}+2} >$, i.e., Q is the path $a_1b_{l+1}a_2b_la_3$ $b_{l-1}\ldots a_{\frac{l}{2}-1}b_{\frac{l}{2}+3}a_{\frac{l}{2}}b_{\frac{l}{2}+2}a_{\frac{l}{2}+2}$, and R, $< b_3 > < a_{i+1}b_{l+3-i} >_{i=1}^{\frac{l}{2}-1} < a_{\frac{l}{2}+1}$ $b_{\frac{l}{2}+2} >$, i.e., R is the path $b_3a_2b_{l+2}a_3b_{l+1}a_4b_l\ldots a_{\frac{l}{2}-1}b_{\frac{l}{2}+5}a_{\frac{l}{2}}b_{\frac{l}{2}+4}a_{\frac{l}{2}+1}b_{\frac{l}{2}+2}$. It is not hard to see that both Q and R are paths of length l, where Q consists of edges with the following labels in order of l, $l-1$, $l-2, \ldots,$ 4, 3, 2, 0, and R, in order of 1, l, $l-1$, $l-2, \ldots,$ 5, 4, 3, 1. Then H can be decomposed into the following paths: $Q, Q+2, Q+4, \ldots, Q+(n-2)$, $R, R+2, R+4, \ldots,$ and $R+(n-2)$. Thus H has a P_l decomposition.

Case 2.2.2. $l \equiv 2 \pmod 4$.

Let Q be the path $< a_i b_{l+2-i} >_{i=1}^{\frac{l}{2}-1} < a_{\frac{l}{2}} b_{\frac{l}{2}+1} a_{\frac{l}{2}+1} >$, i.e., Q is the path $a_1 b_{l+1} a_2 b_l a_3 b_{l-1} \dots a_{\frac{l}{2}-2} b_{\frac{l}{2}+4} a_{\frac{l}{2}-1} b_{\frac{l}{2}+3} a_{\frac{l}{2}} b_{\frac{l}{2}+1} a_{\frac{l}{2}+1}$, and R, $< b_4 >< a_{i+1} b_{l+3-i} >_{i=1}^{\frac{l}{2}-1} < a_{\frac{l}{2}+1} b_{\frac{l}{2}+2} >$, i.e., R is the path $b_4 a_2 b_{l+2} a_3 b_{l+1} a_4 b_l \dots a_{\frac{l}{2}-1} b_{\frac{l}{2}+5} a_{\frac{l}{2}} b_{\frac{l}{2}+4} a_{\frac{l}{2}+1} b_{\frac{l}{2}+2}$. It is not hard to see that both Q and R are paths of length l, where Q consists of edges with the following labels in order of $l, l-1, l-2, \dots, 6, 5, 4, 3, 1, 0$, and R, in order of $2, l, l-1, l-2, \dots, 5, 4, 3, 1$. Then H can be decomposed into $Q, Q+2, Q+4, \dots, Q+(n-2)$, $R, R+2, R+4, \dots$, and $R+(n-2)$. Thus H has a P_l-decomposition.

Finally, we show that $C_{n,k}$ has a P_l decomposition by giving a P_l decomposition of $C_{n,k} - E(H)$. Let $G = C_{n,k} - E(H)$. Suppose $l = k-1$. Since $nk \equiv 0 \pmod l$, $l < k$, we have $n = sl$ for some integer $s \geq 2$. Clearly $E(G) = \{a_i b_j\colon 1 \leq i \leq n,\ \text{i is odd},\ j = i, i+2, \pmod n\}$. Thus the edges of G form a cycle of length n, and hence, can be decomposed into s paths with l edges. Suppose $l \leq k-3$. Then G which is defined in Lemma 3 is exactly the spanning subgraph of $C_{n,k}$. Since $nk \equiv 0 \pmod l$ and H has a P_l-decomposition, , then $|E(G)| \equiv 0 \pmod l$. By Lemma 3, G has a P_l decomposition, since $g(G) = k-1 > l$. This completes Case 2.2, and Case 2. □

The following corollary follows immediately from the main theorem.

Corollary 4 *$K_{n,n}$ has a P_l decomposition if and only if the following conditions are satisfied:*

A. $n^2 \equiv 0 \pmod l$.

B. $l \leq \begin{cases} 2 & \text{if } n=2, \\ 2n-3 & \text{if } n \geq 4 \text{ and even}, \\ n & \text{if } n \text{ is odd.} \end{cases}$ □

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References

[1] J. Bosák, Decompositions of graphs, Kluwer Academic Publishers, Boston (1990).

[2] S. H. Y. Hung and N. S. Mendelsohn, Handcuffied designs, *Discrete Math.* 18 (1977), 23–33.

[3] C. Lin, J. J. Lin and T. W. Shyu, Complete bipartite decomposition of crowns, with Applications to complete directed graphs, Combinatorics and computer science, *Lecture Notes in Computer Science*, 1120 (1996), 58-66.

[4] C. Lin, J. J. Lin and T. W. Shyu, Isomorphic star decompositions of multicrowns and the power of cycles, *Ars Combin.*53(1999), 249 256.

[5] C. Lin and J. J. Lin, Cycle decompositions of crowns, *Discrete Math.* 220 (2000), 251-255.

[6] M. Tarsi, Decomposition of complete multigraph into simple paths : Nonbalanced Handcuffied designs, *J. Combin. Theory* A 34 (1983), 60 70.

[7] M. Truszczyński, Note on the decomposition of $K_{m,n}$ $(K^*_{m,n})$ into paths, *Discrete Math.* 55 (1985), 89–96.