Twin Domination in Digraphs

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Abstract

A vertex v in a digraph D out-dominates itself as well as all vertices u such that (v,u) is an arc of D; while v in-dominates both itself and all vertices w such that (w,v) is an arc of D. A set S of vertices of D is a twin dominating set of D if every vertex of D is out-dominated by some vertex of S and in-dominated by some vertex of S. The minimum cardinality of a twin dominating set is the twin domination number $\gamma^*(D)$ of D. It is shown that $\gamma^*(D) \leq 2p/3$ for every digraph D of order p having no vertex of in-degree 0 or out-degree 0. Moreover, we give a Nordhaus-Gaddum type bound for γ^*, and for transitive digraphs we give a sharp upper bound for the twin domination number in terms of order and minimum degree.

For a graph G, the upper orientable twin domination number $DOM^*(G)$ is the maximum twin domination number $\gamma^*(D)$ over all orientations D of G; while the lower orientable twin domination number $dom^*(G)$ of G is the minimum such twin domination number. It is shown that for each graph G and integer c with $dom^*(G) \leq c \leq DOM^*(G)$, there exists an orientation D of G such that $\gamma^*(D) = c$.

1 Introduction

Domination in graphs has been studied extensively in recent years. The book by Haynes, Hedetniemi, and Slater [7] is entirely devoted to this area. Domination in *directed* graphs, however, has not attracted the same amount of attention by researchers (see the survey [6]), except for the study of kernels of digraphs. A natural directed analogue to the domination number of an undirected graph is the out-domination number of a digraph D, defined as the minimum cardinality of a subset S of the vertex set of D such that each vertex of D is either in S or an out-neighbour of some vertex in S. The out-domination number and the in-domination number, which is defined analogously, are studied e.g. in [5]. In this paper we propose a third, equally natural directed analogue to domination in graphs. The *twin domination number* of a digraph D is the minimum cardinality of a subset S of the vertex set of D such that each vertex of D is either in S or both, an out-neighbour of some vertex in S and an in-neighbour of some (possibly distinct) vertex in S. For terminology and notation pertaining to digraphs, we refer to [2].

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Let D be a digraph with no loops or multiple arcs, and let v be a vertex of D. The ***out-neighborhood*** $N^+(v)$ of v consists of all those vertices w of D such that (v, w) is an arc of D, while the ***in-neighborhood*** $N^-(v)$ of v consists of all those vertices u of D such that (u, v) is an arc of D. The cardinality of $N^+(v)$ ($N^-(v)$) is the ***out-degree*** (***in-degree***) of the vertex v, denoted by od$v$ (id$v$). More generally, for a subset S of vertices of D, the ***out-neighborhood*** $N^+(S)$ of S consists of all those vertices w in $D - S$ such that (v, w) is an arc of D for some $v \in S$. The ***in-neighborhood*** $N^-(S)$ consists of all those vertices u in $D-S$ such that (u, v) is an arc of D for some $v \in S$. If $A, B \subset V(D)$ then we denote the set of all arcs of D that join a vertex of A to a vertex of B by $(A, B)_D$, or if D is understood, by (A, B). The number of arcs in (A, B) is denoted by $q(A, B)$. Similarly we use $p(D)$ for the order of D and also $\overline{A}$ for $V(D) - A$. The subdigraph induced by A is denoted by $\langle A \rangle_D$, or if D is understood, by $\langle A \rangle$. The ***complement*** of D, denoted by $\overline{D}$, is the digraph on the vertex set $V(D)$, in which (v, w) is an arc of $\overline{D}$ if and olny if (v, w) is not an arc of D.

A vertex v in a digraph D is said to ***out-dominate*** the vertices in the set $N^+(v) \cup \{v\}$, while v ***in-dominates*** the vertices in $N^-(v) \cup \{v\}$. A set S of vertices of D is an ***out-dominating set*** if every vertex of D is out-dominated by some vertex of S, while S is an ***in-dominating set*** if every vertex of D is in-dominated by some vertex of S. The minimum cardinality of an out-dominating set (in-dominating set) is the ***out-domination number*** $\gamma^+(D)$ (***in-domination number*** $\gamma^-(D)$). The out-domination number of a digraph D is commonly called simply the ***domination number*** of D, and the in-domination number of D has been referred to as the ***converse domination number*** of D (see Fu [5]). Let D' denote the converse of D (obtained by reversing the direction of every arc of D). Then clearly $\gamma^-(D) = \gamma^+(D')$ and $\gamma^+(D) = \gamma^-(D')$ for every digraph D.

An out-dominating set of vertices in a digraph need not be an in-dominating set, and vice versa. A set S of vertices in a digraph D is a ***twin dominating set*** if S is both an out-dominating set and an in-dominating set. The ***twin domination number*** $\gamma^*(D)$ of D is the cardinality of a minimum twin dominating set of D. Certainly, for every digraph D,

$$\max\{\gamma^+(D), \gamma^-(D)\} \leq \gamma^*(D) \leq \gamma^+(D) + \gamma^-(D).$$

2 Bounds

We now present some bounds on the twin domination number of a digraph in terms of its order and other parameters as well. First, if a vertex u in a digraph D has in-degree 0, then u necessarily belongs to every out-dominating set. If a vertex v has out-degree 0, then v necessarily belongs to every in-dominating set. On the other hand, if w is a vertex of D with positive in-degree and positive out-degree, then $V(D) - \{w\}$ is a twin dominating set of D. While clearly

$\gamma^*(D) \leq p$ for every digraph D of order p, it follows that $\gamma^*(D) = p$ if and only if every vertex of D has in-degree 0 or out-degree 0. Indeed, if S is an independent set of vertices of D (i.e., no two vertices of S are joined by an arc) such that every vertex of S has positive in-degree and positive out-degree, then $\gamma^*(D) \leq p - |S|$. Of course, if S is any set of vertices of D such that every vertex of S is both adjacent to some vertex of $V(D) - S$ and adjacent from some vertex of $V(D) - S$, then $\gamma^*(D) \leq p - |S|$.

An improved bound can be given in terms of order alone if D is a digraph with no vertex of in-degree 0 or out-degree 0. The following result of Chvátal and Lovász [4] will be useful to us. The *minimum degree* $\delta(D)$ of a digraph D is defined as the minimum of all in-degrees and all out-degrees of vertices in D. The distance from a vertex u to a vertex v of D, i.e., the length of a shortest directed $u - v$ path, is denoted by $d(u, v)$.

Theorem A ([4]) *Every digraph D has an independent set S of vertices with the property that for each vertex v of D, there exists a vertex $u \in S$ such that $d(u, v) \leq 2$.* □

Theorem 1 *If D is a digraph of order p with $\delta(D) > 0$, then $\gamma^*(D) \leq \lfloor 2p/3 \rfloor$, and this bound is sharp.*

Proof. By Theorem A, there exists an independent set of vertices such that every vertex is at distance at most 2 from this set. Let S be such a set that is maximal independent. Define $N_1 = N^+(S)$ and let $N_2 = V(D) - S - N_1$. Since S is a maximal independent set, it follows that $N_2 \subseteq N^-(S)$. Next, we partition the set N_1 into four subsets. First let $N_1' = N_1 \cap N^-(N_2)$ and note that the choice of S implies that $N_2 \subset N^+(N_1')$. Then define A as the set of vertices $v \in N_1 - N_1'$ such that there exists a directed path from v to some vertex of N_1', whose internal vertices are in $N_1 - N_1'$ (hence in A). Define B as that set of vertices $v \in N_1 - N_1' - A$ such that there exists a directed path from v to some vertex of S, whose internal vertices are in $N_1 - N_1' - A$ (hence in B). Finally let $C = N_1 - N_1' - A - B$. Then clearly $N_1 = N_1' \cup A \cup B \cup C$ is a partition of the set N_1.

Since no vertex of C is adjacent to a vertex of $V(D) - C$, it follows that each vertex v of C has out-degree at least 1 in the subdigraph $\langle C \rangle$ induced by C. Hence every vertex v of $\langle C \rangle$ that is not in a nontrivial strong component of $\langle C \rangle$ is contained in a path from v to some nontrivial strong component of $\langle C \rangle$. Let $C_1, C_2, ..., C_k (k \geq 1)$ denote the nontrivial strong components of $\langle C \rangle$ and let $v_i \in V(C_i)$ for each $i (1 \leq i \leq k)$. Further, let $M = \{v_1, v_2, ..., v_k\}$.

Next, we partition each of the sets A, B and C into two subsets. Recall that for each vertex v of A, there exists a path in $\langle A \cup N_1' \rangle$ from v to a vertex of N_1'. Let A_o denote the set of vertices of A whose distance to N_1' in $\langle A \cup N_1' \rangle$ is odd and let $A_e = A - A_o$, i.e., A_e is the set of those vertices of A whose distance to N_1' in $\langle A \cup N_1' \rangle$ is even. Recall also that for each vertex v of B, there is a path in $\langle B \cup S \rangle$ from v to some vertex of S. Let B_o denote the set of vertices of B whose distance to S in $\langle B \cup S \rangle$ is odd and let $B_e = B - B_o$. Finally, let C_o denote the set of vertices of $C - M$ whose distance to M in $\langle C \rangle$ is odd and let $C_e = C - C_o$.

Now the set $T_1 = N_1 \cup N_2$ is the complement of a maximal independent set and therefore twin dominating. The set $T_2 = S \cup N_1' \cup A_e \cup B_e \cup C_e$ is a twin dominating set of D since (1) each vertex $v \in A_o$ is in-dominated by its successor on the shortest $v - N_1'$ path and out-dominated by some vertex in S, (2) each vertex of B_o is, analogously, in-dominated by some vertex in $B_e \cup S$ and out-dominated by some vertex in S, (3) each vertex of C_o is, analogously, in-dominated by some vertex in C_e and out-dominated by some vertex in S and (4) each vertex of N_2 is out-dominated by some vertex in N_1' and in-dominated by some vertex in S. Similarly, the set $T_3 = S \cup N_2 \cup A_o \cup B_o \cup C_o$ is a twin dominating set.

Since every vertex v of D is contained in exactly two of the sets T_1, T_2, T_3, we have

$$3\gamma^*(D) \leq |T_1| + |T_2| + |T_3| = 2p, \tag{1}$$

and so $\gamma^*(D) \leq 2p/3$.

The sharpness of the bound is demonstrated by the graph obtained from the disjoint union of $\lfloor p/3 \rfloor - 1$ directed 3-cycles, and, depending on the value of $p(mod3)$, a directed 3-cycle, 4-cycle or 5-cycle. □

From the proof of Theorem 1 it follows that if $\gamma^*(D) = 2p/3$, then we have equality in (1) and thus each of the three sets T_1, T_2 and T_3 is a minimum twin dominating set of D. Since every vertex of D is in exactly two of the three sets, we obtain the following corollary.

Corollary 1 *Let D be a digraph of order p with $\delta(D) > 0$ and $\gamma^*(D) = 2p/3$. For every two vertices u and v of D, there exists a minimum twin dominating set of D containing both u and v.* □

Corollary 2 *Let D be a digraph of order p with $\delta(D) > 0$ and $\gamma^*(D) = (2p - 1)/3$. For each vertex v of D, there exists a minimum twin dominating set of D containing v.*

Proof. Suppose to the contrary that no minimum twin dominating set of D contains v. Let D_1 be the digraph obtained from D by adding a new vertex v' and joining it to and from v. Since no minimum twin dominating set of D contains v, we have

$$\gamma^*(D_1) = \gamma^*(D) + 1 = \frac{2p(D) - 1}{3} + 1 = \frac{2p(D_1)}{3}.$$

Hence, by Corollary 1, D_1 has a minimum twin dominating set T_1 containing both v and v'. This implies that $T_1 - \{v'\}$ is a minimum twin dominating set of D containing v, a contradiction. □

Corollary 1 cannot be extended to designate any three vertices in a twin dominating set of a digraph D of order $p \equiv 0 (mod 3)$ with $\gamma^*(D) = 2p/3$, even if D is strong. For example, if D consists of the disjoint union of three directed 3-cycles a_1, a_2, a_3, b_1, b_2, b_3 and c_1, c_2, c_3 and three additional arcs (a_1, b_1), (b_1, c_1) and (c_1, a_1), then the three vertices a_1, a_2, and a_3 do not all belong to one minimum twin dominating set.

We now know that every digraph D of order p with $\delta(D) > 0$ and $\gamma^*(D) = 2p/3$ has a minimum twin dominating set containing any two prescribed vertices u and v of D. Similarly one can prove that there is a minimum twin dominating set of D avoiding any prescribed vertex w of D. What we do not know, however, is whether there is a minimum twin dominating set of D that possesses these two characteristics simultaneously. That is, if D is a digraph of order p with every vertex having positive in-degree and positive out-degree and such that $\gamma^*(D) = 2p/3$, is it true for any three vertices u, v, and w of D, that there exists a minimum twin dominating set of D containing u and v and avoiding w?

We remark that Chartrand, Harary, and Quan-Yu [1] proved a related result for the out-domination and the in-domination number of a directed graph with no isolates. They proved that for every digraph of order p with no isolated vertices

$$\gamma^+(D) + \gamma^-(D) \leq \frac{4}{3}p.$$

For the case $\delta(D) > 0$, this result is implied by Theorem 1.

If the digraph D is transitive, then the bound given in Theorem 1 can be improved to $\gamma^*(D) \leq p/2$. This result can be generalised to transitive digraphs of given minimum degree as follows.

Theorem 2 *If D is a transitive digraph of order p and minimum degree δ, then*

$$\gamma^*(D) \leq \frac{p}{\delta + 1} \qquad (2)$$

and the bound is sharp.

Proof. Let $S \subset V(D)$ be a minimum twin dominating set of D. Consider a strong component C of D. First we show that $V(C) \cap S \neq \emptyset$ if and only if $min\{q(V(C), \overline{V(C)}), q(\overline{V(C)}, V(C))\} = 0$. If $(V(C), \overline{V(C)}) = \emptyset$ (or $(\overline{V(C)}, V(C)) = \emptyset$), then the vertices in C are in-dominated (or out-dominated) only by a vertex in C and thus $S \cap V(C) \neq \emptyset$. Now let C be a strong component of D such that $(V(C), \overline{V(C)}) \neq \emptyset$ and $(\overline{V(C)}, V(C)) \neq \emptyset$. We show that $V(C) \cap S = \emptyset$. Suppose that $V(C) \cap S$ contains a vertex c. Since the condensation of D (the digraph obtained from D by collapsing each strong component into a single vertex) is transitive, there exist strong components C_1, C_2, C_3 of D such that $(V(C_i), \overline{V(C_{i+1})})$ is not empty for $i = 1, 2$, $C = C_2$, $(\overline{V(C_1)}, V(C_1)) = \emptyset$, and $(V(C_3), \overline{V(C_3)}) = \emptyset$. Hence $S \cap V(C_3)$ contains a vertex v and by the transitivity of D every vertex in-dominated by c is also in-dominated by v. Similarly, $S \cap V(C_1)$ contains a vertex v' such that every vertex out-dominated by c is also out-dominated by v'. Hence $S - \{c\}$ is also a twin dominating set of D, contradicting the minimality of S. This proves that for every strong component C of D,

$$V(C) \cap S \neq \emptyset \quad \text{iff} \quad min\{q(V(C), \overline{V(C)}), q(\overline{V(C)}, V(C))\} = 0. \tag{3}$$

Since D is transitive, each strong component C is complete and symmetric and all vertices of C are adjacent to (and from) exactly the same vertices in D. Therefore $S \cap V(C)$ is either empty or contains exactly one vertex.

If C contains a vertex $v \in S$, then by (3) the out-neighbourhood or the in-neighbourhood of v is completely contained in $V(C)$. Hence $|V(C)| \geq \delta + 1$. Summation over all such strong components C of D yields

$$p \geq \sum_{C:V(C)\cap S\neq\emptyset} |V(C)| \geq \sum_{C:V(C)\cap S\neq\emptyset} (\delta + 1) = |S|(\delta + 1) = \gamma^*(D)(\delta + 1),$$

as desired.

To prove the sharpness of the above bound let k be any positive integer and consider the digraph D consisting of the disjoint union of k copies of the complete symmetric digraph of order $\delta + 1$. Then

$$\gamma^*(D) = k = \frac{p}{\delta + 1},$$

where $p = k(\delta + 1)$ is the order of D. □

We conclude this section with a Nordhaus-Gaddum type bound for $\gamma^*(D)$. In the proof we use the concept of a ***private neighbour***. Let S be a twin dominating set of a digraph D and let $v \in S$. A vertex v' of D is a ***private out-neighbour*** of v if v' is not out-dominated by any vertex of $S - v$. (Note that v can be a private out-neighbour of itself.) A ***private in-neighbour*** of v is defined analogously. A ***private neighbour*** of v is a vertex that is a private in-neighbour or a private out-neighbour of v. It is easy to see that every vertex of a minimum twin dominating set of a digraph D has a private neighbour.

Theorem 3 *Let D be a digraph of order $p \geq 2$ and let $\overline{D}$ be its complement. Then*

$$3 \leq \gamma^*(D) + \gamma^*(\overline{D}) \leq p + 2, \tag{4}$$

$$2 \leq \gamma^*(D)\gamma^*(\overline{D}) \leq \left\lfloor \frac{p+2}{2} \right\rfloor \left\lceil \frac{p+2}{2} \right\rceil, \tag{5}$$

and both inequalities are sharp.

Proof. (i) We first prove the lower bound in (4). Note that $\gamma^*(D) = 1$ if and only if D has a vertex that is adjacent to and from every other vertex in D. Since D and $\overline{D}$ cannot both have this property, we have $\gamma^*(D) \geq 2$ or $\gamma^*(\overline{D}) \geq 2$, implying the lower bound in (4).

We now prove the upper bound in (4). Suppose D is a counter example. Without loss of generality we can assume that $\gamma^*(D) \geq \gamma^*(\overline{D})$.

In the sequel let $k = \gamma^*(D)$ and $S = \{s_1, s_2, ..., s_k\}$ denote a minimum twin dominating set of D. For every $s_i \in S$ there exists a private neighbour t_i. Note that $s_i = t_i$ is possible. Let $T = \{t_1, t_2, \ldots, t_k\}$.

CASE 1: There exist $s_i, s_j \in S$ such that s_i has a private out-neighbour t_i and s_j has a private in-neighbour t_j in D.

No vertex in S except s_i out-dominates t_i and no vertex in S except s_j in-dominates t_j. Thus in $\overline{D}$, the set $\{t_i, t_j\}$ twin dominates every vertex in $S - \{s_i, s_j\}$. Hence $(V - S) \cup \{t_i, t_j, s_i, s_j\}$ twin dominates $\overline{D}$.

We claim that the addition of the set $\{t_i, t_j, s_i, s_j\}$ increases the number of elements of $V - S$ by at most 2. Either $t_i \neq s_i$ and thus $t_i \in V - S$ or $t_i = s_i$. In either case, t_i and s_i (and similarly t_j and s_j) contribute at most one new element to $V - S$. Therefore

$$\gamma^*(D) + \gamma^*(\overline{D}) \leq |S| + |(V - S) \cup \{t_i, t_j, s_i, s_j\}| \leq |S| + |V - S| + 2 = p + 2.$$

CASE 2: No vertex of S has a private in-neighbour.

If $|S \cap T| \leq 1$ then, by our assumption $\gamma^*(\bar{D}) \leq \gamma^*(D)$,

$$\gamma^*(D) + \gamma^*(\bar{D}) \leq 2\gamma^*(D) = 2|S| \leq |S \cup T| + 1 \leq p + 1.$$

Hence we can assume that $|S \cap T| \geq 2$. Every vertex $t_i \in T - S$ is adjacent from s_i and no other vertex in S. Every vertex $t_i \in T \cap S$ is, since $t_i = s_i$, its own private out-neighbour in S and therefore not adjacent from any other vertex in S. This proves

$$(S \cap T, T)_D = \emptyset. \tag{6}$$

We first show that

$$V - S \text{ in-dominates } S \text{ in } \bar{D}. \tag{7}$$

It suffices to show that $|T - S| \geq 2$, since each vertex $t_i \in T - S \subset V - S$ in-dominates every vertex of $S - \{s_i\}$ in $\bar{D}$. Suppose $|T - S| \leq 1$. Then $|S - T| \leq 1$ and there exists a vertex $s_i \in S \cap T$. By (6), s_i twin dominates each vertex of $S \cap T$ in $\bar{D}$. Thus the set $(V - S) \cup \{s_i\} \cup (S - T)$ twin dominates $\bar{D}$, which implies that

$$\gamma^*(D) + \gamma^*(\bar{D}) \leq |S| + |(V - S) \cup \{s_i\} \cup (S - T)| = p + 2,$$

and D is not a counter example. This contradiction proves $|T - S| \geq 2$ and hence (7).

We next show that each vertex of D is in-dominated by at least 3 vertices of S, or, in other words,

$$|N_{\bar{D}}^+[v] \cap S| \geq 3 \text{ for each } v \in V(D). \tag{8}$$

Suppose there exists a vertex v with $|N_{\bar{D}}^+[v] \cap S| \leq 2$. Then v out-dominates each vertex of $S - N_{\bar{D}}^+[v]$ in $\bar{D}$ and thus, by (7), $(V - S) \cup N_{\bar{D}}^+[v]$ twin-dominates S in $\bar{D}$ and is thus a twin-dominating set of $\bar{D}$. Hence

$$\gamma^*(D) + \gamma^*(\bar{D}) \leq |S| + |(V - S) \cup (N_{\bar{D}}^+[v] \cap S)| \leq p + 2,$$

a contradiction. This proves (8).
If, for any two vertices $s_i, s_j \in S \cap T$, the set $\{s_i, s_j\}$ twin-dominates T in $\bar{D}$, then $(V - T) \cup \{s_i, s_j\}$ is a twin-dominating set of $\bar{D}$ with $p - \gamma^*(D) + 2$ vertices, which contradicts our assumption $\gamma^*(D) + \gamma^*(\bar{D}) > p + 2$. Hence there exists a vertex $t_{i,j} \in T$ which is, in D, either a common in-neighbour or a common out-neighbour of s_i and s_j. By (6) we have $t_{i,j} \in T - S$ and $t_{i,j} \in N_D^-(s_i) \cap N_D^-(s_j)$. Consider the set $S_{i,j} = S - \{s_i, s_j\} \cup \{t_{i,j}\}$. Since by (8) each vertex of D is in-dominated by at least three vertices, $S_{i,j}$ is an in-dominating set of D. Since $|S_{i,j}| < \gamma^*(D)$, there exists a vertex $v_{i,j}$ which is not out-dominated in D by $S_{i,j}$. Obviously $v_{i,j} \notin S \cup T$. Since $v_{i,j}$ is adjacent in D from s_i and s_j and from no other vertex, we have $v_{i,j} \neq v_{i',j'}$ if $\{s_i, s_j\} \neq \{s_{i'}, s_{j'}\}$. Hence

$$|V - (S \cup T)| \geq \left|\{v_{i,j} \mid \{s_i, s_j\} \subset S \cap T,\ s_i \neq s_j\}\right| = \binom{|S \cap T|}{2},$$

which in turn implies

$$\begin{aligned} p &= |S \cup T| + |V - (S \cup T)| \\ &\geq 2\gamma^*(D) - |S \cap T| + \binom{|S \cap T|}{2} \\ &= 2\gamma^*(D) + \frac{1}{2}|S \cap T|(|S \cap T| - 3) \\ &\geq 2\gamma^*(D) - 1. \end{aligned}$$

The last estimate follows from $|S \cap T| \geq 2$. In conjunction with our assumption $\gamma^*(\bar{D}) \leq \gamma^*(D)$, we obtain $\gamma^*(D) + \gamma^*(\bar{D}) \leq p + 1$. This final contradiction proves Case 2.

CASE 3: No vertex of S has a private out-neighbour.

This case is analogous to Case 2.

(ii) The lower bound of (5) follows immediately from the fact that $\gamma^*(D) \geq 2$ or $\gamma^*(\overline{D}) \geq 2$. Using (4) we obtain the upper bound:

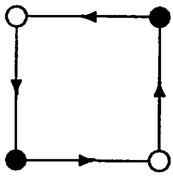
$$\gamma^*(D)\gamma^*(\overline{D}) \leq \left\lfloor \frac{\gamma^*(D) + \gamma^*(\overline{D})}{2} \right\rfloor \left\lceil \frac{\gamma^*(D) + \gamma^*(\overline{D})}{2} \right\rceil \leq \left\lfloor \frac{p+2}{2} \right\rfloor \left\lceil \frac{p+2}{2} \right\rceil$$

Both lower bounds are attained by the digraph D obtained from the star $K_{1,p-1}$ by replacing each edge by two arcs in opposite directions. Both upper bounds are attained by the digraph D obtained from the disjoint union of a complete graph $K_{\lfloor p/2 \rfloor}$ and an empty graph $\lceil p/2 \rceil K_1$ by replacing each edge by two arcs in opposite directions and adding arcs from every vertex in the complete graph to every vertex in the empty graph. □

3 Orientations of Graphs

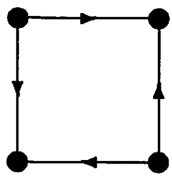
It is not difficult to see that the twin domination number of the directed 4-cycle is 2 (any two nonadjacent vertices form a minimum twin dominating set); while the twin domination number of the antidirected 4-cycle (in which every vertex has in-degree 0 or out-degree 0) is 4 (see below).

D_1 :



$\gamma^*(D_1) = 2$

D_2 :



$\gamma^*(D_2) = 4$

This example shows that distinct orientations of a graph can have distinct twin domination numbers. This suggests the following concepts. For a graph G, the ***lower orientable twin domination number*** $dom^*(G)$ of G is defined by

$$dom^*(G) = \min\{\gamma^*(D) | D \text{ is an orientation of } G\},$$

while the ***upper orientable twin domination number*** $DOM^*(G)$ of G is defined by

$$DOM^*(G) = \max\{\gamma^*(D) | D \text{ is an orientation of } G\}.$$

Corresponding concepts have been defined and studied for orientable domination (out-domination) in graphs [3]. Clearly, $dom^*(G) \leq DOM^*(G)$ for every graph G. Obviously, then, $dom^*(C_4) = 2$ and $DOM^*(C_4) = 4$.

If G is a graph of order p, then certainly $DOM^*(G) \leq p$. It is not hard to prove that equality holds if and only if G is bipartite. The key argument in the proof is the fact that every orientation of an odd cycle contains a vertex v whose out-degree and in-degree equal 1, so that $V - \{v\}$ is a twin-dominating set of order less than p. The same argument proves that if G contains k vertex disjoint odd cycles, then $DOM^*(G) \leq p - k$. We note that equality does not necessarily hold, as demonstrated by the Petersen graph P, which contains only two vertex disjoint odd cycles.

$$dom^*(P) = 4 \text{ and } DOM^*(P) = 7.$$

In general, determining the lower orientable twin domination number and the upper orientable twin domination number seems to be a difficult, possibly NP-hard, problem.

Finally, we note that an "Intermediate Value Theorem" for orientable twin domination holds: For every integer c with $dom^*(G) \leq c \leq DOM^*(G)$ there exists an orientation D of G such that $\gamma^*(D) = c$. Its proof is identical to the proof of the corresponding result for out-domination given in [3].

References

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