Change in Additive Bandwidth When an Edge is Added

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Abstract

Let $G = (V, E)$ be an n-vertex graph and $f : V \to \{1, 2, \ldots, n\}$ be a bijection. The additive bandwidth of G, denoted $B^+(G)$, is given by $B^+(G) = \min_f \max_{uv \in E} |f(u) + f(v) - (n+1)|$, where the minimum ranges over all possible bijections f. The additive bandwidth cannot decrease when an edge is added, but it can increase to a value which is as much as three times the original additive bandwidth. The actual increase depends on $B^+(G)$ and n and is completely determined.

1 Introduction

The general problem of determining how the values of graphical invariants can change when an edge or vertex is added or deleted from a given graph has been studied for a variety of parameters. Harary [7] categorizes all such research under the heading "changing and unchanging" of invariants.

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One such invariant, the *bandwidth* of a graph, is a measure of how close the ones in the graph's adjacency matrix can be placed to the main diagonal, which is relevant to the storage requirements of the graph. Wang, West, and Yao [9] showed that the addition of an edge can, in the worst case, double the bandwidth of a graph. They further established the largest possible increase in all cases. In particular, they proved the following theorem.

Bandwidth Theorem: [Wang, West, and Yao] *Let G be an n-vertex graph whose bandwidth is given by $B(G) = b$, and let $g(n, b)$ denote the maximum value of $B(G + e)$, where e is an edge in the complement of G. Then*

$$g(n,b) = \begin{cases} b+1 & \text{if } n \leq 3b+4 \\ \lceil \frac{n-1}{3} \rceil & \text{if } 3b+5 \leq n \leq 6b-2 \\ 2b & \text{if } n \geq 6b-1 \end{cases}$$

This paper studies the analogous problem for an invariant called *additive bandwidth*, which measures how close the ones can be placed to the main contradiagonal. This concept was first introduced by Bascuñán, Ruiz, and Slater [1], and later examined by Bascuñán, Brigham, Caron, Carrington, Dutton, Hackett, Rogers, Ruiz, Slater, Vitray, and Vogt [2, 3, 4, 5, 6, 8].

A (*vertex*) *labeling* of an n-vertex graph $G = (V, E)$ is a bijection $f : V \to \{1, 2, \ldots n\}$. For a given labeling f of G, the *additive bandwidth with respect to f* of G, denoted $B_f^+(G)$, is given by $B_f^+(G) = \max_{uv \in E} |f(u) + f(v) - (n+1)|$. The *additive bandwidth* of G is $B^+(G) = \min_f B_f^+(G)$, where the minimum is taken over all possible labelings f of G.

Our paper concerns the relationship, for any graph G, between $B^+(G)$ and the maximum possible value of $B^+(G + e)$, where e is an edge joining any pair of non-adjacent vertices of G. For integers b and n, let $h(n, b)$ denote $\max_{e \notin E} B^+(G + e)$, taken over all n-vertex graphs G with additive bandwidth $B^+(G) = b$.

The main purpose of this paper is to prove the following theorem, which expresses $h(n, b)$ as a function of b. Notice that the function is considerably more complicated than its ordinary bandwidth counterpart. For reasons of clarity, it is presented as three cases according to the value of $7b + 2$ mod 3. The bracketed numbers in bold indicate the assertions that show the corresponding result. These are presented only for Case 1 but apply analogously to the other cases.

Additive Bandwidth Theorem: *Let G be an n-vertex graph whose additive bandwidth is given by $B^+(G) = b$, $b \geq 2$, and let $h(n, b)$ denote the maximum value of $B^+(G + e)$, where e is an edge in the complement of G. Then*

Case 1: $7b+2=0 \pmod 3$

$$h(n,b)=\begin{cases} b & \text{if } n=b+2 \ \textbf{[8]} \\ b+1 & \text{if } b+3\le n\le b+5 \ \textbf{[9]} \\ b+2 & \text{if } b+6\le n\le 3b+6 \ \textbf{[11]} \\ \lceil \frac{n}{3}\rceil & \text{if } 3b+7\le n\le 7b-4 \ \textbf{[10,30]} \\ \frac{7b-1}{3} & \text{if } 7b-3\le n\le 7b-1 \text{ or } n=7b+1 \\ & \qquad \textbf{[10,30,36]} \\ \frac{7b+2}{3} & \text{if } n=7b \text{ or } 7b+2\le n\le 7b+6 \\ & \qquad \textbf{[10,35,36]} \\ \frac{7b+5}{3}+2i & \text{if } 7b+7+12i\le n\le 7b+11+12i, \\ & \qquad i=0,\dots,\frac{b-4}{3} \ \textbf{[35,36]} \\ \frac{7b+5}{3}+2i+1 & \text{if } 7b+12+12i\le n\le 7b+18+12i, \\ & \qquad i=0,\dots,\frac{b-7}{3} \ \textbf{[35,36]} \\ 3b & \text{if } n\ge 11b-4 \ \textbf{[6,35,36]} \end{cases}$$

Case 2: $7b+2=1 \pmod 3$

$$h(n,b)=\begin{cases} b & \text{if } n=b+2 \\ b+1 & \text{if } b+3\le n\le b+5 \\ b+2 & \text{if } b+6\le n\le 3b+6 \\ \lceil \frac{n}{3}\rceil & \text{if } 3b+7\le n\le 7b-2 \\ \frac{7b+1}{3} & \text{if } 7b-1\le n\le 7b+3 \\ \frac{7b+4}{3}+2i & \text{if } 7b+4+12i\le n\le 7b+10+12i, \\ & \qquad i=0,\dots,\frac{b-5}{3} \\ \frac{7b+4}{3}+2i+1 & \text{if } 7b+11+12i\le n\le 7b+15+12i, \\ & \qquad i=0,\dots,\frac{b-5}{3} \\ 3b & \text{if } n\ge 11b-4 \end{cases}$$

Case 3: $7b+2=2 \pmod 3$

$$h(n,b)=\begin{cases} b & \text{if } n=b+2 \\ b+1 & \text{if } b+3\le n\le b+5 \\ b+2 & \text{if } b+6\le n\le 3b+6 \\ \lceil \frac{n}{3}\rceil & \text{if } 3b+7\le n\le 7b-3 \\ \frac{7b}{3} & \text{if } 7b-2\le n\le 7b+2 \\ \frac{7b+3}{3}+2i & \text{if } 7b+3+12i\le n\le 7b+7+12i, \\ & \qquad i=0,\dots,\frac{b-3}{3} \\ \frac{7b+3}{3}+2i+1 & \text{if } 7b+8+12i\le n\le 7b+14+12i, \\ & \qquad i=0,\dots,\frac{b-6}{3} \\ 3b & \text{if } n\ge 11b-4 \end{cases}$$

Section 2 presents preliminary results, Section 3 establishes a universal upper bound for $h(n,b)$, which is shown to be best possible, Section 4 presents some special cases, and Section 5 determines lower bounds, which are then used to establish exact values of $h(n,b)$ for all n and b.

We assume throughout that G is an n-vertex simple graph with vertex-set $V = \{v_1, v_2 \ldots, v_n\}$ and edge-set E.

2 Preliminaries

For a nonnegative integer b, a labeling f of graph G is called a *b-labeling* if $B_f^+(G) \le b$. Thus, if f is a b-labeling, then $n+1-b \le f(u)+f(v) \le n+1+b$ for any edge uv of G. We refer to the quantity $f(u)+f(v)$ as the *endpoint sum* of edge uv.

If f is a labeling of graph G, the *complementary labeling* $\bar{f}$ is defined by $\bar{f}(v_i) = n+1-f(v_i)$ for $1 \le i \le n$. Since $B_{\bar{f}}^+(G) = B_f^+(G)$, it follows that f is a b-labeling if and only if its complementary labeling $\bar{f}$ is a b-labeling. Referring to the complementary labeling will make it easier in later sections to dispose of certain symmetric arguments.

Observe that, in computing $h(n,b)$, we need consider only the *edge-maximum* n-vertex graphs having additive bandwidth b. Moreover, these edge-maximum graphs are unique up to isomorphism because, if bijection f is a b-labeling of such a graph, then vertices u and v are adjacent if and only if $n+1-b \le f(u)+f(v) \le n+1+b$. The edge-maximum n-vertex graph with additive bandwidth b is denoted $G_{n,b}$, and accordingly,

$$h(n,b) = \max_{e \notin E(G_{n,b})} B^+(G_{n,b}+e)$$

It will be convenient to subscript the vertices of $G_{n,b}$ according to some b-labeling f, that is, $f(v_i) = i$, $i = 1, \ldots, n$. Also, we partition the vertices into sets $L = \{v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\}$, $R = \{v_{\lceil \frac{n}{2} \rceil + 1}, v_{\lceil \frac{n}{2} \rceil + 2}, \ldots, v_n\}$, and C, where $C = \emptyset$ if n is even, and $C = \{v_{\lceil \frac{n}{2} \rceil}\}$ if n is odd.

Figure 1 shows graphs $G_{15,3}$ and $G_{16,3}$. Typically, as in the figure, only the subscripts of the vertices are displayed.

For $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, vertex v_i (in L) and vertex v_{n+1-i} (in R) are called *partner vertices*. For the case n odd, the lone vertex $v_{\lceil \frac{n}{2} \rceil} \in C$ is regarded as its own partner. For a given labeling g, the labels $g(x)$ and $g(y)$ are said to be *complementary* if $g(x)+g(y) = n+1$. A labeling g of the graph $G_{n,b}+e$ is *balanced* if all partner vertices receive complementary labels, that is, $g(v_i)+g(v_{n+1-i}) = n+1$ for $1 \le i \le \lceil \frac{n}{2} \rceil$. Arguments involving the additive bandwidth with respect to a given labeling tend to be simpler if that labeling is balanced.

The first proposition establishes the value of $h(n,b)$ when $b = 0$.

Proposition 1 $h(n,0) = 1$ *for* $n \ge 3$.

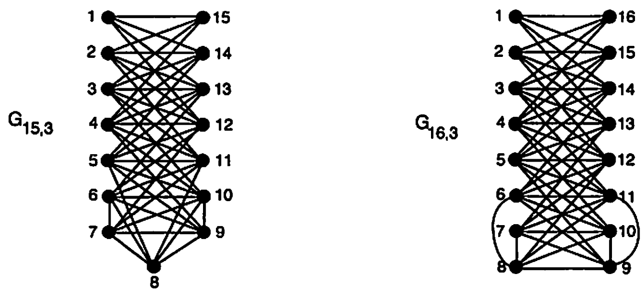

Figure 1: The edge-maximum graphs $G_{15,3}$ and $G_{16,3}$

Proof: Figure 2 shows a 1-labeling for each of the three isomorphically distinct possibilities for $G_{n,b} + e$. □

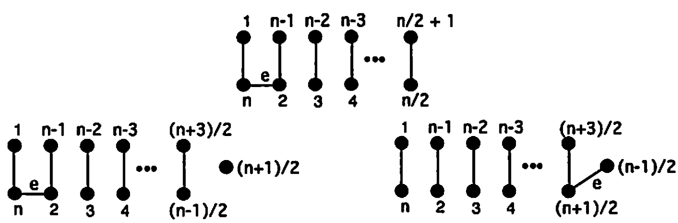

Figure 2: A 1-labeling for each of the three possibilities for $G_{n,0} + e$

We close this section with a monotonicity result for $h(n, b)$, which will be used in subsequent sections. The result hinges on the fact that $G_{n,b}$ appears as an induced subgraph of $G_{n+2,b}$. In particular, if the vertex-set of $G_{n+2,b}$ is $\{v_1, v_2, \ldots, v_{n+2}\}$, then the subgraph induced on the vertices $v_2, v_3, \ldots, v_{n+1}$ is $G_{n,b}$. To avoid confusion when referring to this embedded version of $G_{n,b}$ in the argument that follows, we let its vertices be $w_1, \ldots, w_n$, where $w_i = v_{i+1}$ for $i = 1, \ldots, n$.

Lemma 2 *Let e be an edge joining non-adjacent vertices of the embedded version of graph $G_{n,b}$ within $G_{n+2,b}$, and let g be any labeling of $G_{n+2,b} + e$. Then there exists a labeling $\hat{g}$ of $G_{n,b} + e$ such that $B^+_{\hat{g}}(G_{n,b} + e) \leq B^+_g(G_{n+2,b} + e)$.*

Proof. Let $l = g(v_1)$ and $m = g(v_{n+2})$, and assume without loss of generality that $l < m$. We construct a labeling $\hat{g}$ of $G_{n,b} + e$ as follows: for $i = 1, \ldots, n$, let

$$\hat{g}(w_i) = \begin{cases} g(v_{i+1}) & \text{if } g(v_{i+1}) \leq l-1 \\ g(v_{i+1})-1 & \text{if } l+1 \leq g(v_{i+1}) \leq m-1 \\ g(v_{i+1})-2 & \text{if } g(v_{i+1}) \geq m+1 \end{cases}$$

Let $B_g^+(G_{n+2,b}+e) = k$, and let w_iw_j ($= v_{i+1}v_{j+1}$) be any edge of $G_{n,b}+e$ (and hence, of $G_{n+2,b}+e$). We must show that the endpoint sum $\hat{g}(w_i)+\hat{g}(w_j)$ is between $n+1-k$ and $n+1+k$.

Since g is a k-labeling of $G_{n+2,b}+e$, it follows that $n+3-k \leq g(v_{i+1})+g(v_{j+1}) \leq n+3+k$. Also, since v_1v_{n+2} is an edge of $G_{n+2,b}+e$, we have $n+3-k \leq l+m \leq n+3+k$. We assume $g(v_{i+1}) < g(v_{j+1})$. The argument in the opposite case is analogous.

Case 1: $g(v_{i+1}), g(v_{j+1}) \leq l-1$.
Then $\hat{g}(w_i)+\hat{g}(w_j) = g(v_{i+1})+g(v_{j+1})$, and

$$\begin{aligned} n+1-k < n+3-k &\leq \hat{g}(w_i)+\hat{g}(w_j) \leq 2l-3 < l+m-2 \\ &\leq n+3+k-2 = n+1+k \end{aligned}$$

Case 2: $l+1 \leq g(v_{i+1}), g(v_{j+1}) \leq m-1$ or $g(v_{i+1}) \leq l-1$ and $g(v_{j+1}) \geq m+1$.
Then $\hat{g}(w_i)+\hat{g}(w_j) = g(v_{i+1})+g(v_{j+1})-2$, and

$$n+1-k = n+3-k-2 \leq \hat{g}(w_i)+\hat{g}(w_j) \leq n+3+k-2 = n+1+k$$

Case 3: $g(v_{i+1}), g(v_{j+1}) \geq m+1$.
Then $\hat{g}(w_i)+\hat{g}(w_j) = g(v_{i+1})+g(v_{j+1})-4$, and

$$\begin{aligned} n+1-k = n+3-k-2 &\leq l+m-2 < 2m-2 < \hat{g}(w_i)+\hat{g}(w_j) \\ &\leq n+3+k-4 < n+1+k \end{aligned}$$

Case 4: $g(v_{i+1}) \leq l-1$ and $l+1 \leq g(v_{j+1}) \leq m-1$.
Then $\hat{g}(w_i)+\hat{g}(w_j) = g(v_{i+1})+g(v_{j+1})-1$, and

$$n+1-k < n+3-k-1 \leq \hat{g}(w_i)+\hat{g}(w_j) \leq l+m-3 \leq n+3+k-3 < n+1+k$$

Case 5: $l+1 \leq g(v_{i+1}) \leq m-1$ and $g(v_{j+1}) \geq m+1$.
Then $\hat{g}(w_i)+\hat{g}(w_j) = g(v_{i+1})+g(v_{j+1})-3$, and

$$n+1-k < (l+1)+(m+1)-3 \leq \hat{g}(w_i)+\hat{g}(w_j) \leq n+3+k-3 < n+1+k$$

□

Corollary 3 [2-step monotonicity] *For any n and b such that $n \geq b+2$,*

$$h(n,b) \leq h(n+2,b)$$

Proof. As before, let $G_{n,b}$ be embedded in $G_{n+2,b}$. Let e be an edge between non-adjacent vertices of this embedded version of $G_{n,b}$ such that $h(n,b) = B^+(G_{n,b}+e)$, and let g be a labeling of $G_{n+2,b}+e$ such that $B_g^+(G_{n+2,b}+e) = B^+(G_{n+2,b}+e)$. By Lemma 2, there exists a labeling $\hat{g}$ of $G_{n,b}+e$ such that $B_{\hat{g}}^+(G_{n,b}+e) \leq B_g^+(G_{n+2,b}+e)$. The following chain of inequalities completes the proof.

$$\begin{aligned} h(n,b) = B^+(G_{n,b}+e) &\leq B_{\hat{g}}^+(G_{n,b}+e) \leq B_g^+(G_{n+2,b}+e) \\ &= B^+(G_{n+2,b}+e) \leq h(n+2,b) \end{aligned}$$

□

3 A Universal Upper Bound

In this section we show that adding an edge to a graph with additive bandwidth b can increase the additive bandwidth significantly, to a maximum of at most $3b$, that is, $h(n,b) \leq 3b$.

Lemma 4 *Let g be a balanced labeling of graph $G_{n,b}$ such that $|g(v_i) - g(v_{i+1})| \leq 3$ for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$. Then g is a $3b$-labeling of $G_{n,b}$.*

Proof. Observe that the premises imply that $|g(v_i) - g(v_{i+1})| \leq 3$ for all $i = 1, \ldots, n-1$. Let vertices v_i and v_j be adjacent in $G_{n,b}$. We must show $|n+1-(g(v_i)+g(v_j))| \leq 3b$. We know $|n+1-(i+j)| \leq b$, or equivalently, $j = n+1-i+k$, where $|k| \leq b$. Since g is a balanced labeling, $g(v_{n+1-i}) + g(v_i) = n+1$. The following chain establishes the desired inequality.

$$\begin{aligned} |n+1-(g(v_i)+g(v_j))| &= |g(v_{n+1-i}) + g(v_i) - (g(v_i)+g(v_j))| \\ &= |g(v_{n+1-i}) - g(v_j)| \\ &= |g(v_{n+1-i}) - g(v_{n+1-i+k})| \leq 3|k| \leq 3b \end{aligned}$$

□

It will be convenient to call vertex v_s larger than vertex v_t if $s > t$.

Lemma 5 *Let l and m be positive integers with $l < m \leq \lfloor \frac{n}{2} \rfloor$. Then there exists a balanced $3b$-labeling g of $G_{n,b}$ such that $g(v_l) = m$, $g(v_m) = m-1$, and $g(v_t) = t$ for $m < t < n+1-m$.*

Proof. The construction of g begins with the assignments:

$$g(v_l) = m; \quad g(v_m) = m-1$$

Labels are then assigned iteratively in descending order, starting with $m-2$. In each complete iteration, three labels are assigned in the order (1) to the largest unlabeled vertex in the set $\{v_1, v_2, \ldots, v_{l-1}\}$; (2) to the smallest unlabeled vertex in the set $\{v_{l+1}, v_{l+2}, \ldots, v_{m-1}\}$; and (3) to the largest unlabeled vertex in the set $\{v_{l+1}, v_{l+2}, \ldots, v_{m-1}\}$. When one of these two sets has all its vertices labeled, the scheme continues until the vertices in the other set have all been labeled. The kth full iteration, $k \geq 1$, consists of:

$$g(v_{l-k}) = m + 1 - 3k; \quad g(v_{l+k}) = m - 3k; \quad g(v_{m-k}) = m - 1 - 3k$$

It is easy to see that the g-labels that have been assigned to vertices $v_1, v_2, \ldots, v_m$ satisfy $|g(v_i) - g(v_{i+1})| \leq 3$ for $i = 1, 2, \ldots, m-1$. Furthermore, since g has simply permuted the labels $1, 2, \ldots, m$ among the vertices $v_1, v_2, \ldots, v_m$, the complementary labels $n, n-1, \ldots, n+1-m$ are still available for the partner vertices. Thus, we can let $g(v_{n+1-i}) = n + 1 - g(v_i)$, $i = 1, 2, \ldots, m$. Clearly, these assignments also satisfy $|g(v_i) - g(v_{i+1})| \leq 3$.

To complete the construction of labeling g, we let $g(v_i) = i$ for $i = m+1, m+2, \ldots, n-m$. Then $|g(v_m) - g(v_{m+1})| = |g(v_{n+1-m}) - g(v_{n-m})| = 2$, and all other newly assigned g-labels satisfy $|g(v_i) - g(v_{i+1})| = 1$. Since g is a balanced labeling, it follows by Lemma 4 that g is a $3b$-labeling of $G_{n,b}$. □

The construction outlined in the proof of Lemma 5 for the case $n = 17$, $l = 5$, and $m = 8$ is shown in Figure 3.

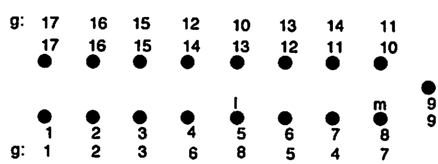

Figure 3: Construction illustrating Lemma 5 with $n = 17, l = 5$, and $m = 8$

Proposition 6 *For $b \geq 1$, $h(n, b) \leq 3b$.*

Proof. Let e be an edge between non-adjacent vertices v_r and v_s of $G_{n,b}$, and assume $r < s$. We must show that $B^+(G_{n,b} + e) \leq 3b$. There are two essentially different cases to consider, according to where vertices v_r and v_s lie. We may assume that $r < n + 1 - s$ since, otherwise, we can work with the complementary labeling of $G_{n,b}$ and reverse the roles of r and s.

Case 1: $v_r \in L$ and $v_s \in R \cup C$ $\quad (1 \leq r \leq \lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n+1}{2} \rceil \leq s \leq n - s)$

If n is odd and $s = \frac{n+1}{2}$, then apply Lemma 5 with $l = r$ and $m = \frac{n-1}{2}$ (as in Figure 3). The balanced 3b-labeling g of $G_{n,b}$ satisfies $g(v_r) = \frac{n-1}{2}$ and $g(v_s) = s = \frac{n+1}{2}$. Hence, the endpoint sum $g(v_r) + g(v_s)$ of the edge e equals n. Otherwise, apply Lemma 5 with $l = r$ and $m = n + 1 - s$ (as in Figure 4 appearing at the end of the proof). For this situation, $g(v_r) = n + 1 - s$ and $g(v_{n+1-s}) = n - s$. Since g is balanced, $g(v_s) = n + 1 - (n - s) = s + 1$. Hence, the endpoint sum $g(v_r) + g(v_s)$ of edge e equals $n + 2$, and g is a 3b-labeling of $G_{n,b} + e$.

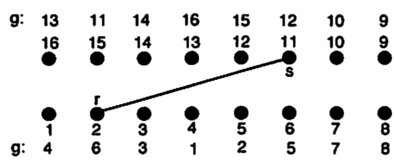

Figure 4: Case 1 of Proposition 6 with $n = 16$, $b = 2$, $r = 2$, $s = 11$

Case 2: $v_r, v_s \in L \quad (1 \le r \le \lfloor \frac{n}{2} \rfloor - 1$ and $r + 1 \le s \le \lfloor \frac{n}{2} \rfloor)$

If n is odd, then set $g(v_{\frac{n+1}{2}}) = \frac{n+1}{2}$, and for n even, set $g(\frac{n}{2}) = \frac{n}{2} + 2$ and $g(\frac{n}{2} + 1) = \frac{n}{2} - 1$. The remainder of the construction of the 3b-labeling g depends on which of the two quantities $s - \lfloor \frac{r+s}{2} \rfloor$ and $\frac{n}{2} - s$ is larger.

Subcase a. $\frac{n}{2} - s \ge s - \lfloor \frac{r+s}{2} \rfloor$

Let $x = \lfloor \frac{r+s}{2} \rfloor$. We start with the assignments:

$$g(v_x) = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad g(v_{n+1-x}) = n + 1 - \left\lfloor \frac{n}{2} \right\rfloor$$

Then g assigns labels iteratively, using the labels described below, in order (see Figure 5), (1) to the smallest unlabeled vertex greater than v_x, (2) to the largest unlabeled vertex less than v_x, (3) to the largest unlabeled vertex less than $v_{\lceil \frac{n}{2} \rceil}$, and (4) to the partner vertices in set R. When the vertices less than v_x or the vertices between v_x and $v_{\lfloor \frac{n}{2} \rfloor}$ have all been assigned g-labels, the labeling scheme continues until all the vertices in the other range (and their partners in R) have been labeled. Each full iteration consists of:

(i) $g(v_{x+i}) = \lfloor \frac{n}{2} \rfloor + 3i$ and $g(v_{n+1-(x+i)}) = \lceil \frac{n}{2} \rceil - 3i + 1$
(ii) $g(v_{x-i}) = \lfloor \frac{n}{2} \rfloor - 3i$ and $g(v_{n+1-(x-i)}) = \lceil \frac{n}{2} \rceil + 3i + 1$
(iii) $g(v_{\lceil \frac{n}{2} \rceil - i}) = \lfloor \frac{n}{2} \rfloor + 3i + 1 + (n-1) \bmod 2$ and
$g(v_{\lfloor \frac{n}{2} \rfloor + 1 + i}) = \lceil \frac{n}{2} \rceil - 3i - (n-1) \bmod 2$

A straightforward induction argument can be used to show that after k full iterations the set of consecutive integers $\{\lceil \frac{n}{2} \rceil - 3k - 1, \ldots, \lfloor \frac{n}{2} \rfloor, \ldots, \lfloor \frac{n}{2} \rfloor + 3k + 1 + (n-1) \bmod 2\}$ have been assigned to the vertices in the set $\{v_{x-k}, v_{x-k+1}, \ldots, v_x, \ldots, v_{x+k}\} \cup \{v_{\lfloor \frac{n}{2} \rfloor - k}, v_{\lfloor \frac{n}{2} \rfloor - k+1}, \ldots, v_{\lceil \frac{n}{2} \rceil}\}$ and their

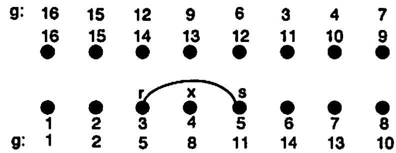

Figure 5: Case 2a of Proposition 6 with $n = 16$, $b = 2$, $r = 3$, $s = 5$

partner vertices in R, and that this partial labeling is balanced. Clearly, the pattern of assigning consecutive integers to the vertices in each iteration continues even after the vertices in one of the ranges have all been labeled. At that point, consecutive vertices receive either consecutive labels or labels that differ by 2. Thus, the labeling g satisfies $|g(v_i) - g(v_{i+1})| \leq 3$ for all $i = 1, \ldots, n-1$, and hence, by Lemma 4, g is a balanced $3b$-labeling of $G_{n,b}$.

Since x is the midpoint between r and s (or "near midpoint" if $r+s$ is odd), we have, because of the condition for this subcase, that v_r and v_s receive labels in the same iteration or in consecutive interations. Therefore, $|n+1-(g(v_r)+g(v_s))| \leq 3$, and hence, g is a $3b$-labeling of $G_{n,b}+e$.

Subcase b. $\frac{n}{2} - s < s - \lfloor \frac{r+s}{2} \rfloor$

Let $x = 2s - \lfloor \frac{n}{2} \rfloor$, and let $k = \lfloor \frac{n}{2} \rfloor - s$. By the condition defining this subcase, we can perform $k-1$ full iterations of the labeling scheme used in Subcase a. Moreover, at the end of the kth iteration, $g(v_{x-k}) = \lfloor \frac{n}{2} \rfloor - 3k = x - k$. Observe by the condition for this case that $x = 2s - \lfloor \frac{n}{2} \rfloor > \frac{n}{2} + \lfloor \frac{r+s}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \geq \lfloor \frac{r+s}{2} \rfloor$, so $x \geq \lfloor \frac{r+s}{2} \rfloor + 1$. Thus, $x - r \geq \lfloor \frac{r+s}{2} \rfloor + 1 - r \geq s - \lfloor \frac{r+s}{2} \rfloor \geq \frac{n}{2} - s + 1 \geq \lfloor \frac{n}{2} \rfloor - s + 1 = k+1$, implying $x - k - 1 \geq r$, where equality is possible. In the case of equality, the vertices $v_1, v_2, \ldots, v_r$ and their partners retain the standard labeling. If $x - k - 1 > r$, apply Lemma 5 with $l = r$ and $m = x - k - 1$ and restrict the labeling obtained to v_1 through v_{x-k-1} and their partners (see Figure 6).

Arguing as in Subcase a, it is not hard to show that the resulting labeling g is a balanced $3b$-labeling of $G_{n,b}$. Furthermore, $g(v_r) = \lfloor \frac{n}{2} \rfloor - 3k - 1$ and $g(v_s) = \lfloor \frac{n}{2} \rfloor + 3k$, from which it follows that $|n+1-(g(v_r)+g(v_s))| \leq 3 \leq 3b$. Therefore, g is a $3b$-labeling of $G_{n,b}+e$, which completes the proof. □

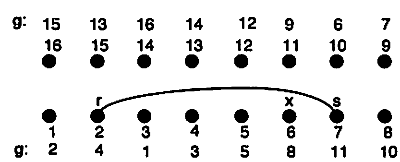

Figure 6: Case 2b of Proposition 6 with $n = 16$, $b = 2$, $r = 2$, $s = 7$

4 Special Cases

Part of the strategy for establishing lower bounds is based on the restrictions that three mutually adjacent vertices impose on the labels they can be assigned. Such vertices are called *triangle vertices.*

We begin this section by determining the value of $h(n,b)$ when $b=1$.

Proposition 7 *For all $n \geq 4$,*

$$h(n,1) = \begin{cases} 2 & \text{if } n \text{ is even or } n=5 \\ 3 & \text{if } n \geq 7 \text{ and odd} \end{cases}$$

Proof: *Case 1:* n is even or $n=5$.
The 2-labeling of $G_{5,1}+e$ shown in Figure 7 establishes $h(5,1)=2$.

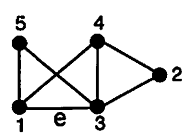

Figure 7: A 2-labeling showing $h(5,1)=2$

To complete Case 1, we construct a 2-labeling of $G_{n,1}+e$ for n even and at least 4. The labeling depends on whether one or both of v_r and v_s, $r<s$, are L-vertices or are R-vertices. By symmetry, we need consider only the following two subcases.

Subcase a. $v_r, v_s \in L$

Let $x=\lfloor \frac{r+s}{2} \rfloor$, $g(v_x)=\frac{n}{2}$, and label the remaining L-vertices iteratively by $g(v_{x-i})=\frac{n}{2}-2i$ and $g(v_{x+i})=\frac{n}{2}+2i$ until either (i) all L-vertices smaller than v_x have been labeled or (ii) all those larger than v_x have been labeled, whichever occurs first (see Figure 8). Let t be the value of i when this occurs. At that point, the remaining unlabeled L-vertices are assigned consecutive labels as follows: in case (i), $g(v_{x+t+j})=\frac{n}{2}+2t+j+1)$, $j=1,2,\ldots,\frac{n}{2}-(x+t)$, and in case (ii), $g(v_j)=j$, $j=1,2,\ldots,x-t-1$. It is straightforward to show that no pair of L-vertices receive complementary labels. Therefore, a balanced 2-labeling is completed by assigning for each L-vertex v_l, the complementary label $n+1-g(v_l)$ to v_{n+1-l}, the partner of v_l in R.

Subcase b. $v_r \in L$ and $v_s \in R$

The labeling scheme is similar to the one in Subcase a. Here, we let x be the midpoint between r and the partner of s, that is, $x=\lfloor \frac{r+(n+1-s)}{2} \rfloor$, and let $g(v_x)=\frac{n}{2}$. Proceed iteratively to label the L-vertices smaller than v_x and the R-vertices smaller than v_{n+1-x} as follows (see Figure

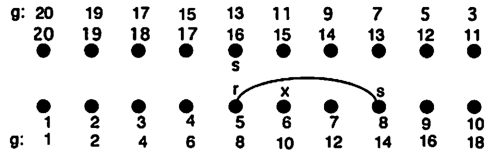

Figure 8: Subcase a of Proposition 7 with $n = 20$, $r = 5$, and $s = 8$

9): $g(v_{x-i}) = \frac{n}{2} - 2i$ and $g(v_{n+1-x-i}) = \frac{n}{2} + 2i$ until either (i) all L-vertices smaller than v_x have been labeled or (ii) all R-vertices smaller than v_{n+1-x} have been labeled, whichever occurs first. Again, let t be the value of i when this occurs. In case (i), the remaining unlabeled R-vertices smaller than v_{n+1-x} are assigned by letting $g(v_{n+1-x-t-j}) = \frac{n}{2}+2t+j+1$, $j = 1, 2, \ldots, \frac{n}{2}-(x+t)$, and in case (ii), the remaining unlabeled L-vertices smaller than v_x are assigned by letting $g(v_j) = j$, $j = 1, 2, \ldots, x - t - 1$. It is straightforward to show that no pair of vertices labeled so far receive complementary labels. Therefore, a balanced 2-labeling is completed by letting $g(v_{n+1-t}) = n + 1 - g(v_t)$ for each labeled vertex v_t.

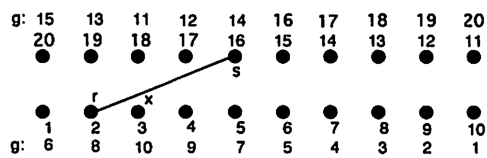

Figure 9: Subcase b of Proposition 7 with $n = 20$, $r = 2$, and $s = 16$

Case 2: $n \geq 7$ and odd.
We first show $h(7,1) > 2$. Suppose that g is a 2-labeling of $G_{7,1} + v_1v_2$. Observe that every vertex of $G_{7,1}+v_1v_2$ is a triangle vertex. Let $x = g^{-1}(1)$, and let x, y, and z be mutually adjacent vertices. Then $g(y) \geq 5$ and $g(z) \geq 5$, which implies, since $g(y) \neq g(z)$, that $g(y) + g(z) > 10$. This exceeds the upper range for an endpoint sum in a 2-labeling and shows that no such 2-labeling exists. Thus, $h(7,1) \geq 3$ and hence, by Corollary 3, $h(n,1) \geq 3$. Equality follows by Proposition 6. □

The remainder of this section determines the conditions for which $h(n,b) = b, b+1, b+2$.

Proposition 8 *For all $b \geq 1$, $h(n,b) = b$ if and only if $n = b+2$.*

Proof: The graph $G_{n,b}$ is the complete graph K_n if and only if $n = b+2$. □

We assume throughout the rest of the paper that $b \geq 2$.

Proposition 9 *For all $b \geq 2$, $h(n,b) = b+1$ if and only if $b+3 \leq n \leq b+5$.*

Proof: Suppose $b+3 \leq n \leq b+5$. The endpoint sums in a $(b+1)$-labeling of $G_{b+5,b}+e$ must lie in $[5, 2b+7]$. By symmetry, the only new edges we need to consider are $e = v_1v_2$ and $e = v_1v_3$ since the standard labeling suffices for all other added edges. In the first case, let g be the standard labeling except for $g(v_2) = 4$ and $g(v_4) = 2$. In the second case, let g be the standard labeling except for $g(v_3) = 4$ and $g(v_4) = 3$. In either case, it is straightforward to check that g is a $(b+1)$-labeling. Thus, $h(b+5,b) \leq b+1$ and hence, by Proposition 8, $h(b+5,b) = b+1$. Corollary 3 and Proposition 8 imply $h(b+3,b) = b+1$. It remains to show $h(b+4,b) = b+1$. Observe that the only case we need to consider for $G_{b+4,b}+e$ is $e = v_1v_2$. In this case, let g be the $(b+1)$-labeling obtained by using the standard labeling except for $g(v_2) = 3$ and $g(v_3) = 2$.

Next suppose that $h(n,b) = b+1$. By Corollary 3, Proposition 8, and the fact that $h(b+5,b) = b+1$, it suffices to show that $h(b+6,b) > b+1$ and $h(b+7,b) > b+1$. The individual cases $b = 2$ and $b = 3$ require separate, straightforward arguments, so we assume that $b \geq 4$. The range of the endpoint sums for a $(b+1)$-labeling of $G_{b+6,b}+v_1v_2$ is $[6, 2b+8]$. Since $b \geq 4$, every L-vertex is adjacent to every R-vertex. Moreover, the vertices assigned labels 1, 2, and 3 must be independent, so we may assume without loss of generality that they are assigned to L-vertices. However, there is no independent set of size three in L of $G_{b+6,b}+v_1v_2$, which shows that $h(b+6,b) > b+1$. A similar argument may be used to show $h(b+7,b) > b+1$. □

Proposition 10 *Let b, k, and n be positive integers such that $b+2 \leq k < 3b$ and $n \leq 3k$. Then $h(n,b) \leq k$.*

Proof: It suffices to show that each of the following bijections, g, is a k-labeling of the graph $G_{n,b}+e$ for any edge $e = v_rv_s$, $r < s$.

Case 1: $n+1-k \leq r+s \leq n+1+k$.

Then the standard labeling, $g(v_i) = i$, is a k-labeling.

Case 2: $r+s < n+1-k$

Subcase 2a. $s \geq \lfloor \frac{n+1-k}{2} \rfloor + 1$.

Then let

$$g(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq r-1 \\ n+1-k-s & \text{if } i = r \\ i-1 & \text{if } r+1 \leq i \leq n+1-k-s \\ i & \text{if } n+2-k-s \leq i \leq n \end{cases}$$

To show that the bijection g is a k-labeling, consider any edge v_iv_j of $G_{n,b}+e$, where $i<j$.

(i) if $i \neq r$, then, since $n+1-b \leq i+j \leq n+1+b$, we have

$$n+1-k \leq n+1-b-2 \leq i+j-2 \leq g(v_i)+g(v_j) \leq i+j \leq n+1+b \leq n+1+k$$

(ii) if $i=r$ and $j=s$, we have $g(v_i)+g(v_j)=n+1-k$ since $s \geq \frac{n+2-k}{2}$, that is, $s \geq n+2-k-s$.

(iii) if $i=r$ and $j \neq s$, the following chain of inequalities demonstrates the result, where the last inequality follows because $n \leq 3k$.

$$\begin{aligned}
n+1-k &< n+1-b \leq r+j = r+1+j-1 \leq n+1-k-s+j-1 \\
&\leq g(v_r)+g(v_j) = n+1-k-s+g(v_j) \\
&\leq n+1-k-\left(\left\lfloor\frac{n+1-k}{2}\right\rfloor+1\right)+g(v_j) \\
&= \left\lceil\frac{n+1-k}{2}\right\rceil-1+g(v_j) \\
&\leq \left\lceil\frac{n+1-k}{2}\right\rceil-1+n \leq n+1+k
\end{aligned}$$

Subcase 2b: $s \leq \lfloor\frac{n+1-k}{2}\rfloor$

Let

$$g(v_i)=\begin{cases} i & \text{if } 1 \leq i \leq r-1 \\ \lfloor\frac{n+1-k}{2}\rfloor & \text{if } i=r \\ i-1 & \text{if } r+1 \leq i \leq s-1 \\ \lfloor\frac{n+1-k}{2}\rfloor+1 & \text{if } i=s \\ i-2 & \text{if } s+1 \leq i \leq \lfloor\frac{n+1-k}{2}\rfloor+1 \\ i & \text{if } \lfloor\frac{n+1-k}{2}\rfloor+2 \leq i \leq n \end{cases}$$

Let $v_iv_j \in E(G_{n,b})$, $i<j$. Then $(j-1)+j = 2j-1 \geq i+j \geq n+1-b = n+1-b-2+2 \geq n+1-k+2$, which implies $j \geq \lceil\frac{n+4-k}{2}\rceil = \lfloor\frac{n+1-k}{2}\rfloor+2$, implying $g(v_j)=j$.

(i) for $i \neq r,s$, we have

$$n+1-k \leq n+1-b-2 \leq i+j-2 \leq g(v_i)+g(v_j) \leq i+j \leq n+1+b < n+1+k$$

(ii) for $i=r$ and $j \neq s$, or $i=s$, we have, using $n \leq 3k$,

$$\begin{aligned}
n+1-k < n+1-b \leq i+j &\leq \left\lfloor\frac{n+1-k}{2}\right\rfloor+j \leq g(v_i)+g(v_j) \\
&\leq \left\lfloor\frac{n+1-k}{2}\right\rfloor+1+n \leq n+1+k
\end{aligned}$$

(iii) for $i = r$ and $j = s$, we have $g(v_i)+g(v_j) = \lfloor \frac{n+1-k}{2} \rfloor + \lfloor \frac{n+1-k}{2} \rfloor + 1$, which is $n+1-k$ or $n+2-k$, depending on the parities of n and k.

Case 3: $r+s > n+1+k$.

Use the complementary labeling and interchange the roles of L and R and of r and s to transform this case into Case 2. □

Corollary 11 *If* $b+6 \leq n \leq 3b+6$, *then* $h(n,b) = b+2$.

Proof: Since $n \geq b+6$, it follows from Propositions 8 and 9 that $h(n,b) \geq b+2$. But Proposition 10 with $k = b+2$ implies $h(n,b) \leq b+2$. □

5 Lower Bounds for $h(n, b)$

We know from earlier results that $b \leq h(n,b) \leq 3b$. In this section we establish sharp lower bounds that lead eventually to exact values for $h(n,b)$.

5.1 On the Structure of $G_{n,b}$

It will be important to understand which vertices of $G_{n,b}$ lie in triangles. The first result of this analysis is straightforward, and its proof is omitted.

Lemma 12 *Let v_i and v_j be adjacent vertices of $R \cup C$ of $G_{n,b}$, where $i < j$. Then v_i is adjacent to vertex v_k for all k, $i < k \leq j$. In particular, v_i is adjacent to v_{i+1}.*

Lemma 13 *Let v_i and v_j, $i < j$, be adjacent vertices of set L of $G_{n,b}$. Then each is adjacent to at least one pair of adjacent vertices of set R (and hence, are triangle vertices).*

Proof: Recall that $b \geq 2$. By the symmetry of $G_{n,b}$, v_{n+1-i} and v_{n+1-j} are adjacent vertices of R. Moreover, v_i is adjacent to v_{n+1-j} since $n+1-b \leq i+j < i+n+1-j < n+1 < n+1+b$, and, by symmetry, v_j is adjacent to v_{n+1-i}. The result follows since each vertex in L is adjacent to its partner in R. □

The following proposition is an immmediate consequence of Lemmas 12 and 13.

Proposition 14 *An L-vertex x is a triangle vertex of $G_{n,b}$ if and only if x is adjacent to a pair of consecutive, adjacent non-L-vertices.*

Proposition 15 *The smallest triangle vertex of L in $G_{n,b}$ is adjacent to the largest pair of consecutive, adjacent non-L-vertices.*

Proof: Let v_l be the smallest triangle vertex in L, and let r be the largest integer such that v_r and v_{r+1} are adjacent non-L-vertices that are both adjacent to v_l (v_r exists by Proposition 14). Since the assertion is trivially true if $l = 1$, we assume that $l \geq 2$. Then v_{l-1} is adjacent to v_{r+1} and v_{r+2}, and hence, v_{r+1} is not adjacent to v_{r+2} (by definitions of r and l). Thus, v_r and v_{r+1} form the largest pair of consecutive, adjacent non-L-vertices. □

Proposition 16 *Let b and n be integers such that $n \geq 3b - 1$. Then the index of the smallest triangle vertex of L equals $n + 1 - b - \lfloor \frac{n+b}{2} \rfloor$, and the index of the largest triangle vertex (in R) equals $b + \lfloor \frac{n+b}{2} \rfloor$.*

Proof: By Proposition 15, it suffices to determine the largest r such that v_r and v_{r+1} are adjacent non-L-vertices. If v_r and v_{r+1} are adjacent, then $n + 1 - b \leq 2r + 1 \leq n + 1 + b$, which implies that $r \leq \lfloor \frac{n+b}{2} \rfloor$. Thus, we seek the smallest index l such that $l + \lfloor \frac{n+b}{2} \rfloor \geq n + 1 - b$. It follows that $l = n + 1 - b - \lfloor \frac{n+b}{2} \rfloor$, which is positive since $n \geq 3b - 1$. The index of the largest triangle vertex is the complement of this, that is, $b + \lfloor \frac{n+b}{2} \rfloor$. □

It is now possible to count the number of triangle vertices and to deduce how many are in at least two triangles.

Corollary 17 *Let b and n be integers such that $n \geq 3b - 1$. Then the number of triangle vertices in set L equals $\lfloor \frac{3b}{2} \rfloor$ if n is even, and equals $\lfloor \frac{3b-1}{2} \rfloor$ if n is odd.*

Proof: By Proposition 16, the number of triangle vertices in L equals $\lfloor \frac{n}{2} \rfloor - (n + 1 - b - \lfloor \frac{n+b}{2} \rfloor) + 1$, from which a parity argument yields the result. □

Proposition 18 *Let b and n be integers such that $n \geq 3b - 1$. If n and b have opposite parity, then every triangle vertex in $G_{n,b}$ is in at least two triangles. If n and b have the same parity, then the smallest triangle vertex (in L) and the largest triangle vertex (in R) are the only triangle vertices that are in exactly one triangle.*

Proof: As in the proof of Proposition 16, if v_l is the smallest triangle vertex in L, and v_r, v_{r+1} is the largest pair of consecutive, adjacent non-L vertices that are adjacent to v_l, then $r = \lfloor \frac{n+b}{2} \rfloor$. It follows that v_r is adjacent to v_{r+2} if and only if n and b have opposite parity. Thus, vertex v_l is in a second triangle $\{v_l, v_r, v_{r+2}\}$ if and only if n and b have opposite parity. It is not hard to show that each of the other triangle vertices in L is in at least two triangles, regardless of the parities of n and b. A symmetric argument may be applied to the triangle vertices of R. □

Corollary 19 *Let b and n be integers such that $n \geq 3b - 1$. Then the number of vertices in set L that are in at least two triangles equals $\lfloor \frac{3b-1}{2} \rfloor$ if n is even, and equals $\lfloor \frac{3b-2}{2} \rfloor$ if n is odd.*

Proof: Follows directly from Corollary 17 and Proposition 18. □

The final result of this subsection is fundamental to subsequent analysis.

Corollary 20 *Let b and n be integers such that $n \geq 3b - 1$. Then there are at least b vertices in each of the sets L and R that are in at least two triangles.*

5.2 Properties of c-Labelings of n-Vertex Graphs

In this subsection, we establish various properties of c-labelings, which will be necessary in order to determine exact values for $h(n, b)$. Throughout, G is an n-vertex graph and $c \geq b$.

Proposition 21 *Let g be a c-labeling of a graph, and let x, u, v, and y be vertices such that $\langle x, u, v, y \rangle$ is a path from x to y. Then*

$$n + 1 - 3c \leq g(x) + g(y) \leq n + 1 + 3c$$

Proof: Since g is a c-labeling, we have

$$\begin{aligned} n+1-c &\leq g(x) + g(u) \leq n + 1 + c \\ n+1-c &\leq g(v) + g(y) \leq n + 1 + c \\ n+1-c &\leq g(u) + g(v) \leq n + 1 + c \end{aligned}$$

Subtracting the third chain of inequalities from the sum of the first two yields the result. □

Proposition 22 *Let g be a c-labeling of graph G, and let t be an integer satisfying either $1 \leq t \leq \frac{n+1-3c}{2}$ or $\frac{n+1+3c}{2} \leq t \leq n$. Then $g^{-1}(t)$ is not a triangle vertex of G.*

Proof: Suppose that x, y, and z are mutually adjacent vertices of G, and that $1 \leq g(x) \leq \frac{n+1-3c}{2}$. Since y and z are both adjacent to x and g is a c-labeling, we have $g(y) \geq \frac{n+1+c}{2}$ and $g(z) \geq \frac{n+1+c}{2}$. But $g(y) \neq g(z)$, which implies that $g(y) + g(z) > n + 1 + c$, which contradicts the definition of a c-labeling and shows that $g^{-1}(t)$ is not a triangle vertex when $1 \leq t \leq \frac{n+1-3c}{2}$. A similar argument shows that $g^{-1}(t)$ cannot be a triangle vertex when $\frac{n+1+3c}{2} \leq t \leq n$. □

Proposition 23 *Let g be a c-labeling of graph G, where n and c have the same parity. Then $g^{-1}(\frac{n+2-3c}{2})$ cannot be in two triangles.*

Proof: Suppose that $\{x, y, z\}$ is a set of three mutually adjacent vertices and that $g(x) = \frac{n+2-3c}{2}$. Then $g(y) \geq \frac{n+c}{2}$ and $g(z) \geq \frac{n+c}{2}$. But $g(y) + g(z) \leq n + 1 + c$, implying $\{g(y), g(z)\} = \{\frac{n+c}{2}, \frac{n+2+c}{2}\}$. This precludes x from being in another triangle. □

Corollary 24 *Let n, b, and c be integers such that $n \geq 3b - 1$ and b has parity opposite to the parities of n and c. Let g be a c-labeling of graph $G = G_{n,b}$. Then $g^{-1}(\frac{n+2-3c}{2})$ is not a triangle vertex of G.*

Proof: This follows directly from Propositions 18 and 23. □

Lemma 25 *Let g be a c-labeling of graph G, and let u and v be vertices that are both adjacent to a vertex x. Then $|g(u) - g(v)| \leq 2c$.*

Proof: Subtract the two chains of inequalities $n + 1 - c \leq g(u) + g(x) \leq n + 1 + c$ and $n + 1 - c \leq g(v) + g(x) \leq n + 1 + c$ to obtain the result. □

For subsets U and W of vertices in a graph G, let $N(W)$ denote the set of vertices adjacent to at least one vertex in W, and $N_U(W)$ denotes the set $N(W) \cap U$.

Proposition 26 *Let g be a c-labeling of graph G, and let t be an integer satisfying $1 \leq t < \frac{n+1-2c}{2}$. Then*

$$N(g^{-1}\{1, 2, \ldots, t\}) \cap N(g^{-1}\{n + 1 - t, n + 2 - t, \ldots, n\}) = \emptyset$$

Proof: Suppose $x \in N(g^{-1}\{1, 2 \ldots, t\}) \cap N(g^{-1}\{n+1-t, n+2-t, \ldots, n\})$. Then there exist vertices u and v that are both adjacent to x such that $g(u) \leq t$ and $g(v) \geq n + 1 - t$. But then $|g(u) - g(v)| \geq n + 1 - 2t > n + 1 - 2 \cdot \frac{n+1-2c}{2} = 2c$, which contradicts Lemma 25. □

The next result shows that for sufficiently small u, the vertices of $G_{n,b}$ that are assigned the u smallest labels are either all in set L or all in set R, and likewise for the vertices assigned the u largest labels.

Proposition 27 *Let g be a c-labeling of $G_{n,b}$, where $n \geq 3b - 1$ and $c < 3b$. Let $u = \lfloor \frac{n+2-3c}{2} \rfloor$. Then $g^{-1}\{1, 2, \ldots, u\}$ is a subset of L or R, and $g^{-1}\{n + 1 - u, n + 2 - u, \ldots, n\}$ is a subset of L or R.*

Proof: Observe that when n is odd, the lone C-vertex $v_{\frac{n+1}{2}}$ is in at least two triangles (since $b \geq 2$). Thus, in all cases, $g^{-1}\{1, 2, \ldots, u\} \subseteq R \cup L$ by Propositions 22 and 23. Since the assertion is trivially true for $u \leq 1$, we assume that $u \geq 2$.

Suppose that the sets $P = g^{-1}\{1,2,\ldots,u\} \cap L$ and $Q = g^{-1}\{1,2,\ldots,u\} \cap R$ are both nonempty, and let p_{min}, p_{max}, q_{min}, and q_{max} be the smallest and largest indices of vertices in each of the sets P and Q, respectively. Then by Propositions 22 and 23, none of these four vertices can be in more than one triangle. By Corollary 20, there are at least b triangle vertices in L whose indices are larger than p_{max}, and at least b triangle vertices in R whose indices are smaller than q_{min}. Thus, $N_R(v_{p_{max}})$ contains at least b vertices that are not partners of any vertex in P, and $N_L(v_{q_{min}})$ contains at least b vertices that are not partners of any vertex in Q.

Moreover, $v_{p_{min}}$ is not adjacent to $v_{q_{max}}$, since $g(v_{p_{min}}) + g(v_{q_{max}}) \leq 2u - 1 \leq n + 1 - 3c < n + 1 - c$. Hence, at least one of the inequalities $p_{min} \geq b + 1$ and $q_{max} \leq n - b$ must hold. Thus, there are at least $3b$ neighbors of $P \cup Q$ that are not partners of any vertices in $P \cup Q$. In addition, each partner of a vertex in P and each partner of a vertex in Q are elements of $N_R(P)$ and $N_L(Q)$, respectively. It follows that $|N(P \cup Q)| \geq |N_R(P)| + |N_L(Q)| \geq u + 3b$. But a neighbor of a vertex in $P \cup Q$ must receive a label between $n+1-u-c$ and n, of which there are only $u+c < u+3b$ different values, showing such a labeling is impossible. A similar argument demonstrates that $g^{-1}\{n+1-u,\ldots,n\}$ is a subset of L or R. □

Corollary 28 *Let g be a c-labeling of $G_{n,b}$, where $n \geq 3b-1$, $c < 3b$, and $n \geq 6c - 7b + 1$. Let $u = \lfloor \frac{n+2-3c}{2} \rfloor$. Then $g^{-1}\{1,2,\ldots,u\} \subseteq L$ and $g^{-1}\{n+1-u, n+2-u, \ldots, n\} \subseteq R$, or vice versa.*

Proof: We may assume that $u \geq 1$. Let $S = g^{-1}\{1,2,\ldots,u\}$ and $T = g^{-1}\{n+1-u, n+2-u, \ldots, n\}$. By Proposition 27, it suffices to show that $S \cup T$ is neither a subset of L nor a subset of R.

Suppose $S \cup T \subseteq L$. Since $c \geq b \geq 2$, Proposition 26 implies that $N(S) \cap N(T) = \emptyset$, and hence, there must be at least $2b$ vertices between any vertex in S and any vertex in T. The argument proceeds by considering the number of non-triangle vertices in $S \cup T$.

Case 1: n and c have opposite parity (and hence, $u = \frac{n+1-3c}{2}$)

By Proposition 22, all of the vertices in $S \cup T$ are non-triangle vertices. Moreover, no vertex between an S-vertex and a T-vertex can be a triangle vertex. Hence, there must be at least $2u + 2b$ non-triangle vertices in L. Thus, by Proposition 16, $n - b - \lfloor \frac{n+b}{2} \rfloor \geq 2u + 2b = n + 1 - 3c + 2b$. It follows that

$$n \leq \begin{cases} 6c - 7b - 2 & \text{if } n \text{ and } b \text{ have same parity} \\ 6c - 7b - 1 & \text{if } n \text{ and } b \text{ have opposite parity} \end{cases}$$

which contradicts the lower bound on n.

Case 2: n and c have the same parity

If n and b have opposite parity, then by Proposition 16 and Corollary

24, $n - b - \lfloor\frac{n+b}{2}\rfloor \geq 2u + 2b = n + 2 - 3c + 2b$. If n and b have the same parity, Proposition 22 implies that all but at most two of the vertices in $S \cup T$ are non-triangle vertices ($g^{-1}(u)$ and $g^{-1}(n+1-u)$ can be triangle vertices). Thus, $n - b - \lfloor\frac{n+b}{2}\rfloor \geq 2u - 2 + 2b = n - 3c + 2b$. It follows that

$$n \leq \begin{cases} 6c - 7b & \text{if } n \text{ and } b \text{ have same parity} \\ 6c - 7b - 3 & \text{if } n \text{ and } b \text{ have opposite parity} \end{cases}$$

which again contradicts the lower bound on n.

A symmetric argument can be used to show that $S \cup T \not\subseteq R$. □

Given a labeling g, two vertices v_i and v_j are said to be *successive vertices* in $g^{-1}\{1, 2, \ldots, w\}$ if $v_t \notin g^{-1}\{1, 2, \ldots, w\}$ for all t strictly between i and j.

Proposition 29 *Let g be a c-labeling of $G_{n,b}$, where $n \geq 3b-1$ and $c < 3b$. Let $w \leq \lfloor\frac{n+2-3c}{2}\rfloor$. Then for every pair of successive vertices $v_i, v_j \in g^{-1}\{1, 2, \ldots, w\}$, $|i - j| \leq 2b$. The same inequality holds for successive vertices in $g^{-1}\{n+1-w, n+2-w, \ldots, n\}$.*

Proof: By Proposition 27 and the symmetry of $G_{n,b}$, we may assume without loss of generality that $g^{-1}\{1, 2, \ldots, w\} \subseteq R$. Suppose v_i, v_j are successive vertices in $g^{-1}\{1, 2, \ldots, w\}$ such that $j - i \geq 2b+1$. Let $P = \{v_t \in g^{-1}\{1, 2, \ldots, w\} \mid t \leq i\}$ and $Q = \{v_t \in g^{-1}\{1, 2, \ldots, w\} \mid t \geq j\}$. Then $N_L(P) \cap N_L(Q) = \emptyset$, and hence, arguing in a manner similar to the proof of Proposition 27, $|N_L(P \cup Q)| \geq w + 3b > w + c$, which is greater than the number of labels available for $N_L(P \cup Q)$. A similar argument establishes the inequality for successive vertices in $g^{-1}\{n+1-w, n+2-w, \ldots, n\}$. □

5.3 Establishing Lower Bounds for $h(n, b)$

Proposition 30 *Let n, b, and c be integers such that $3c + 1 \leq n \leq 7b$ and $c < 3b$. Then $h(n, b) > c$.*

Proof: Since $n \leq 7b$, Corollary 17 implies that there are at most $2b$ non-triangle vertices in each of the sets L and R in $G_{n,b}$. By Proposition 16, the indices of the non-triangle L-vertices are $1, 2, \ldots, (n - b - \lfloor\frac{n+b}{2}\rfloor)$. Let $e = v_b v_{b+1}$. Each of the non-triangle vertices of R is adjacent to both v_b and v_{b+1}. It follows that every R-vertex in $G_{n,b} + e$ is now a triangle vertex. Suppose g is a c-labeling of $G_{n,b} + e$. Since $n \geq 3c + 1$, Proposition 22 implies that $g^{-1}(1)$ and $g^{-1}(n)$ are both non-triangle vertices, and hence are two of the L-vertices with indices between 1 and $n - b - \lfloor\frac{n+b}{2}\rfloor$. Thus, there is an R-vertex, v_x, adjacent to both of these vertices. It follows that

$x+1 \geq n+1-b$ and $x+n \leq n+1+b$. Subtracting these inequalities shows $n \leq 2b+1$, which contradicts $n \geq 3c+1$. □

For the remainder of this section, we assume that $n > 7b$.

Proposition 31 *Let n, b, and c be integers such that $c < 3b$ and $n \geq 6c-7b+1$. Let $z = \lfloor \frac{n+1-3c}{2} \rfloor$, and let e be an edge joining a pair of non-adjacent consecutive L-vertices in $G_{n,b}$. If $2b$ new triangle vertices of set R in $G_{n,b}+e$ are created so that the remaining non-triangle R-vertices are separated into two nonempty sets, each of size $\leq z-1$, then $B^+(G_{n,b}+e) > c$.*

Proof: Let R^+ be the set of non-triangle R-vertices whose indices are larger than those of the new triangle vertices in $G_{n,b}+e$, and let R^- be those vertices whose indices are smaller. Suppose $G_{n,b}+e$ has a c-labeling g. Corollary 28 implies that $g^{-1}\{1,2,\ldots,z\} \subseteq R$ or $g^{-1}\{n+1-z, n+2-z, \ldots, n\} \subseteq R$. Assume, without loss of generality, that $g^{-1}\{1,2,\ldots,z\} \subseteq R$. Then, by Proposition 22, $g^{-1}\{1,2,\ldots,z\} \subseteq R^+ \cup R^-$. But both sets $R^+$and R^- have size at most $z-1$, which implies that the sets $R^+ \cap g^{-1}\{1,2,\ldots,z\}$ and $R^- \cap g^{-1}\{1,2,\ldots,z\}$ are both nonempty. It follows that the indices of the smallest vertex in $R^+ \cap g^{-1}\{1,2,\ldots,z\}$ and the largest vertex in $R^- \cap g^{-1}\{1,2,\ldots,z\}$ differ by at least $2b+1$, which contradicts Proposition 29. □

Corollary 32 *Let n, b, and c be integers such that $c < 3b$, $n \geq 6c-7b+1$, and $\left\lceil \frac{n-3b-\lfloor \frac{n+b}{2} \rfloor}{2} \right\rceil \leq \lfloor \frac{n-1-3c}{2} \rfloor$. Then $h(n,b) > c$.*

Proof: By Proposition 16, there are $n-b-\lfloor \frac{n+b}{2} \rfloor$ non-triangle R-vertices in $G_{n,b}$, so the indices of their partner vertices in L range from 1 to $n-b-\lfloor \frac{n+b}{2} \rfloor$. Thus, if $e = v_s v_{s+1}$, where $s = \left\lfloor \frac{n-b-\lfloor \frac{n+b}{2} \rfloor}{2} \right\rfloor$, then $G_{n,b}+e$ contains $2b$ new triangle vertices in R. Moreover, these vertices separate the remaining $n-3b-\lfloor \frac{n+b}{2} \rfloor$ non-triangle R-vertices into two sets of sizes $\left\lfloor \frac{n-3b-\lfloor \frac{n+b}{2} \rfloor}{2} \right\rfloor$ and $\left\lceil \frac{n-3b-\lfloor \frac{n+b}{2} \rfloor}{2} \right\rceil$. Since $n \geq 7b+1$ if n and b have opposite parity and $n \geq 7b+2$ if they have the same parity, these sets are nonempty and, by hypothesis, no larger than $z-1$, where $z = \lfloor \frac{n+1-3c}{2} \rfloor$. Hence, by Proposition 31, $B^+(G_{n,b}+e) > c$. □

Proposition 33 *Let n, b, and c be integers such that b has parity opposite to the parities of n and c, $c < 3b$, $\frac{n+2-3c}{2} \geq 3$, and $n \geq 6c-7b+1$. Let $u = \frac{n+2-3c}{2}$, and let e be an edge joining non-adjacent L-vertices v_s and v_{s+2} in $G_{n,b}$. Suppose that $G_{n,b}+e$ contains $2b-1$ new triangle vertices in set R, and that these vertices separate the remaining non-triangle R-vertices into two sets, each of size $\leq u-1$. Then $B^+(G_{n,b}+e) > c$.*

Proof: Let R^+ and R^- be defined as in the proof of Proposition 31, and assume that g is a c-labeling of $G_{n,b} + e$. By Proposition 22 and Corollary 28, we may assume without loss of generality that $g^{-1}\{1, 2, \ldots, u-1\} \subseteq R^+ \cup R^-$. The rest of the proof hinges on the observation that the two smallest vertices in R^+ are adjacent to v_s, and the two largest vertices in R^- are adjacent to v_{s+2}. Figure 10 illustrates this for $G_{20,2}$ and $s = 3$.

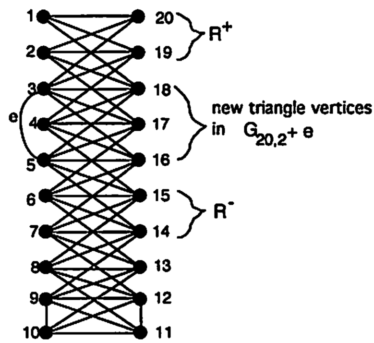

Figure 10: Illustrating Proposition 33 for $G_{20,2}$ and $s = 3$

There are two cases to consider, according to where the vertices of $g^{-1}\{1, 2, \ldots, u-1\}$ lie.

Case 1: $g^{-1}\{1, 2, \ldots, u-1\} = R^+$ or $g^{-1}\{1, 2, \ldots, u-1\} = R^-$
Assume that the first equality holds. Then at least one of the two vertices of R^+ that are adjacent to v_s was assigned a label $\leq u-2$ (since $u \geq 3$). It follows by Proposition 21 that $g^{-1}(u)$ could not be either of the two largest vertices of R^-, nor could it be any of the new triangle vertices (since they too are adjacent to v_{s+2}). Thus, the difference between the index of the smallest vertex in $g^{-1}\{1, 2, \ldots, u-1\}$ and the index of $g^{-1}(u)$ is greater than $2b$, which contradicts Proposition 29. A contradiction is similarly derived under the assumption that $g^{-1}\{1, 2, \ldots, u-1\} = R^-$.

Case 2: $g^{-1}\{1, 2, \ldots, u-1\} \cap R^+ \neq \emptyset$ and $g^{-1}\{1, 2, \ldots, u-1\} \cap R^- \neq \emptyset$
Let $S^+ = g^{-1}\{1, 2, \ldots, u-1\} \cap R^+$ and $S^- = g^{-1}\{1, 2, \ldots, u-1\} \cap R^-$. By Proposition 21, either the two smallest vertices in R^+ are not in S^+ or the two largest vertices in R^- are not in S^-. In either case, the difference between the indices of the smallest vertex in S^+ and the largest vertex in S^- is greater than $2b$, which contradicts Proposition 29 (taking $w = u-1$).
□

Corollary 34 *Let n, b, and c be integers such that b has parity opposite to the parities of n and c, $c < 3b$, $n \geq 3c+4$, $n \geq 6c - 7b + 1$, and $\lceil \frac{n-7b+3}{4} \rceil \leq \frac{n-3c}{2}$. Then $h(n, b) > c$.*

Proof: The proof is similar to that of Corollary 32, although now we let the edge $e = v_s v_{s+2}$. Then $G_{n,b} + e$ contains $2b - 1$ new triangle vertices in R, and these vertices separate the remaining $n - 3b - \lfloor \frac{n+b}{2} \rfloor + 1 = \frac{n-7b+3}{2}$ non-triangle R-vertices into two nonempty sets of sizes $\lfloor \frac{n-7b+3}{4} \rfloor$ and $\lceil \frac{n-7b+3}{4} \rceil$, which, by hypothesis, are no larger than $u - 1$, where $u = \frac{n+2-3c}{2} \geq 3$. The conclusion follows by Proposition 33. □

Observe that the values of n for which Corollary 34 implies $h(n, b) > c$ include all those satisfying the hypotheses of Corollary 32.

Proposition 35 *Let n, b, and c be integers such that $c < 3b$, and suppose that one of the following conditions holds:*
(1) n, b, and c have the same parity, and $n \geq 6c - 7b + 4$.
(2) n has opposite parity to b and c, and $n \geq 6c - 7b + 3$.
(3) c has opposite parity to b and n, and $n \geq 6c - 7b + 2$.
(4) b has opposite parity to n and c, $n \leq 3c + 2$, and $n \geq 6c - 7b + 5$.
(5) b has opposite parity to n and c, $n \geq 3c + 4$, and $n \geq 6c - 7b + 3$.

Then $h(n, b) > c$.

Proof: If any one of Conditions (1) through (4) holds, then the result follows by a straightforward parity argument applied to Corollary 32. For instance, if n, b, and c have the same parity, then the inequality $\left\lceil \frac{n-3b-\lfloor \frac{n+b}{2} \rfloor}{2} \right\rceil \leq \lfloor \frac{n-1-3c}{2} \rfloor$ of Corollary 32 reduces to $n \geq 6c - 7b + 4$ if $n - 7b = 0$ (mod 4), and it reduces to $n \geq 6c - 7b + 6$ if $n - 7b = 2$ (mod 4). But if n, b, and c have the same parity and $n - 7b = 2$ (mod 4), then $n \neq 6c - 7b + 4$. Thus, Condition (1) implies that the hypothesis of Corollary 32 is satisfied. A similar argument can be used to show that if Condition (5) holds, the result follows by Corollary 34. □

6 Sharpness of the Lower Bound

In this section, we show that the lower bound for $h(n, b)$ for each of the five cases in Proposition 35 is sharp. In particular, we prove the following.

Proposition 36 *Let n, b, and c be integers such that $n \geq 3b-1$, $n \geq 3c+1$, $b \geq 2$, and $\frac{7b-1}{3} \leq c < 3b$.*
Assume further that one of the following conditions holds:
(1) n, b, and c have the same parity, and $n = 6c - 7b + 2$.
(2) n has opposite parity to b and c, and $n = 6c - 7b + 1$.
(3) c has opposite parity to b and n, and $n = 6c - 7b$.
(4) b has opposite parity to n and c, $n = 3c + 2$, and $n = 6c - 7b + 3$.

(5) b has opposite parity to n and c, $n \geq 3c+2$, *and* $n = 6c - 7b + 1$.

Then $h(n,b) \leq c$.

There is no loss of generality in including the condition $n \geq 3c+1$ in the hypothesis of Proposition 36, since otherwise, $h(n,b) \leq c$ by Proposition 10.

The proof of Proposition 36 consists of several cases, each of which is presented in a separate subsection. In each case our task is to construct a c-labeling g of $G_{n,b} + e$, where e is any edge joining non-adjacent vertices v_r and v_s of $G_{n,b}$. The construction of g depends on the intervals in which r and s lie. We assume that $r+1 \leq s \leq n-b-r$ and $v_r \in L$. If these conditions are not satisfied, we may interchange the roles of r and s and/or of R and L accordingly. There are six cases to consider. In each case, the c-labeling g must be shown to satisfy $n+1-c \leq g(v_i) + g(v_j) \leq n+1+c$ for each edge $v_i v_j$ of $G_{n,b} + e$.

The following lemma is convenient for the arguments which follow.

Lemma 37 *The hypothesis of Proposition 36 implies* $c \geq b+3$.

Proof: We have $c \geq \lceil \frac{7b-1}{3} \rceil = b + 2 + \lceil \frac{4b-7}{3} \rceil \geq b+3$, where the last inequality follows because $b \geq 2$. □

We now present the various cases.

6.1 $n - 2c + 1 \leq s \leq n - b - r$

Subcase a: $r \geq c+1$. Then the standard labeling $g(v_t) = t$ may be used since

$$n+1-c = c + (n-2c+1) < r+s \leq r+n-b-r < n+1+c$$

Subcase b: $r \leq c$. Then define the labeling g by

$$g(v_t) = \begin{cases} c+1 & \text{if } t = r \\ t-1 & \text{if } r+1 \leq t \leq c+1 \\ t & \text{otherwise} \end{cases}$$

Verification of the endpoint sum of each edge $v_i v_j$ follows:

1. $v_r v_s$: $n+1-c = (c+1) + (n-2c) \leq g(v_r) + (s-1) \leq g(v_r) + g(v_s) \leq c+1+s < n+1+c$

2. $v_i v_j$, $i,j \neq r$: $n+1-c < n+1-b-2 \leq i+j-2 \leq g(v_i) + g(v_j) \leq i+j \leq n+1+b < n+1+c$.

3. $v_r v_j$: $n+1-c < n+1-b \leq r+j \leq c+j = (c+1)+(j-1) \leq g(v_r) + g(v_j) \leq c+1+j \leq n+1+c$.

6.2 $\lfloor\frac{n+1-c}{2}\rfloor \le r < s \le n-2c$

The following chain of inequalities shows that the standard labeling suffices. The last inequality in the chain holds because $n \le 6c - 7b + 3$ and $c < 3b$.

$$\begin{aligned} n+1-c &= \left\lfloor\frac{n+1-c}{2}\right\rfloor + \left\lceil\frac{n+1-c}{2}\right\rceil \\ &\le \left\lfloor\frac{n+1-c}{2}\right\rfloor + \left\lfloor\frac{n+1-c}{2}\right\rfloor + 1 \\ &\le r + s \le (n-2c-1) + (n-2c) \le n+1+c \end{aligned}$$

6.3 $\left\lfloor\frac{n-3c+1}{2}\right\rfloor + b \le r \le \left\lfloor\frac{n-c-1}{2}\right\rfloor$ and $s \le n-2c$

Subcase a: $s \ge \lfloor\frac{n-c+3}{2}\rfloor$. Define the labeling g by

$$g(v_t) = \begin{cases} \lfloor\frac{n-c+1}{2}\rfloor & \text{if } t = r \\ t-1 & \text{if } r+1 \le t \le \lfloor\frac{n-c+1}{2}\rfloor \\ t & \text{otherwise} \end{cases}$$

Subcase b: $s \le \lfloor\frac{n-c+1}{2}\rfloor$. Define g by

$$g(v_t) = \begin{cases} \lfloor\frac{n-c+1}{2}\rfloor & \text{if } t = r \\ \lfloor\frac{n-c+3}{2}\rfloor & \text{if } t = s \\ t-1 & \text{if } r+1 \le t \le s-1 \\ t-2 & \text{if } s+1 \le t \le \lfloor\frac{n-c+3}{2}\rfloor \\ t & \text{otherwise} \end{cases}$$

The following arguments, which show that the endpoint sums satisfy the requirements of a c-labeling, apply to both subcases.

1. $v_r v_s$: Clearly, in either subcase, $g(v_r) + g(v_s) \ge n - c + 1$. Also, $g(v_r) + g(v_s) \le \lfloor\frac{n-c+1}{2}\rfloor + (n-2c) = n+1+c+\lfloor\frac{n-7c-1}{2}\rfloor \le n+1+c+\lfloor\frac{-c-7b+2}{2}\rfloor < n+1+c$, where the second to the last inequality follows because $n \le 6c - 7b + 3$.

2. $v_x v_t$, $x \in \{r, s\}$ and $t \notin \{r, s\}$: Observe that $g(v_x) \ge x$ and $g(v_s) - s \le g(v_r) - r \le \lfloor\frac{n-c+1}{2}\rfloor - (\lfloor\frac{n-3c+1}{2}\rfloor + b)$. It follows from the first inequality that $g(v_x)+g(v_t) \ge x+t-2 \ge (n+1-b)-2 \ge n+1-c$. The second inequality shows that $g(v_x)+g(v_t) \le t+[x+\lfloor\frac{n-c+1}{2}\rfloor - (\lfloor\frac{n-3c+1}{2}\rfloor + b)] \le n+1+b+\lfloor\frac{n-c+1}{2}\rfloor - (\lfloor\frac{n-3c+1}{2}\rfloor + b) = n+1+\lfloor\frac{n-c+1}{2}\rfloor - (\lfloor\frac{n-c+1}{2}\rfloor - c) = n+1+c$.

3. v_iv_j, $i,j \neq r,s$: Observe that $t = \lfloor\frac{n-c+3}{2}\rfloor$ is the largest t such that $g(v_t) < t$. It follows that any two vertices whose labels have been reduced cannot be adjacent in $G_{n,b}$ since $\lfloor\frac{n-c+3}{2}\rfloor+\lfloor\frac{n-c+3}{2}\rfloor-1 \leq n-c+2 < n+1-b$. Thus,
$n+1-c \leq n+1-b-2 \leq i+j-2 \leq g(v_i)+g(v_j) \leq i+j \leq n+1+b < n+1+c$.

6.4 $1 \leq r \leq \left\lfloor\frac{n-3c-1}{2}\right\rfloor + b$, $s \leq n-2c$, and $n \geq 3c+2$

To simplify the notation when referring to a vertex v_t, we frequently use its index t, for instance, $g(t)$ means $g(v_t)$. Also, since a labeling g often assigns labels sequentially to various subsets of consecutive vertices, we let the expression $g\langle t_1, t_2, \ldots, t_l\rangle = \langle a_1, a_2, \ldots, a_l\rangle$ mean $g(t_i) = a_i$, $i = 1, 2, \ldots, l$. Moreover, the indices $t_1, t_2, \ldots, t_l$ will always be consecutive and increasing, whereas the assigned labels $a_1, a_2, \ldots, a_l$ are consecutive but may be increasing or decreasing. Also, $\langle a_1, a_2, \ldots, \overline{a_t}, \ldots, a_m\rangle$ denotes the sequence $\langle a_1, a_2, \ldots, a_{t-1}, a_{t+1}, \ldots, a_m\rangle$. The set notation $g\{t_1, t_2, \ldots, t_l\} = \{a_1, a_2, \ldots, a_l\}$ has its usual meaning: $g(t_i) \in \{a_1, a_2, \ldots, a_l\}$, $i = 1, 2, \ldots, l$.

6.4.1 Specifying the c-Labeling g

The labeling g is a piecewise-defined function on six subintervals of $\langle 1, 2, \ldots, n\rangle$, determined by partition points d_1, d_2, d_3, d_4, d_5, where

$$d_1 = \lfloor\tfrac{n-5c-3}{2}\rfloor + 3b, \qquad d_2 = d_1 + (c+2), \qquad d_3 = d_2 + (n-3c-2),$$
$$d_4 = d_3 + (9c - 5b - 2n + 4) - [(n-c) \bmod 2], \qquad d_5 = d_4 + (n-3c-1)$$

The following lemma shows that the d_i's do indeed partition the interval $\langle 1, 2, \ldots, n\rangle$ into six subintervals (one of which is empty when $n = 3c+2$).

Lemma 38 *Let integers n, b, and c satisfy the hypothesis of Proposition 36 and the inequality $n \geq 3c+2$. Then the d_i's defined above satisfy $0 < d_1 < d_2 \leq d_3 < d_4 < d_5 < n$.*

Proof: (a) $d_1 > 0$: If n and c have opposite parity, then $d_1 = \frac{n-5c-3+6b}{2}$ and one of Conditions (2) or (3) of Proposition 36 hold. Thus, $d_1 \geq \frac{(6c-7b)-5c-3+6b}{2} = \frac{c-b-3}{2} \geq 0$ by Lemma 37. But if $c = b+3$, $3c = 3b+9 \geq 7b-1$, which would imply $b = 2$, $c = 5$, and $n = 16$, contradicting our assumption that $n \geq 3c+2$. Thus, $c \neq b+3$ and $d_1 > 0$. Similar arguments may be used when n and c have the same parity to complete the proof that $d_1 > 0$.
(b) $d_3 < d_4$: If n and c have opposite parity, then $n \leq 6c - 7b + 1$ and $d_4 - d_3 = 9c - 5b - 2n + 3 \geq 9c - 5b - 2(6c - 7b + 1) + 3 = 3(3b - c) + 1 > 0$. If n and c have the same parity, then $n \leq 6c - 7b + 3$ and $d_4 - d_3 = 9c - 5b - 2n + 4 \geq 9c - 5b - 2(6c - 7b + 3) + 4 = 3(3b - 1 - c) + 1 > 0$.

(c) $d_5 < n$: Since $d_5 = d_1 + 4c - 5b + 3 - [(n-c) \bmod 2]$, we have $n-d_5 = n-d_1-4c+5b-3+[(n-c) \bmod 2] \geq n-\lfloor\frac{n-5c-3}{2}\rfloor-3b-4c+5b-3 \geq n - \frac{n-5c-3}{2} - 4c + 2b - 3 = \frac{n-3c+4b-3}{2} \geq \frac{(6c-7b)-3c+4b-3}{2} = \frac{3(c-b-1)}{2} > 0$, where the last inequality follows from Lemma 37.
(d) The remaining inequalities are obvious consequences of $c > 0$ or $n \geq 3c+2$. □

Although there are only five non-empty subintervals when $n = 3c+2$, the numbering (1) through (6) of the six subintervals will be retained. We now completely specify the action of g on all but the second subinterval:

Subinterval (1): $g\langle 1, 2, \ldots, d_1\rangle = \langle d_3, d_3 - 1, \ldots, n - 2c + 1\rangle$

Subinterval (2): $g\{d_1 + 1, \ldots, d_2\} = \{1, 2, \ldots, c+1\} \cup \{n - 2c\}$

Subinterval (3) (if $n > 3c+2$): $g\langle d_2 + 1, \ldots, d_3\rangle = \langle c+2, c+3, \ldots, n-2c-1\rangle$

Subinterval (4): $g\langle d_3 + 1, \ldots, d_4\rangle = \langle d_3 + 1, d_3 + 2, \ldots, d_4\rangle$

Subinterval (5): $g\langle d_4 + 1, \ldots, d_5\rangle = \langle 3c + 2, 3c + 3, \ldots, n\rangle$

Subinterval (6): $g\langle d_5 + 1, \ldots, n\rangle = \langle 3c + 1, 3c, \ldots, d_4 + 1\rangle$

6.4.2 Specifying the Action of g on Subinterval (2)

Except for one, two, or three special assignments that depend on the subintervals in which r and s lie, the vertices in Subinterval (2) are labeled with $1, 2, \ldots, c+1$ in increasing order if $d_4 \geq n-c-1$, and are labeled in two segments, both consecutive, one increasing and one decreasing, if $d_4 \leq n-c-2$. The labeling of the two segments also depends on the comparison of s with the value of $d_1 + (n - 3c + 2)$, which we denote by p. Observe that the conditions $3c+2 \leq n \leq 6c-7b+3$ and $c \geq \frac{7b-1}{3}$ ensure that $d_1 < p < d_2$. Also note that the upper bound on r for Case 4, $\lfloor\frac{n-3c-1}{2}\rfloor+b = d_2-2b-1$.

Subinterval (2) - Case *i*: $d_4 \geq n - c - 1$.

Subcase a: $1 \leq r \leq d_1$ and ($s \leq d_1$ or $s \geq d_2 + 1$).

$g(d_1 + 1) = n - 2c, \quad g\langle d_1 + 2, d_1 + 3, \ldots, d_2\rangle = \langle 1, 2, \ldots, c+1\rangle$

Subcase b: $1 \leq r \leq d_1$ and $d_1 + 1 \leq s \leq d_2 - 2b - 1$.

$g(s) = n - 2c, \quad g\langle d_1 + 1, d_1 + 2, \ldots, \overline{s}, \ldots, d_2\rangle = \langle 1, 2, \ldots, c+1\rangle$

Subcase c: $1 \leq r \leq d_1$ and $d_2 - 2b \leq s \leq d_2$.

$$g(d_1 + 1) = n - 2c, \quad g(s) = c + 1,$$
$$g\langle d_1 + 2, d_1 + 3, \ldots, \overline{s}, \ldots, d_2\rangle = \langle 1, 2, \ldots, c\rangle$$

Subcase d: $d_1 + 1 \leq r < s \leq d_2$.

$$g(r) = n - 2c, \quad g(s) = c + 1,$$
$$g\langle d_1 + 1, d_1 + 2, \ldots, \overline{r}, \ldots, \overline{s}, \ldots, d_2\rangle = \langle 1, 2, \ldots, c\rangle$$

Subcase e: $d_1 + 1 \leq r \leq d_2 - 2b - 1$ and $s \geq d_2 + 1$.

$$g(r) = n - 2c, \quad g\langle d_1 + 1, d_1 + 2, \ldots, \overline{r}, \ldots, d_2\rangle = \langle 1, 2, \ldots, c + 1\rangle$$

Subinterval (2) - Case *ii*: $d_4 \leq n - c - 2$ and $s \leq p$.

Subcase a: $1 \leq r < s \leq d_1$.

$$g(d_1 + 1) = n - 2c, \quad g\langle d_1 + 2, d_1 + 3, \ldots, p\rangle = \langle n - 3c + 1, n - 3c, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, d_2\rangle = \langle n - 3c + 2, n - 3c + 3, \ldots, c + 1\rangle$$

Subcase b: $1 \leq r \leq d_1$ and $d_1 + 1 \leq s \leq d_2 - 2b - 1$.

$$g(s) = n - 2c, \quad g\langle d_1 + 1, d_1 + 2, \ldots, \overline{s}, \ldots, p\rangle = \langle n - 3c + 1, n - 3c, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, d_2\rangle = \langle n - 3c + 2, n - 3c + 3, \ldots, c + 1\rangle$$

Subcase c: $1 \leq r \leq d_1$ and $d_2 - 2b \leq s \leq d_2$.

$$g(d_1 + 1) = n - 2c, \quad g(s) = c + 1,$$
$$g\langle d_1 + 2, d_1 + 3, \ldots, \overline{s}, \ldots, p\rangle = \langle n - 3c, n - 3c - 1, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, d_2\rangle = \langle n - 3c + 1, n - 3c + 2, \ldots, c\rangle$$

Subcase d: $d_1 + 1 \leq r < s \leq d_2$.

$$g(r) = n - 2c, \quad g(s) = c + 1,$$
$$g\langle d_1 + 1, d_1 + 2, \ldots, \overline{r}, \ldots, \overline{s}, \ldots, p\rangle = \langle n - 3c, n - 3c - 1, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, d_2\rangle = \langle n - 3c + 1, n - 3c + 2, \ldots, c\rangle$$

Subinterval (2) - Case *iii*: $d_4 \leq n - c - 2$ and $s \geq p + 1$.

Subcase a: $1 \leq r \leq d_1$ and $s \geq d_2 + 1$.

$$g(d_1 + 1) = n - 2c, \quad g\langle d_1 + 2, d_1 + 3, \ldots, p\rangle = \langle n - 3c + 1, n - 3c, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, d_2\rangle = \langle n - 3c + 2, n - 3c + 3, \ldots, c + 1\rangle$$

Subcase b: $1 \leq r \leq d_1$ and $d_1 + 1 \leq s \leq d_2 - 2b - 1$.

$$g(s) = n - 2c, \quad g\langle d_1 + 1, d_1 + 2, \ldots, p\rangle = \langle n - 3c + 2, n - 3c + 1, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, \overline{s}, \ldots, d_2\rangle = \langle n - 3c + 3, n - 3c + 4, \ldots, c + 1\rangle$$

Subcase c: $1 \leq r \leq d_1$ and $d_2 - 2b \leq s \leq d_2$.

$$g(d_1 + 1) = n - 2c, \quad g(s) = c + 1,$$
$$g\langle d_1 + 2, d_1 + 3, \ldots, p\rangle = \langle n - 3c + 1, n - 3c, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, \overline{s}, \ldots, d_2\rangle = \langle n - 3c + 2, n - 3c + 3, \ldots, c\rangle$$

Subcase d: $d_1 + 1 \leq r < s \leq d_2$.

$$g(r) = n - 2c, \quad g(s) = c + 1,$$
$$g\langle d_1 + 1, d_1 + 2, \ldots, \overline{r}, \ldots, p\rangle = \langle n - 3c + 1, n - 3c, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, \overline{s}, \ldots, d_2\rangle = \langle n - 3c + 2, n - 3c + 3, \ldots, c\rangle$$

Subcase e: $d_1 + 1 \leq r \leq d_2 - 2b - 1$ and $s \geq d_2 + 1$.

$$g(r) = n - 2c, \quad g\langle d_1 + 1, d_1 + 2, \ldots, \overline{r}, \ldots, p\rangle = \langle n - 3c + 1, n - 3c, \ldots, 1\rangle,$$
$$g\langle p + 1, p + 2, \ldots, d_2\rangle = \langle n - 3c + 2, n - 3c + 3, \ldots, c + 1\rangle$$

6.4.3 Lemmas Used for the Verification that g is a c-Labeling

The following lemmas help determine the subintervals in which adjacent vertices of $G_{n,b}$ may lie.

Lemma 39 *Let $d_1, d_2, \ldots, d_5$ be defined as above. Then the following three equations hold:*
(a) $d_2 + d_3 = 2n - 6c + 6b - 2 + [(n - c) \bmod 2]$
(b) $d_3 + d_4 = n + b$
(c) $d_2 + d_5 = n + 1 + b$

Proof: These are all immediate consequences of the definitions of the d_i's together with parity arguments involving n and c. For instance, for (b) we observe that $d_3 = d_1 + (n-2c)$ and $d_4 = d_1 - n + 7c - 5b + 4 - [(n-c) \bmod 2]$, from which it follows that $d_3 + d_4 = 2\left\lfloor\frac{n-5c-3}{2}\right\rfloor + b + 5c + 4 - [(n-c) \bmod 2]$. This last quantity equals $n + b$ in both cases for the relative parities of n and c. □

Lemma 40 *Let $d_1, d_2, \ldots, d_5$ be defined as above, and let $v_i v_j$ be an edge of $G_{n,b}$ such that $i < j$ and $1 \le i \le d_2 - 2b - 1$. Then index j is in Subinterval (6).*

Proof: Since $v_i v_j \in E(G_{n,b})$, we have $j \ge n + 1 - b - i \ge n + 1 - b - (d_2 - 2b - 1) = n + 1 + b - d_2 + 1 = d_5 + 1$ by Lemma 39(c). □

Lemma 41 *Let $d_1, d_2, \ldots, d_5$ be defined as above, let $v_i v_j$ be an edge of $G_{n,b}$ such that $i < j$ and $i \le d_2$, and let n, b, and c satisfy the hypotheses of Proposition 36. Then $j \ge d_3 + 1$.*

Proof: Since $v_i v_j$ is an edge of $G_{n,b}$, we have the following chain of inequalities, the first equality of which follows from Lemma 39(a).

$$\begin{aligned} j &\ge n + 1 - b - i \ge n + 1 - b - d_2 \\ &= n + 1 - b - (2n - 6c + 6b - 2 + [(n - c) \bmod 2] - d_3) \\ &= d_3 + 1 + (6c - 7b + 2 - n - [(n - c) \bmod 2]) \end{aligned}$$

If n and c have the same parity and $n > 3c+2$, then $n \le 6c - 7b + 2$, in which case the last expression in the chain is no smaller than $d_3 + 1$. If n and c have opposite parity, then $n \le 6c - 7b + 1$ and again, the last expression is at least $d_3 + 1$. The remaining possibility is $n = 3c + 2$ with $d_2 = d_3$. By way of contradiction, suppose $j \le d_2$. Since $i < j$, we have $i + j \le 2d_2 - 1 = 2(d_1 + c + 2) - 1 = 2\left\lfloor\frac{n-5c-3}{2}\right\rfloor + 6b + 2c + 4 - 1 = 6b + 1 = 7b + 1 - b \le n - b$, where the last inequality follows from $n = 3c + 2 \ge 7b - 1 + 2 = 7b + 1$. This contradicts the premise $v_i v_j \in E(G_{n,b})$, and the proof is complete. □

6.4.4 Verification that g is a c-labeling

We now verify for any edge v_iv_j in $G_{n,b}+v_rv_s$ that $n+1-c \leq g(v_i)+g(v_j) \leq n+1+c$. Assume without loss of generality that $i < j$.

1. v_rv_s: Since $r < d_2$ and $s \leq n-2c \leq d_3$, we have $n-2c \leq g(r) \leq d_3$ and $c+1 \leq g(s) \leq d_3$. By Lemma 39(b), $n+1-c \leq g(r)+g(s) \leq 2d_3 < d_3+d_4 = n+b < n+1+c$.

2. v_iv_j, $1 \leq i \leq d_1$: Index j is in Subinterval (6) by Lemma 40. Among the edges incident on vertex v_1, v_1v_{n-b} is the one with the largest endpoint sum of value $g(1)+g(n-b) = d_3+d_4+1+b$, according to the action of g on Subintervals (1) and (6) and the fact that Subinterval (6) has at least $b+1$ elements. Moreover, since $\langle g(1), g(2), \ldots, g(d_1)\rangle$ is a consecutive and decreasing sequence, and $\langle g(n), g(n-1), \ldots, g(d_5+1)\rangle$ is consecutive and increasing, we have $g(1)+g(n-b) = g(2)+g(n-b-1) = g(3)+g(n-b-2)\ldots$, as long as the larger vertex remains in Subinterval (6). It follows that d_3+d_4+1+b is the largest possible endpoint sum of an edge joining vertices with one index in Subinterval (1) and the other index in Subinterval (6). Hence, by Lemma 39(b), $g(i)+g(j) \leq d_3+d_4+1+b = n+2b+1 < n+1+c$ since $c \geq \frac{7b-1}{3}$. Similar reasoning shows that $g(d_1)+g(n)$ is the smallest possible endpoint sum involving Subintervals (1) and (6), from which it follows that $g(i)+g(j) \geq g(d_1)+g(n) = n-2c+1+d_4+1 = n+1-c+(d_4+1-c) > n+1-c$.

3. v_rv_j, $d_1+1 \leq r \leq d_2-2b-1$ and $j \neq s$: We have $g(r) = n-2c$ by the action of g on Subinterval (2), and, by Lemma 40, index j is in Subinterval (6). Thus, $n+1-c < n+1-c+(d_4-c) = (n-2c)+d_4+1 \leq g(r)+g(j) \leq (n-2c)+(3c+1) = n+1+c$.

4. v_sv_j, $d_1+1 \leq s \leq d_2$: Then $g(s) = n-2c$ or $c+1$. If $g(s) = n-2c$, then $s \leq d_2-2b-1$, and again, index j is in Subinterval (6). Hence, $n+1-c < n+1-c+(d_4-c) = (n-2c)+d_4+1 \leq g(s)+g(j) \leq (n-2c)+(3c+1) = n+1+c$. If $g(s) = c+1$, then clearly $g(s)+g(j) \leq n+c+1$. It remains to show $g(s)+g(j) \geq n+1-c$, or equivalently $g(j) \geq n-2c$. By Lemma 41, index j is in Subinterval (4), (5), or (6), and hence, by the action of g on these three subintervals, $g(j) \geq d_3+1$. But $d_3 = d_1+(n-2c) > n-2c$.

5. v_iv_j, $i,j \notin \{r,s\}$, $d_1+1 \leq i \leq d_2$. The case references below correspond to those in Section 6.4.2.

Case i: $d_4 \geq n-c-1$.

If $g(i) = n-2c$, then $i = d_1+1 \leq d_2-2b-1$ and hence by Lemma 40, index j is in Subinterval (6). As in paragraph 4 above, $n+1-c < g(i)+g(j) \leq n+1+c$. Thus we may assume that $1 \leq g(i) \leq c+1$. By Lemma 41, index j lies in Subinterval (4), (5), or (6). If j is in Subinterval (5) or (6), then $g(j) \geq d_4+1$, and hence $g(i)+g(j) \geq d_4+2 \geq n+1-c$.

If j is in Subinterval (4), then, using Lemma 39b, we have $i \geq n+1-b-d_4 = (n+b-d_4)-2b+1 = d_3-2b+1 = d_1+(n-2c)-2b+1 = d_1+4+n-2c-2b-3 \geq d_1+4+c+2-2b-3 \geq d_1+4$, where the last two inequalities follow since $n \geq 3c+2$ and $c \geq 2b+1$. Thus, v_{d_1+4} is the smallest possible vertex that could be adjacent to a vertex in Subinterval (4). Moreover, $2 \leq g(d_1+4) \leq 4$ depending on the values of r and s.

It follows that, if $j = d_4$, $g(i)+g(j) \geq 2+d_4 \geq n+1-c$. If $j = d_4-t$ is any other index in Subinterval (4), it can be shown similarly that $g(i)+g(j) \geq d_4+2 \geq n+1-c$, which shows that the endpoint sum $g(i)+g(j)$ satisfies the lower bound for a c-labeling. Since $g(i) \leq c+1$, $g(i)+g(j) \leq n+1+c$, which shows that the upper bound is satisfied.

Cases *ii* and *iii*: $d_4 \leq n-c-2$.

The following two lemmas are useful in this case. The first makes use of the easily shown assertion that $d_4 = \lfloor \frac{9c-n+4}{2} \rfloor - 2b$.

Lemma 42 *If $c = 2b+1$, then $d_4 > n-c-2$.*

Proof: If $n = 3c+2 = 6c-7b+3$, then $3c = 7b-1 = 6b+3$. It follows that $b=4$, $c=9$, $n=29$, and $d_4 = 20$, and hence the desired inequality is satisfied. Otherwise, since b and c have opposite parity, the conditions of Proposition 36 imply that $n \leq 6c-7b+1$. Then $n-c-2 \leq 6c-7b+1-c-2 = \frac{9c+c-10b-2}{2} - 2b = \frac{9c+2b+1-10b-2}{2} - 2b = \frac{9c-(8b+4)+3}{2} - 2b < \frac{9c-(5b+7)+3}{2} - 2b = \frac{9c-(12b+6-7b+1)+3}{2} - 2b = \frac{9c-(6c-7b+1)+3}{2} - 2b \leq \frac{9c-n+3}{2} - 2b \leq d_4$.
□

Lemma 43 *The conditions of Proposition 36 imply that $n < 4c+1$.*

Proof: We have $n \leq 6c-7b+3 = 4c+1+2c-7b+2 \leq 4c+1+6b-2-7b+2 = 4c+1-b < 4c+1$. □

If $g(i) = 1$, then $i = p-2$, $p-1$, or p. Then the following chain of inequalities shows that index j is in Subinterval (5) or (6): $j \geq n+1-b-p = n+1-b-(n-3c+2+d_1) = 3c-b-1-d_1 = 3c-b-1+(n-2c)-d_3 = c-b-1+n+b-d_3-b = d_4+1+c-(2b+2) \geq d_4+1$, where the last inequality follows by Lemma 42. By the action of g on Subintervals (5) and (6) and by Lemmas 39(c) and 43, the smallest endpoint sum involving $g(i) = 1$ satisfies $g(i)+g(j) \geq 1+g(n+1+b-(p-2)) = 1+g(n+1+b-(n-3c+d_1)) = 1+g(n+1+b-(d_2+n-3c-2)+c) = 1+g(n+1+b-d_2-n+4c+2) = 1+g(d_5+(4c+2-n)) = 1+3c+2-(4c+2-n) = n-c+1$. It follows by an argument similar to those used previously that for any $i \leq p$, $g(i)+g(j) \geq n+1-c$.

We now establish that the endpoints sums involving $i \geq p+1$ are sufficiently large. The action of g shows that the smallest value of $g(i)$

occurs when $i = p+1$ or $p+2$ and this value equals $n-3c+1$, $n-3c+2$, or $n-3c+3$. From earlier comments, the smallest possible vertex adjacent to v_i is v_{d_4-1}. We may assume that $n \neq 3c+2$ since otherwise, it is easy to show that $d_4 > n-c-2$. It follows that $n \leq 6c-7b+2$. For this value of i, $g(i)+g(j) \geq (n-3c+1)+(d_4-1) \geq \frac{9c-n+3}{2}-2b+n-3c = n-c+1+\frac{9c-n+1}{2}-2b-2c \geq n-c+1+\frac{9c-(6c-7b+2)+1}{2}-2b-2c = n-c+1-\frac{c-(3b-1)}{2} \geq n-c+1$, where the last inequality follows since $c \leq 3b-1$. Again, as in previous arguments, it can be shown that $g(i)+g(j) \geq n+1-c$ for $p+1 \leq i \leq d_2$.

In all cases, since $g(i) \leq c+1$, $g(i)+g(j) \leq n+1+c$.

6. v_iv_j, $d_2+1 \leq i \leq d_3$ (and hence $n > 3c+2$): Since $j > i$, index j cannot lie in Subinterval (1) or (2). Moreover, by Lemma 39(c), $j \leq n+b-d_2 = d_5-1$, and hence is not in Subinterval (6). The index i for which $g(i)$ is smallest is $i = d_2+1$. For this i, $g(i) = c+2$ and $n-b-d_2 \leq j \leq d_5-1$. Thus, by the action of g on Subintervals (3), (4), and (5), the smallest value of $g(j)$, for j in this range, occurs when $j = n-b-d_2$. By Lemma 39(a), we have $n-b-d_2 = n-b-(2n-6c+6b-2+[(n-c) \bmod 2]-d_3) = d_3+(6c-7b+2-n)-[(n-c) \bmod 2]$. But this last expression is no smaller than d_3 (since Condition (4) of Proposition 36 does not apply and n and c have the same parity when $n = 6c-7b+2$). This shows that $g(d_2+1)+g(d_3)$ is the smallest possible endpoint sum for which $i = d_2+1$. Moreover, since g is consecutive and increasing on Subintervals (3), (4), and (5), $g(d_2+1)+g(d_3) \leq g(d_2+2)+g(d_3-1) \leq \cdots$. It follows that $g(d_2+1)+g(d_3)$ is the smallest endpoint sum involving *any* index i in Subinterval (3). Thus, $g(i)+g(j) \geq g(d_2+1)+g(d_3) = (c+2)+(n-2c-1) = n+1-c$. It remains to show that $g(i)+g(j) \leq n+1+c$. The largest value of $g(i)$ occurs when $i = d_3$. For this i, we have $d_3+1 \leq j \leq n+1+b-d_3 = d_4+1$, where the last equality follows from Lemma 39(b). Using an argument similar to the one for the smallest endpoint sum, the largest endpoint sum is $g(d_3)+g(d_4+1) = n-2c-1+3c+2 = n+1+c$.

7. v_iv_j, $d_3+1 \leq i \leq d_4$: Then $i+1 \leq j \leq n+1+b-(d_3+1) = d_4$, by Lemma 39(b). Thus, indices i and j are both in Subinterval (4), on which $g(x) = x$. It follows that $n+1-c < n+1-b \leq g(i)+g(j) \leq n+1+b < n+1+c$.

8. v_iv_j, $i \geq d_4+1$: There are no such edges since $i+j \geq d_4+1+d_4+2 > d_3+d_4+3 > n+1+b$ by Lemmas 38 and 39(b), and the verification for the Case 4 labeling is complete.

6.5 $1 \leq r \leq \left\lfloor \frac{n-3c-1}{2} \right\rfloor + b$ and $r < s \leq n-2c$; and $n = 3c+1$

It follows that $1 \leq r \leq b$.

The labeling g is a piecewise-defined function on four subintervals of $\langle 1, 2, \ldots, n \rangle$, determined by partition points a_1, a_2, a_3, where

$$a_1 = 3b - c - 1, \qquad a_2 = 3b + 1, \qquad a_3 = 3c - 2b$$

Lemma 44 *Let n, b, and c satisfy $n = 3c+1$ and the hypotheses of Proposition 36, and let a_1, a_2, a_3 be defined as above. Then $0 \le a_1 < b < n-2c = c+1 < a_2 < a_3 < n$.*

Proof: These inequalities follow directly from the definitions and conditions on n, b, and c. For instance, $a_3 = 3c - 2b = n - 1 - 2b < n$. □

We can immediately specify the action of g on all but the second subinterval:

Subinterval (1) (if $c < 3b - 1$): $g\langle 1, 2, \ldots, a_1\rangle = \langle a_2, a_2 - 1, \ldots, c + 3\rangle$

Subinterval (2): $g\{a_1 + 1, a_1 + 2, \ldots, a_2\} = \{1, 2, \ldots, c + 2\}$

Subinterval (3): $g\langle a_2 + 1, a_2 + 2, \ldots, a_3\rangle = \langle a_2 + 1, a_2 + 2, \ldots, a_3\rangle$

Subinterval (4): $g\langle a_3 + 1, a_3 + 2, \ldots, n\rangle = \langle n, n - 1, \ldots, n - 2b = a_3 + 1\rangle$

Specifying the action of g on Subinterval (2)

The action of g on Subinterval (2) depends on which one of the following four subcases occurs. Observe that if $c = 3b-1$, then $a_1 = 0$ and Subinterval (1) is empty. In that special case, only Subcase 4 applies.

Subcase 1. $1 \le r < s \le a_1$

$$g(a_1 + 1) = c + 2 \quad g\langle a_1 + 2, a_1 + 3, \ldots, a_2\rangle = \langle 1, 2, \ldots, c + 1\rangle$$

Subcase 2. $1 \le r \le a_1$ and $a_1 + 1 \le s \le b$

$$g(s) = c + 2, \quad g\langle a_1 + 1, a_1 + 2, \ldots, \overline{s}, \ldots, a_2\rangle = \langle 1, 2, \ldots, c + 1\rangle$$

Subcase 3. $1 \le r \le a_1$ and $b + 1 \le s \le n - 2c$

$$g(a_1 + 1) = c + 2, \quad g(s) = c + 1,$$
$$g\langle a_1 + 2, a_1 + 3, \ldots, \overline{s}, \ldots, a_2\rangle = \langle 1, 2, \ldots, c\rangle$$

Subcase 4. $a_1 + 1 \le r \le b$ and $r + 1 \le s \le n - 2c$

$$g(r) = c + 2, \quad g(s) = c + 1,$$
$$g\langle a_1 + 1, a_1 + 2, \ldots, \overline{r}, \ldots, \overline{s}, \ldots, a_2\rangle = \langle 1, 2, \ldots, c\rangle$$

The following lemmas are analogous to Lemmas 40 and 41.

Lemma 45 *Let n, b, and c satisfy $n = 3c+1$ and the hypotheses of Proposition 36. Let a_1, a_2, and a_3 be defined as above, and let v_iv_j be an edge of $G_{n,b}$ such that $1 \le i \le b$. Then $j \ge a_3 + 2$.*

Proof: Since $v_iv_j \in G_{n,b}$, $j \ge n + 1 - b - i \ge n + 1 - b - b = n + 1 - 2b = 3c + 2 - 2b = a_3 + 2$. □

Lemma 46 *Let n, b, and c satisfy $n = 3c+1$ and the hypotheses of Proposition 36. Let a_1, a_2, and a_3 be defined as above, and let v_iv_j be an edge of $G_{n,b}$ such that $b+1 \leq i \leq a_2$. Then $j \geq a_2 + 1$.*

Proof: Suppose not. Then $i+j \leq 2a_2-1 = 2(3b+1)-1 = 7b+1-b \leq n-b$, which contradicts the premise that v_i is adjacent to v_j. □

Verification that g is a c-labeling

We show for any edge v_iv_j in $G_{n,b} + v_rv_s$, $i < j$, that $n + 1 - c \leq g(v_i) + g(v_j) \leq n + 1 + c$.

1. $i = r$ and $j = s$: We have $c+2 \leq g(r) \leq a_2 = 3b+1$ and $c+1 \leq g(s) \leq 3b$. It follows that $n + 1 - c = 2c + 2 < g(r) + g(s) \leq 6b + 1 \leq 7b - 1 \leq 3c = n - 1 < n + 1 + c$.

2. $i \neq r$ or $j \neq s$, $1 \leq i \leq b$: By Lemma 45, index j is in Subinterval (4). Let $x = g^{-1}(1)$ and let y be smallest index such that v_y is adjacent to v_x. Then $x = a_1+1, a_1+2,$ or a_1+3; and $y = n+1-b-x \geq n+1-b-(a_1+3) = 3c+2-b-a_1-3 = 3c-2b+1+(c-2b-1) = a_3+1+(c-2b-1)$. Since g is decreasing on Subinterval (4), $g(x)+g(y) \geq 1+g(a_3+1+c-2b-1) = 1+[n-(c-2b-1)] = n+1-c+2b+1 > n+1-c$. Moreover, for $1 \leq d \leq b-x$, we have $g(x+d) \geq 1+d$ and $g(y-d) = g(y)+d$, and $g(x-d) \geq 1+d$ and $g(y+d) = g(y)-d$. It follows that $g(i)+g(j) \geq g(x)+g(y) > n+1-c$. Since g is decreasing and consecutive on both Subintervals (1) and (4), we have $g(1)+g(n-b) = g(2)+g(n-b-1) = \cdots = g(a_1)+g(n-b-(a_1-1))$, and this common value is the largest possible endpoint sum of an edge joining vertices in Subintervals (1) and (4). Clearly, the values of g on Subinterval (2) are all smaller than those on Subinterval (1), and hence, the largest endpoint sum is $g(1)+g(n-b) = (3b+1)+(n-2b+b) = n+1+2b < n+1+c$.

3. $b+1 \leq i \leq a_2$: We have that $g(i) \leq c+1$, which implies that $g(i)+g(j) \leq n+1+c$. By Lemma 46, index j is in Subinterval (3) or (4). Observe that v_{b+1} is adjacent to v_x, where x ranges from $n - 2b = a_3 + 1$ to n. It follows that $g(b+1) + g(n)$ is the smallest possible endpoint sum involving v_{b+1}. Using arguments similar to those employed previously, it can be shown that as index i increases from $b + 1$, the endpoint sums involving v_i are no smaller than $g(b + 1) + g(n)$. By the definitions of a_1 and g, $g(b + 1) = g(a_1 + 2 + c - 2b) \geq 1 + c - 2b$ and $g(n) = n - 2b$. Thus, $g(i)+g(j) \geq (1+c-2b)+(n-2b) = n-4b+c+1 = n+1-c+(2c-4b) > n+1-c$.

4. $a_2+1 \leq i \leq a_3$: We have $j \leq n+1+b-(a_2+1) = n+b-3b-1 = 3c-2b = a_3$. Thus, since $j > i$, index j is in Subinterval (3) and $g(i) + g(j) = i + j$. Hence, $n + 1 - c < n + 1 - b \leq g(i) + g(j) \leq n + 1 + b < n + 1 + c$.

5. $i \geq a_3 + 1$: Since $i < j$, we have $i + j \geq 2a_3 + 3 = 2(3c - 2b) + 3 = 6c - 4b + 3 = n + 1 + b + (3c - 5b + 1) > n + 1 + b$, which contradicts the premise that vertices v_i and v_j are adjacent in $G_{n,b}$. This completes the verification that g is a c-labeling and completes the proof of Proposition 36. □

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