The anatomy of magic squares

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Abstract

We give a parametric representation for generic magic squares. This makes it relatively easy to construct magic squares having desired properties. It also suggests a convenient method for generating and classifying all the magic squares of every given order.

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1. Magic squares.

A magic square is a square array of order greater than two whose entries are taken from a set of consecutive whole numbers – beginning from 1 – with the property that the numbers in any row, column or diagonal of the array add up to the same sum.

Magic squares were discovered in China as far back as 2200 BC from where they were subsequently introduced into India, Japan and – much later – Europe. For centuries, magic squares have been a source of exciting mathematical amusements and challenging problems, many of which have remained unsolved to this day (cf. [1]).

Classical methods for constructing magic squares can be found in [2–5,8,11,12,14,15]. These methods usually start off with some initial square array of numbers, from which they generate magic squares by moving numbers from one cell to another according to some predetermined rule.

It is well known that there are only 8 magic squares of order three, each of which can be obtained as reflections and/or rotations of any of the others.

Magic squares of order four were first listed and counted by Bernard Frénicle de Bessy in 1693 who gave the final census figure as 7040. This census was repeated in more recent times by Lehrer [13] who obtained the same figure.

A census of magic squares of order 5 was carried out by Richard Schroeppel in 1973 with the aid of a computer program that ran for 100 hours on a PDP-11 Computer. His final census figure – as reported in [9] – was 2,202,441,792 magic squares of order five.

We must remark here that many authors follow the practice – which we do not adopt here, because of the algebraic context we work in – of identifying a magic square with all other magic squares obtainable from it by reflections and rotations. Such authors will obviously give magic-square census figures which are smaller than ours by a factor of 8.

It appears that the census of magic squares of order six and above are open problems.

In the next section of this paper we define Euler squares and show how they can be used to characterize a class of magic squares which we refer to as normal magic squares. In the last section of the paper we introduce the concept of quasi-Euler squares and show that generic magic squares can be decomposed in terms of complementary orthogonal pairs of such squares. We derive a parametric representations for generic quasi-Euler squares, which leads in a natural way to a parametric representation for generic magic squares. We also indicate how these representations could be used to design efficient algorithms for the census of magic squares of any desired order. In addition to making it possible to construct quickly magic squares having desired properties, this approach also suggests a convenient method for classifying magic squares.

In the sequel, given any integers a and b, $[a]$ will designate the largest integer less than or equal to a, a Mod b will designate the remainder when b divides a, and a Div b will denote the integer $[a/b]$.

2. Magic squares and Euler squares.

A $p \times p$ square array is called a Latin square if each entry is an element of the set $Z_p = \{0, 1, \ldots, p-1\}$, with each entry occuring exactly once in each row and once in each column. If, in addition, the numbers in any row, column or diagonal of the array add up to the same sum, then we refer to the Latin square as a special Latin square. Two $p \times p$ Latin squares $A = (a_{ij})$ and $B = (b_{ij})$ are said to be orthogonal if

$$\{(a_{ij}, b_{ij}) : i, j = 1, 2, \ldots, p\} = Z_p \times Z_p. \tag{2.1}$$

Euler [7] seems to have been the first person to realize that every pair (A, B) of orthogonal special Latin squares gives rise to a magic square

$$M = E + A + pB, \tag{2.2}$$

where E is the $p \times p$ matrix having 1 in all entries.

Consider the orthogonal pair of special Latin squares

$$A = \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array} \qquad B = \begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array}\,. \tag{2.3}$$

The associated magic square obtained by substituting these special Latin squares in Euler's formula (2.2) is the array

4	9	2
3	5	7
8	1	6

.

Conversely, if we express this magic square in the form $M = (m_{ij})$, we can recover the orthogonal special Latin square pair $A = (a_{ij})$ and $B = (b_{ij})$ in (2.3) from the equations

$$a_{ij} = (m_{ij} - 1) \text{ Mod } p \tag{2.4}$$

$$b_{ij} = (m_{ij} - 1) \text{ Div } p. \tag{2.5}$$

So we can generate orthogonal pairs of special Latin squares from some magic squares.

That this cannot be done which all magic squares is evident on considering the order-6 magic square

$M =$

31	2	3	34	5	36
30	26	9	10	29	7
24	23	16	15	14	19
13	17	22	21	20	18
12	8	28	27	11	25
1	35	33	4	32	6

.

When applied to this magic square, equations (2.4) and (2.5) yield the arrays

$A =$

0	1	2	3	4	5
5	1	2	3	4	0
5	4	3	2	1	0
0	4	3	2	1	5
5	1	3	2	4	0
0	4	2	3	1	5

$B =$

5	0	0	5	0	5
4	4	1	1	4	1
3	3	2	2	2	3
2	2	3	3	3	2
1	1	4	4	1	4
0	5	5	0	5	0

which are not Latin squares. However, each of them contains every element of Z_6 six times, and the pair (A, B) satisfies the orthogonality relation (2.1). Furthermore, the numbers in each row, column or diagonal of A and B add up to the same sum. In the sequel we will refer to such arrays as Euler squares.

Thus an Euler square of order p is a $p \times p$ array whose entries are taken from Z_p in such a way that each entry occurs p times, with the further property that the numbers in any row, column, or diagonal of the array add up to the same sum. Every orthogonal pair (A, B) of such squares yields a magic square through the formula in (2.2).

In the sequel, given a $p \times p$ magic square, we will refer to the matrices A, B defined in (2.4) and (2.5) as its canonical components, and to the expression in (2.2) as its canonical form.

Are the canonical components of a magic square always Euler squares? This question can be answered in the negative by considering the order-4 magic square

$$M = \begin{array}{|c|c|c|c|}\hline 16 & 3 & 10 & 5 \\ \hline 1 & 12 & 7 & 14 \\ \hline 8 & 13 & 2 & 11 \\ \hline 9 & 6 & 15 & 4 \\ \hline \end{array}.$$

The canonical components of this magic square are the arrays

$$A = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 0 \\ \hline 0 & 3 & 2 & 1 \\ \hline 3 & 0 & 1 & 2 \\ \hline 0 & 1 & 2 & 3 \\ \hline \end{array}, \qquad B = \begin{array}{|c|c|c|c|}\hline 3 & 0 & 2 & 1 \\ \hline 0 & 2 & 1 & 3 \\ \hline 1 & 3 & 0 & 2 \\ \hline 2 & 1 & 3 & 0 \\ \hline \end{array}$$

which are *not* Euler squares.

Which magic squares have Euler squares as their canonical components? This class – referred to as normal magic squares – has been studied in [16]. Such magic square occur in associated pairs, for if A and B are an orthogonal pair of Euler squares, then both $E + A + pB$ and $E + B + pA$ are normal magic squares.

3. Magic squares and quasi-Euler squares.

In this section we study the canonical decomposition (2.2) in the general case when M is a generic magic square. The following result gives an insight into the nature of the canonical components A and B.

Theorem 1. Let $M = (m_{ij})$ be a $p \times p$ magic square with canonical form (2.2). Then there exist integers $r_1, r_2, \ldots, r_{2p+2}$ such that

$$|r_i| < (p-1)/2, \quad i = 1, 2, \ldots, 2p+2 \tag{3.1}$$

$$\sum_{i=1}^{p} r_i = \sum_{i=1}^{p} r_{p+1+i} = 0, \tag{3.2}$$

$$\begin{aligned} pr_i + \textstyle\sum_{j=1}^{p} a_{ij} &= pr_{p+1+i} + \textstyle\sum_{j=1}^{p} a_{ji} = pr_{2p+2} + \textstyle\sum_{j=1}^{p} a_{jj} \\ &= pr_{p+1} + \textstyle\sum_{j=1}^{p} a_{j\,p+1-j} = p(p-1)/2. \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
-r_i + \textstyle\sum_{j=1}^{p} b_{ij} &= -r_{p+1+i} + \textstyle\sum_{j=1}^{p} b_{ji} = -r_{2p+2} + \textstyle\sum_{j=1}^{p} b_{jj} \\
&= -r_{p+1} + \textstyle\sum_{j=1}^{p} b_{j\ p+1-j} = p(p-1)/2.
\end{aligned} \tag{3.4}
$$

Moreover, each element of the set $Z_p = \{0, 1, \ldots, p-1\}$ occurs p times in $A = (a_{ij})$ and p times in $B = (b_{ij})$, and A and B satisfy the orthogonality condition (2.1).

Proof: It is evident from the definitions of $A = (a_{ij})$ and $B = (b_{ij})$ that

$$\{(a_{ij}, b_{ij}) : i, j = 1,\, 2,\, \ldots,\, p\} \subseteq Z_p^2.$$

However, given any $(u, v) \in Z_p^2$, we have $1 + u + pv \in \{1, 2, \ldots, p^2\}$. Since M is a magic square, there exists a unique pair (i, j) of indices such that $m_{ij} = 1+u+pv = 1+a_{ij}+pb_{ij}$. Consequently, we see that $(a_{ij}, b_{ij}) = (u, v)$. That shows that $Z_p^2 \subseteq \{(a_{ij}, b_{ij}) \mid i, j = 1,\, 2,\, \ldots,\, p\}$ and implies that (2.1) holds, and hence that A and B are orthogonal.

It follows from equation (2.2) that

$$
\begin{aligned}
&p + \sum_{j=1}^{p} a_{ij} + p\sum_{j=1}^{p} b_{ij} = \sum_{j=1}^{p} m_{ij} = p(p^2+1)/2, \quad i = 1, \ldots, p, \\
&p + \sum_{j=1}^{p} a_{ji} + p\sum_{j=1}^{p} b_{ji} = \sum_{j=1}^{p} m_{ji} = p(p^2+1)/2, \quad i = 1, \ldots, p, \\
&p + \sum_{j=1}^{p} a_{jj} + p\sum_{j=1}^{p} b_{jj} = \sum_{j=1}^{p} m_{jj} = p(p^2+1)/2, \\
&p + \sum_{j=1}^{p} a_{j\ p+1-j} + p\sum_{j=1}^{p} b_{j\ p+1-j} = \sum_{j=1}^{p} m_{j\ p+1-j} = p(p^2+1)/2,
\end{aligned}
$$

and hence that

$$
\begin{aligned}
\sum_{j=1}^{p} a_{ij} + p\sum_{j=1}^{p} b_{ij} &= p(p-1)/2 + p^2(p-1)/2, \quad i = 1, \ldots, p, \\
\sum_{j=1}^{p} a_{ji} + p\sum_{j=1}^{p} b_{ji} &= p(p-1)/2 + p^2(p-1)/2, \quad i = 1, \ldots, p, \\
\sum_{j=1}^{p} a_{jj} + p\sum_{j=1}^{p} b_{jj} &= p(p-1)/2 + p^2(p-1)/2, \\
\sum_{j=1}^{p} a_{j\ p+1-j} + p\sum_{j=1}^{p} b_{j\ p+1-j} &= p(p-1)/2 + p^2(p-1)/2.
\end{aligned}
$$

On setting

$$\begin{aligned} r_i &= \sum_{j=1}^{p} b_{ij} - p(p-1)/2, \quad i = 1, \ldots, p, \\ r_{p+1+i} &= \sum_{j=1}^{p} b_{ji} - p(p-1)/2, \quad i = 1, \ldots, p, \\ r_{2p+2} &= \sum_{j=1}^{p} b_{jj} - p(p-1)/2, \\ r_{p+1} &= \sum_{j=1}^{p} b_{j\ p+1-j} - p(p-1)/2, \end{aligned}$$

in these equations we immediately obtain the equations in (3.3) and (3.4).

Since $\{m_{ij} \mid i, j = 1, 2, \ldots, p\} = \{1, 2, \ldots, p^2\}$, it follows from the definitions of $A = (a_{ij})$ and $B = (b_{ij})$ in (2.4) and (2.5) that each element of Z_p occurs p times in $\{a_{ij}\}$ and p times in $\{b_{ij}\}$. Consequently, we have

$$\sum_{i=1}^{p} \sum_{j=1}^{p} b_{ij} = p(0 + 1 + \ldots + p - 1) = p^2(p-1)/2.$$

On the other hand, (3.4) implies that

$$\sum_{i=1}^{p} \sum_{j=1}^{p} b_{ij} = p^2(p-1)/2 + \sum_{i=1}^{p} r_i.$$

from which we conclude that $\sum_{i=1}^{p} r_i = 0$, proving one part of (3.2). The second part of (3.2) follows similarly.

For any fixed $1 \le i \le p$, it is evident that $\sum_{j=1}^{p} a_{ij} \le p(p-1)$. Therefore, it follows from (3.3) that

$$-p(p-1)/2 = p(p-1)/2 - p(p-1) \le pr_i = p(p-1)/2 - \sum_{j=1}^{p} a_{ij} \le p(p-1)/2$$

and hence that $|r_i| \le (p-1)/2$. If $r_i = (p-1)/2$, then it follows from (3.3) and (3.4) that $\sum_{j=1}^{p} a_{ij} = 0$ and $\sum_{j=1}^{p} b_{ij} = (p^2-1)/2$ and hence that the ith row of A consists of zeros. By orthogonality, the ith row of B must contain each element of Z_p once, and hence we must have $\sum_{j=1}^{p} b_{ij} = 0 + 1 + \ldots + (p-1) = p(p-1)/2 = (p^2-1)/2$, an evident contradiction. In a similar way, it can be shown that $r_i \neq -(p-1)/2$, and hence, that $|r_i| < (p-1)/2$. The inequalities $|r_{p+1+i}| < (p-1)/2$, $|r_{2p+2}| < (p-1)/2$

and $|r_{p+1}| < (p-1)/2$ are proved similarly. This proves (3.1) and completes the proof of the Theorem. □

The equations in (3.3) and (3.4) are of the form

$$\begin{aligned} zr_i + \textstyle\sum_{j=1}^p l_{ij} &= zr_{p+1+i} + \textstyle\sum_{j=1}^p l_{ji} = zr_{2p+2} + \textstyle\sum_{j=1}^p l_{jj} \\ &= zr_{p+1} + \textstyle\sum_{j=1}^p l_{j\ p+1-j} = p(p-1)/2. \end{aligned} \tag{3.5}$$

where $z = -1$ or $z = p$ and the r_i satisfy (3.2). This is a system of $2p+2$ linear equations. However, these equations are not independent, for if

$$zr_i + \sum_{j=1}^{p} l_{ij} = zr_{p+1+k} + \sum_{j=1}^{p} l_{jk} = p(p-1)/2, \quad i = 1, \ldots, p,\ k = 2, \ldots, p,$$

then

$$\begin{aligned} \sum_{j=1}^{p} l_{j1} &= \sum_{i=1}^{p}\sum_{j=1}^{p} l_{ij} - \sum_{k=2}^{p}\sum_{j=1}^{p} l_{jk} \\ &= p^2(p-1)/2 - p(p-1)(p-1)/2 + \sum_{k=2}^{p} zr_{p+1+k} \\ &= p(p-1)/2 - zr_{p+2}. \end{aligned}$$

This implies that the equation $\sum_{j=1}^{p} l_{j1} = p(p-1)/2 - zr_{p+2}$ is redundant, and hence, that (3.5) contains only $2p+1$ independent linear equations for the p^2 unknowns (l_{ij}). The general solution will depend on z, $r = (r_1, r_2, \ldots, r_{2p+2})$ and on some free parameters of the form $s = (s_1, s_2, \ldots, s_q)$, where $q = p^2 - 2p - 1$.

In the sequel, we will refer to any matrix solution $L(r; s; z) = (l_{ij})$ of (3.5) whose entries are elements of Z_p as a quasi-Euler square. Two matrix solutions corresponding to the two distinct values $z = -1$ and $z = p$ of the z parameter will be said to be complementary. The main conclusion of Theorem 1 can then be expressed by saying that the canonical components (2.3) – (2.4) of every magic square are a complementary orthogonal pair of quasi-Euler squares.

If $L(r; s; z)$ is the generic literal quasi-Euler square that solves equation (3.5), then the generic magic square of order p will be expressible in the form

$$M(r; s; t) = E + L(r; s; p) + pL(r; t; -1) \tag{3.6}$$

where $s = (s_1, \ldots, s_q)$ and $t = (t_1, \ldots, t_q)$ are taken from the set Z_p^q, and the components of $r = (r_1, \ldots, r_{2p+2})$ satisfy (3.1) and (3.2).

The magic square in (3.6) is normal if and only if $r = (0, \ldots, 0)$. It follows from (3.1) that when $p = 3$, this condition always holds, and hence magic squares of order three are always normal.

The canonical decomposition (3.6) expresses the basic structure of the canonical decomposition of generic magic squares. It turns out that the general solution $L(r;s;z) = (l_{ij})$ of (3.5) can be obtained explicitly, thereby getting an even deeper insight into the decomposition (3.6).

When $p = 3$, then $q = 2$ and $2p + 2 = 8$, and we can easily verify that the general solution of (3.5) is a matrix $L(r_1, \ldots, r_8; s_1, s_2; z)$ given by the expression:

$2 - s_2 + z(-r_1 - r_3 + r_4 + r_6 - 2r_8)/3$	$2 - s_1 + z(-r_1 - r_3 + r_4 - 2r_6 + r_8)/3$	$-1 + s_1 + s_2 + z(2r_1 + 3r_2 + 5r_3 - 2r_4 - 3r_5 - 2r_6 - 3r_7 + r_8)/2$
$-2 + s_1 + 2s_2 + z(r_1 + 4r_3 - r_4 - 3r_5 - r_6 + 2r_8)/3$	$1 + z(r_1 + r_3 - r_4 - r_6 - r_8)/3$	$4 - s_1 - 2s_2 + z(-2r_1 - 3r_2 - 5r_3 + 2r_4 + 3r_5 + 2r_6 - r_8)/3$
$3 - s_1 - s_2 - zr_3$	s_1	s_2

The general solution of (3.5) when $p > 3$ is given in the next theorem.

Theorem 2: Given a quasi-Euler square $L = L(r;s;z) = (l_{ij})$, of order p (with $p > 3$), let

$$s_{j-1+(p-2)(p-1)} = l_{2j}, \quad j = 2, \ldots, p-2 \qquad (3.7a)$$

$$s_{j-1+(p-i)(p-1)} = l_{ij}, \quad j = 2, \ldots, p, \; i = 3, \ldots, p, \qquad (3.7b)$$

(this defines $s = (s_1, \ldots, s_q)$, with $q = p^2 - 2p - 1$). Then the remaining coefficients of L are given in terms of $s = (s_1, \ldots, s_q)$ by the expressions:

$$\begin{aligned}
l_{i1} &= p(p-1)/2 - zr_i - \sum_{j=2}^{p} s_{j-1+(p-i)(p-1)}, \; i = 3, \ldots, p, \\
l_{11} &= p(p-1)/2 - zr_{2p+2} - \sum_{j=2}^{p} s_{j-1+(p-j)(p-1)} \\
l_{21} &= -p(p-1)(p-2)/2 - z(r_1 + r_2 + r_{p+2} - r_{2p+2}) \\
&+ \sum_{i=3}^{p} s_{i-1+(p-i)(p-1)} + \sum_{i=1}^{q-p+4} s_i \\
l_{1j} &= p(p-1)/2 - zr_{p+1+j} - \sum_{i=2}^{p} s_{j-1+(p-i)(p-1)}, \; j = 2, \ldots, p-2, \\
l_{2\,(p-1)} &= \Bigg[p(p-1)(p-2)/2 - z(r_{p+1} + r_{2p+2} - r_1 - r_p - r_{p+2} - r_{2p+1}) \\
&- \sum_{j=2}^{p-1} s_{j-1+(p-j)(p-1)} - \sum_{j=2}^{p-2} s_{(p-j)(p-1)-1} \\
&- \sum_{j=3}^{p-1} s_{p(p-j)} - \sum_{j=2}^{p-2} \sum_{i=2}^{p-1} s_{j-1+(p-i)(p-1)} \Bigg] /2
\end{aligned}$$

$$
\begin{aligned}
l_{2p} &= \Bigg[p^2(p-1)/2 - z(r_p - r_{p+1} + r_{2p+2} - r_1 - r_{p+2} + r_{2p+1}) \\
&- \sum_{j=2}^{p-1} s_{j-1+(p-j)(p-1)} - \sum_{j=2}^{p-2} s_{(p-j)(p-1)-1} + \sum_{j=3}^{p-1} s_{p(p-j)} \\
&- \sum_{j=2}^{p-2}\sum_{i=2}^{p-1} s_{j-1+(p-i)(p-1)} - 2\sum_{i=1}^{p-1} s_i - 2\sum_{i=3}^{p} s_{(p-i+1)(p-1)}\Bigg] /2 \\
l_{1p} &= \Bigg[p(p-1)(2-p)/2 - z(r_{p+1} - r_{2p+2} + r_1 - r_p + r_{p+2} + r_{2p+1}) \\
&+ \sum_{j=2}^{p-1} s_{j-1+(p-j)(p-1)} + \sum_{j=2}^{p-2} s_{(p-j)(p-1)-1} - \sum_{j=3}^{p-1} s_{p(p-j)} \\
&+ \sum_{j=2}^{p-2}\sum_{i=2}^{p-1} s_{j-1+(p-i)(p-1)} + 2\sum_{i=1}^{p-1} s_i\Bigg] /2 \\
l_{1\,(p-1)} &= \Bigg[p(p-1)(4-p)/2 - z(2r_{2p} - r_{p+1} - r_{2p+2} + r_1 + r_p + r_{p+2} + r_{2p+1}) \\
&+ \sum_{j=2}^{p-1} s_{j-1+(p-j)(p-1)} + \sum_{j=2}^{p-2} s_{(p-j)(p-1)-1} + \sum_{j=3}^{p-1} s_{p(p-j)} \\
&+ \sum_{j=2}^{p-2}\sum_{i=2}^{p-1} s_{j-1+(p-i)(p-1)} - 2\sum_{i=3}^{p} s_{(p-i+1)(p-1)-1}\Bigg] /2
\end{aligned}
$$

Proof: On using the definitions in (3.7) and the fact that L is a quasi-Euler square, we obtain relations:

$$l_{i1} = p(p-1)/2 - zr_i - \sum_{j=2}^{p} l_{ij}, \quad i = 3, \ldots, p, \tag{3.8a}$$

$$l_{11} = p(p-1)/2 - zr_{2p+2} - \sum_{j=2}^{p} l_{jj}, \tag{3.8b}$$

$$l_{21} = p(p-1)/2 - zr_{p+2} - l_{11} - \sum_{i=3}^{p} l_{i1}, \tag{3.8c}$$

$$l_{1j} = p(p-1)/2 - zr_{p+1+j} - \sum_{i=2}^{p} l_{ij}, \quad j = 2, \ldots, p-2. \tag{3.8d}$$

In these relations, each quantity is expressed in terms of other quantities that had – because of (3.7) – been previously expressed in terms of $s = (s_1, \ldots, s_q)$.

The remaining terms $l_{1\,p-1}$, l_{1p}, $l_{2\,p-1}$ and l_{2p} satisfy the equations

$$\begin{aligned} l_{1\,p-1} + l_{2\,p-1} &= p(p-1)/2 - zr_{2p} - \sum_{i=3}^{p} l_{i\,p-1} \equiv \lambda_1, \\ l_{2\,p-1} + l_{2p} &= p(p-1)/2 - zr_2 - \sum_{j=1}^{p-2} l_{2j} \equiv \lambda_2, \\ l_{1p} + l_{2\,p-1} &= p(p-1)/2 - zr_{p+1} - \sum_{i=3}^{p} l_{i\,p+1-i} \equiv \lambda_3, \\ l_{1\,p-1} + l_{1p} &= p(p-1)/2 - zr_1 - \sum_{i=1}^{p-2} l_{1j} \equiv \lambda_4. \end{aligned}$$

On solving the system we obtain the expressions

$$\begin{aligned} l_{1\,p-1} &= (\lambda_1 - \lambda_3 + \lambda_4)/2, \\ l_{1p} &= (-\lambda_1 + \lambda_3 + \lambda_4)/2, \\ l_{2\,p-1} &= (\lambda_1 + \lambda_3 - \lambda_4)/2, \\ l_{2p} &= (-\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_4)/2. \end{aligned}$$

The terms λ_1, λ_2, λ_3, λ_4 can be expressed in terms of $s = (s_1, \ldots, s_q)$ by using (3.7) and (3.8). On substituting the resulting terms in the expressions above and making use of (3.2) we obtain all entries of $L(r; s; z)$ as functions of $s = (s_1, \ldots, s_q)$, exactly as stated in the Theorem. This expression is unique for fixed values of r and z, for if $L = L(r; s; z) = L(r; t; z)$, with $s, t \in Z_p^q$, then it follows from (3.7) that for all $k = 1, \ldots, q$ we have $s_k = t_k = l_{ij}$, with $i = p - [(k-1) \text{ Div } (p-1)]$ and $j = 2 + [(k-1) \text{ Mod } (p-1)]$. Hence $s = t$. That completes the proof of the theorem. □

When $p > 3$, it is convenient to express the literal quasi-Euler square $L(r; s; z) = (l_{ij})$ of order p in the following compact form (useful for purposes of computation):

$$\begin{aligned} l_{ij} &= s_k, \quad k = 1, \ldots, (p-1)^2 - 2 \\ &\qquad i = p - [(k-1) \text{ Div } (p-1)], \\ &\qquad j = 2 + [(k-1) \text{ Mod } (p-1)], \\ l_{i1} &= p(p-1)/2 - zr_i - \sum_{j=2}^{p} l_{ij}, \quad i = 3, \ldots, p \end{aligned}$$

$$l_{11} = p(p-1)/2 - zr_{2p+2} - \sum_{j=2}^{p} l_{jj}$$

$$l_{21} = p(p-1)/2 - zr_{p+2} - l_{11} - \sum_{i=3}^{p} l_{i1}$$

$$l_{1j} = p(p-1)/2 - zr_{p+1+j} - \sum_{i=2}^{p} l_{ij}, \quad j = 2, \ldots, p-2$$

$$l_{2\,(p-1)} = \frac{1}{2}\Bigg\{ p(p-1)/2 - z(r_{2p} + r_{p+1} - r_1) - l_{p1}$$

$$+ \sum_{j=1}^{p-2} l_{1j} - \sum_{j=2}^{p-1} s_{(p-j)(p-1)-1} - \sum_{j=3}^{p-1} s_{p(p-j)} \Bigg\}$$

$$l_{2p} = p(p-1)/2 - zr_2 - l_{21} - \sum_{j=2}^{p-1} l_{2j}$$

$$l_{1j} = p(p-1)/2 - zr_{p+1+j} - \sum_{i=2}^{p} l_{ij}, \quad j = p-1,\ p.$$

It is not difficult to see – by applying these equations – that the literal quasi-Euler square $L(r_1, \ldots, r_{12}; s_1, \ldots, s_{14}; z)$ corresponding to $p = 5$ is given by the array:

$10-s_4-s_7-s_{10}-s_{13}-zr_{12}$	$10-s_1-s_5-s_9-s_{13}-zr_8$	$10-s_2-s_6-s_{10}-s_{14}-zr_9$	$-5-s_3+s_5+s_{13}+(s_6+s_9+3s_{10}-s_{11}+s_{14})/2-z(r_1+r_5-r_6+r_7+2r_{10}+r_{11}-r_{12})/2$	$-15+s_1+s_2+s_3+s_4+s_7+s_{13}+\frac{1}{2}(s_6+s_9+s_{10}+s_{11}+s_{14})-z(r_1-r_5+r_6+r_7+r_{11}-r_{12})/2$
$-30+s_1+s_2+s_3+2s_4+s_5+s_6+2s_7+s_8+s_9+2s_{10}+s_{11}+s_{12}+s_{13}-z(r_1+r_2+r_7-r_{12})$	s_{13}	s_{14}	$15-s_5-s_7-s_{13}-(s_6+s_9+3s_{10}+s_{11}+s_{14})/2+z(r_1+r_5-r_6+r_7+r_{11}-r_{12})/2$	$25-s_1-s_2-s_3-2s_4-s_7-s_8-s_{12}-s_{13}-(s_6+s_9-s_{10}+s_{11}+s_{14})/2+z(r_1-r_5+r_6+r_7-r_{11}-r_{12})/2$
$10-s_9-s_{10}-s_{11}-s_{12}-zr_3$	s_9	s_{10}	s_{11}	s_{12}
$10-s_5-s_6-s_7-s_8-zr_4$	s_5	s_6	s_7	s_8
$10-s_1-s_2-s_3-s_4-zr_5$	s_1	s_2	s_3	s_4

The canonical decomposition (3.6) suggests an efficient algorithm for generating all the magic squares of any given order p. One simply scans through all possible values of the parameters r, s and t, selecting all those for which $L(r; s; p)$ and $L(r; t; -1)$ are orthogonal quasi-Euler squares.

A crude computer implementation of this algorithm easily generated the eight magic squares

$$\begin{array}{ll} M(0,0,0,0,0,0,0,0;0,1;0,2), & M(0,0,0,0,0,0,0,0;0,1;2,0), \\ M(0,0,0,0,0,0,0,0;0,2;0,1), & M(0,0,0,0,0,0,0,0;0,2;2,1), \\ M(0,0,0,0,0,0,0,0;2,0;0,1), & M(0,0,0,0,0,0,0,0;2,0;2,1), \\ M(0,0,0,0,0,0,0,0;2,1;0,2), & M(0,0,0,0,0,0,0,0;2,1;2,0) \end{array}$$

which are known to be the only magic squares of order 3, and all the 7040 known magic squares of order 4. Of the latter number, 896 magic squares correspond to the case $r = (0,0,0,0,-1,0,0,0,0,1)$, 5248 correspond to the (normal) case $r = (0,0,0,0,0,0,0,0,0,0)$ and 896 correspond to the case $r = (0,0,0,0,1,0,0,0,0,-1)$.

However, the implementation of the algorithm in a manner efficient enough to perform the census of higher-order magic squares in an acceptable length of time appears to be a task that calls for advanced skills in backtracking programming techniques. We expect that such an algorithm, when successfully implemented, would be able to perform the (open problem) census of magic squares of order 6, and probably also some other higher orders.

Another oft-discussed problem is that of classifying magic squares in a meaningful way. Elaborate classifications of the 7040 magic squares of order four are found in classical works on magic squares (cf. [2,4,8,12,14]). Henry Ernest Dudeney thought [6] that some of these classifications seemed as useful as dividing people into those who take snuff and those who do not. However, one of the most popular classifications of order-4 magic squares was devised by Dudeney himself [6]. To date, the task of classifying the 2,202,441,792 magic squares of order five in a meaningful way seems to have defeated even the most ardent enthusiasts.

It appears to us that the parametric representation (3.6) provides a natural classification scheme for magic squares into disjoint classes, characterized by different values of r. To find the class of any given magic square all we have to do is express it in the canonical form (3.6) and then read off the value of r. For instance, it is easy to verify that the magic square

2	23	25	7	8
4	16	9	14	22
21	11	13	15	5
20	12	17	10	6
18	3	1	19	24

has canonical expansion $M(r;s;t)$ – as defined in (3.6) – with

$$r = (0,0,0,0,0,0,0,1,0,-1,0,0),$$

$$
\begin{aligned}
s &= (2,0,3,3,1,1,4,0,0,2,4,4,0,3) \\
t &= (0,0,3,4,2,3,1,1,2,2,2,0,3,1).
\end{aligned}
$$

Therefore the magic square is of type $(0,0,0,0,0,0,0,1,0,-1,0,0)$.

Our classification scheme yields three classes of magic squares of order four, corresponding to the r-values:

$$
\begin{gathered}
(0,0,0,0,0,0,0,0,0,0), \\
(0,0,0,0,1,0,0,0,0,-1), \\
(0,0,0,0,-1,0,0,0,0,1).
\end{gathered}
$$

Our magic square code is not yet efficient enough to enable us perform a complete classification of order-five magic squares. However, the cases which we have identified at the moment include the classes defined by the r-values:

$$
\begin{array}{ll}
(0,0,0,0,0,0,0,0,0,0,0,0), & (-1,0,0,0,1,0,-1,0,0,0,1,0), \\
(-1,0,0,0,1,0,-1,0,0,0,-1,0), & (1,0,0,0,-1,0,-1,0,0,0,1,0), \\
(1,0,0,0,-1,0,1,0,0,0,-1,0), & (-1,0,0,0,1,0,0,0,0,0,0,0), \\
(0,0,0,0,0,0,-1,0,0,0,1,0), & (1,0,0,0,-1,0,0,0,0,0,0,0),
\end{array}
$$

and, naturally, the class $(0,0,0,0,0,0,0,1,0,-1,0,0)$ that occured in the example given above.

If one wishes, one may consider the classes

$$
\begin{gathered}
(r_1,\ldots,r_p,r_{p+1},r_{p+2},\ldots,r_{2p+1},r_{2p+2}) \\
(r_{p+2},\ldots,r_{2p+1},r_{2p+2},r_p,\ldots,r_1,r_{p+1}) \\
(r_p,\ldots,r_1,r_{p+1},r_{2p+1},\ldots,r_{p+2},r_{2p+2}) \\
(r_{2p+1},\ldots,r_{p+2},r_{2p+2},r_p,\ldots,r_1,r_{p+1}) \\
(r_1,\ldots,r_p,r_{2p+2},r_{2p+1},\ldots,r_{p+2},r_{p+1}) \\
(r_{2p+1},\ldots,r_{p+2},r_{p+1},r_p,\ldots,r_1,r_{2p+2}) \\
(r_p,\ldots,r_1,r_{2p+2},r_{p+2},\ldots,r_{2p+1},r_{p+1}) \\
(r_{p+2},\ldots,r_{2p+1},r_{p+1},r_1,\ldots,r_p,r_{2p+2})
\end{gathered}
$$

as equivalent, for if (m_{ij}) is a magic square of class $(r_1,r_2,\ldots,r_{2p+2})$, then the arrays $(m_{p+1-i\,j})$, $(m_{j\,p+1-i})$, $(m_{i\,p+1-j})$, $(m_{p+1-j\,i})$, $(m_{p+1-i\,p+1-j})$, (m_{ji}) and $(m_{p+1-j\,p+1-i})$, obtained from (m_{ij}) by performing rotations and/or reflections are also magic squares belonging to the other classes in the list.

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