Connected Resolving Sets in Graphs

Varaporn Saenpholphat and Ping Zhang [1]

Department of Mathematics
Western Michigan University
Kalamozoo, MI 49008, USA

To Gary Chartrand on the occasion of his 65th birthday

Abstract

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is a resolving set for G if distinct vertices of G have distinct representations with respect to W. A resolving set for G containing a minimum number of vertices is a basis for G. The dimension $\dim(G)$ is the number of vertices in a basis for G. A resolving set W of G is connected if the subgraph $\langle W \rangle$ induced by W is a connected subgraph of G. The minimum cardinality of a connected resolving set in a graph G is its connected resolving number $cr(G)$. The relationship between bases and minimum connected resolving sets in a graph is studied. A connected resolving set W of G is a minimal connected resolving set if no proper subset of W is a connected resolving set. The maximum cardinality of a minimal connected resolving set is the upper connected resolving number $cr^+(G)$. The upper connected resolving numbers of some well-known graphs are determined. We present a characterization of nontrivial connected graphs of order n with upper connected resolving number $n - 1$. It is shown that for a pair a, b of integers with $1 \leq a \leq b$ there exists a connected graph G with $cr(G) = a$ and $cr^+(G) = b$ if and only if $(a, b) \neq (1, i)$ for all $i \geq 2$.

Key Words: distance, connected resolving set, connected resolving number, upper connected resolving number.

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1 Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representations with respect to W. A resolving set for G containing a minimum number of vertices is a *minimum resolving set* or a *basis* for G. The *(metric) dimension* $\dim(G)$ is the number of vertices in a basis for G. For a connected graph G, its vertex set $V(G)$ is always a resolving set for G. Moreover, $\langle V(G)\rangle = G$ is a connected graph. Thus, a resolving set W of G is defined to be *connected* if the subgraph $\langle W\rangle$ induced by W is a connected subgraph of G. The minimum cardinality of a connected resolving set W in a graph G is the *connected resolving number* $cr(G)$. A connected resolving set of cardinality $cr(G)$ is called a *cr-set* of G.

To illustrate these concepts, consider the graph G of Figure 1. The set $W = \{u, v\}$ is a basis for G and so $\dim(G) = 2$. The representations for the vertices of G with respect to W are

$$\begin{array}{lll} r(u|W) = (0,2) & r(v|W) = (2,0) & r(w|W) = (1,2) \\ r(x|W) = (1,1) & r(y|W) = (2,1) & \end{array}$$

Since $\langle\{u, v\}\rangle$ is disconnected, W is not a connected resolving set. On the other hand, the set $W' = \{u, v, x\}$ is a connected resolving set. The representations for the vertices of G with respect to W' are

$$\begin{array}{lll} r(u|W') = (0,2,1) & r(v|W') = (2,0,1) & r(w|W') = (1,2,1) \\ r(x|W') = (1,1,0) & r(y|W') = (2,1,1) & \end{array}$$

Since G contains no 2-element connected resolving set, that is, a resolving set consisting of two adjacent vertices, $cr(G) = 3$.

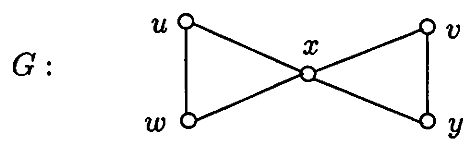

Figure 1: A graph G with $\dim(G) = 2$ and $cr(G) = 3$

The concepts of resolving set and minimum resolving set have previously appeared in the literature. In [6] and later in [7], Slater introduced these ideas and used *locating set* for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its *location number*. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [3] discovered these concepts independently as well but used the term *metric dimension* rather than location number, the terminology that we have adopted. These concepts were rediscovered by Johnson [4] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. Connected resolving sets in graphs were introduced and studied in [5]. We refer to [1] for graph theory notation and terminology not described here.

The following two observations (see [2, 5]) will be useful to us.

Observation 1.1 *Let G be a nontrivial connected graph. Then* $\dim(G) = cr(G)$ *if and only if G contains a connected basis.*

Observation 1.2 *If S is a set of $p \geq 2$ vertices in a connected graph G such that $d(u, x) = d(v, x)$ for all $u, v \in S$ and $x \in V(G) - \{u, v\}$, then every resolving set must contain at least $p - 1$ vertices of S.*

2 Comparison of cr-Sets and Bases in Graphs

In this section, we study the relationship between *cr*-sets and bases in a nontrivial connected graph G. Certainly, if W is a resolving set of G and $W \subseteq W'$, then W' is also a resolving set of G. Therefore, if W is a basis of G such that $\langle W \rangle$ is disconnected, then surely there is a smallest superset W' of W for which $\langle W' \rangle$ is connected. This suggests the following question: For each basis W of a nontrivial connected graph G, does there exist a *cr*-set W' of G such that $W \subseteq W'$? We show that this question has a negative answer.

Proposition 2.1 *There is an infinite class of connected graphs G such that some cr-sets of G contain a basis of G and others contain no basis of G.*

Proof. Let G be the graph obtained from the 4-cycle u_1, u_2, u_3, u_4, u_1 by adding the k (≥ 2) new vertices $v_1, v_2, \cdots, v_k$ and joining each v_i ($1 \leq i \leq k$) to u_1 and u_4. The graph G is shown in Figure 2. Let $V = \{v_1, v_2, \cdots, v_k\}$. Since $W = \{u_2\} \cup (V - \{v_k\})$ is a basis of G, it follows that $\dim(G) = k$.

Next we show that $cr(G) = k + 1$. By Observation 1.2, every resolving set of G contains at least $k-1$ vertices in V. Thus every connected resolving set must contain at least one vertex from $\{u_1, u_4\}$. However, if S is a set of vertices of G consisting of $k-1$ vertices from V and one vertex from $\{u_1, u_4\}$,

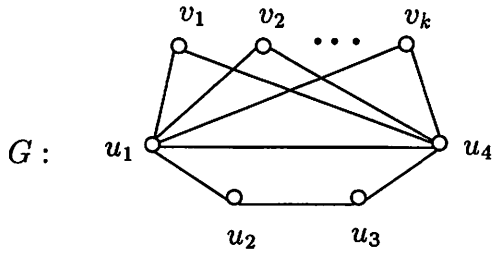

Figure 2: The graph G

say $S = (V - \{v_k\}) \cup \{u_1\}$, then $r(u_2 \mid S) = r(v_k \mid S)$ and so S is not a resolving set. Therefore, G contains no connected basis and so $cr(G) \geq k+1$ by Observation 1.1. On the other hand, $S_1 = \{u_1, u_2\} \cup (V - \{v_k\})$ is a connected resolving set of cardinality $k+1$ and so $cr(G) \leq k+1$. Therefore, $cr(G) = k+1$.

Observe that the cr-set S_1 contains the basis $W = \{u_2\} \cup (V - \{v_k\})$ of G. On the other hand, let $S_2 = \{u_1\} \cup V$. By a similar argument, it can be verified that S_2 is also a cr-set of G. Since every basis of G contains exactly $k-1$ vertices from V and exactly one vertex from $\{u_2, u_3\}$, it follows that S_2 contains no basis of G. ∎

Proposition 2.1 suggests another question. For each connected graph G, does there exist some cr-set W' such that there is some basis W of G for which $W \subseteq W'$? We show that even this question has a negative answer.

Theorem 2.2 *There is an infinite class of connected graphs G such that every cr-set of G is disjoint from every basis of G.*

Proof. For two integers $p, q \geq 3$, let G be that graph obtained from two odd cycles $C_{2p+1} : u_0, u_1, u_2, \cdots, u_p, u'_p, u'_{p-1}, \cdots, u'_1, u_0$ and C_{2q+1}: v_0, v_1, v_2, $\cdots$, v_q, v'_q, v'_{q-1}, $\cdots$, v'_1, v_0 by (1) identifying the vertex u_0 of C_{2p+1} with the vertex v_0 of C_{2q+1}, denoting the identified vertex by x, and (2) adding a pendant edge xy. The graph G is shown in Figure 2.

First we make an observation. Let $U = \{u_1, u_2, \cdots, u_p\}$, $U' = \{u'_1, u'_2, \cdots, u'_p\}$, $V = \{v_1, v_2, \cdots, v_q\}$, and $V' = \{v'_1, v'_2, \cdots, v'_q\}$. If S is a resolving set of G, then S contains at least one vertex from each of $U \cup U'$ and $V \cup V'$. Otherwise, if $S \subseteq U \cup U' \cup \{x, y\}$, then $r(v_1 \mid S) = r(v'_1 \mid S)$. If $S \subseteq V \cup V' \cup \{x, y\}$, then $r(u_1 \mid S) = r(u'_1 \mid S)$. In either case, S is not a resolving set, which is a contradiction.

Let $W_1 = \{u_p, v_q\}$, $W_2 = \{u'_p, v_q\}$, $W_3 = \{u_p, v'_q\}$, and $W_4 = \{u'_p, v'_q\}$. Since G is not a path (which is the only nontrivial connected graph of dimension 1) and each of the sets W_i $(1 \leq i \leq 4)$ is a resolving set, $\dim(G) =$

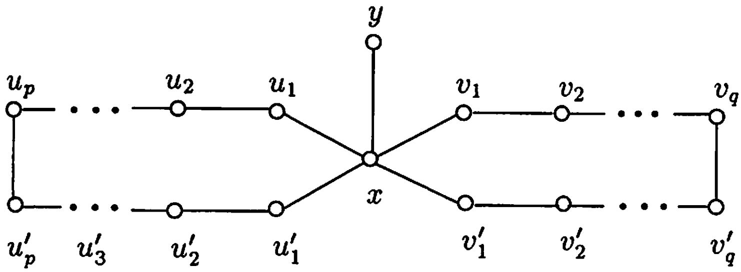

Figure 3: The graph G for $p = 4$ and $q = 3$

2. We show next that the sets W_i $(1 \le i \le 4)$ are the only bases of G. Assume, to the contrary, that G contains a basis W that is distinct from all W_i for $1 \le i \le 4$. Let $W = \{s, t\}$. By the observation above, W contains exactly one vertex from each of $U \cup U'$ and $V \cup V'$, say $s \in U \cup U'$ and $t \in V \cup V'$. We consider two cases.

Case 1. $s = u_p$ *or* $s = u'_p$. Assume, without loss of generality, that $s = u_p$. Since $W \ne W_i$ for $1 \le i \le 4$, it follows that $t = v_j$ or $t = v'_j$ for $1 \le j \le q-1$. If $t = v_j$ for $1 \le j \le q-1$, then $d(v'_1, t) = d(y, t)$. Since $d(v'_1, s) = d(y, s)$, it follows that $r(v'_1 \mid W) = r(y \mid W)$, a contradiction. If $t = v'_j$ for $1 \le j \le q-1$, then $d(v_1, t) = d(y, t)$. Since $d(v_1, s) = d(y, s)$, it follows that $r(v_1 \mid W) = r(y \mid W)$, a contradiction.

Case 2. $s = u_i$ *or* $s = u'_i$, *where* $1 \le i \le p-1$. Assume, without loss of generality, that $s = u_i$ for some i with $1 \le i \le p-1$. Then $t = v_j$ or $t = v'_j$ for $1 \le j \le q$. If $t = v_j$ for $1 \le j \le q-1$, then $r(v'_1 \mid W) = r(y \mid W)$. If $t = v'_j$ for $1 \le j \le q-1$, then $r(v_1 \mid W) = r(y \mid W)$. If $t = v_q$ or $t = v'_q$, then $r(u'_1 \mid W) = r(y \mid W)$. Thus in each case, a contradiction is produced.

Therefore, W is not a basis of G and so the sets W_i $(1 \le i \le 4)$ are the only bases of G.

Next we show that $cr(G) = 5$. Since $S_0 = \{u_1, u'_1, v_1, v'_1, x\}$ is a connected resolving set, $cr(G) \le 5$. Assume, to the contrary, that $cr(G) \le 4$. Since W_i $(1 \le i \le 4)$ are the only bases of G and none of W_i $(1 \le i \le 4)$ are connected, G contains no connected basis. Thus $cr(G) \ge 3$ by Observation 1.1. Let S be a cr-set of G. Then $|S| = 3$ or $|S| = 4$. We consider these two cases.

Case 1. $|S| = 3$. Let $N[x] = \{u_1, u'_1, v_1, v'_1, x, y\}$ be the closed neighborhood of x. By the observation above, S contains at least one vertex from each of $V \cup V'$ and $U \cup U'$. This implies that $x \in S$ and $S \subseteq N[x]$. Since $\langle N[x] \rangle = K_{1,5}$ and $cr(K_{1,5}) = 5$, it follows that S is not a connected resolving set of G, a contradiction.

Case 2. $|S| = 4$. Again, S contains at least one vertex from each of $V \cup V'$ and $U \cup U'$. An argument similar to that used in Case 1 shows that if $S \subseteq N[x]$, then S is not a connected resolving set for G. Thus $S \not\subseteq N[x]$. Since $|S| = 4$ and S is connected, S must contain x and exactly one vertex from each of $\{u_1, u_1'\}$, $\{v_1, v_1'\}$, and $\{u_2, u_2', v_2, v_2'\}$. Assume, without loss of generality, that $S = \{u_2, u_1, x, v_1\}$. However, then $r(u_1' \mid S) = r(v_1' \mid S)$, which is a contradiction.

Therefore, $cr(G) = 5$. Moreover, we claim that each cr-set S is a subset of $\{u_1, u_1', v_1, v_1', x, y\}$. Assume, to the contrary, that there is a cr-set S' such that S' is not a subset of $\{u_1, u_1', v_1, v_1', x, y\}$. Since S' is connected, $|S'| = 5$, and S' must contain at least one vertex from each of $V \cup V'$ and $U \cup U'$, it follows that S' is a subset of $\{u_1, u_2, u_1', u_2', v_1, v_2, v_1', v_2', x, y\}$. Assume, without loss of generality, that $S' = \{u_2, u_1, v_1, x, y\}$ or $S' = \{u_2, u_1, v_1, v_2, x\}$. However, in each case, $r(u_1' \mid S') = r(v_1' \mid S')$, which is a contradiction. Therefore, each cr-set S is a subset of $\{u_1, u_1', v_1, v_1', x, y\}$, as claimed. Since each basis W of G is a subset of $\{u_p, u_p', v_q, v_q'\}$ and $p, q \geq 3$, it follows that every cr-set of G is disjoint from every basis of G. ■

The graph G constructed in the proof of Theorem 2.2 can be extended to show the following result. First we need an additional definition. Let X and Y be two sets of vertices in a connected graph G. The *distance between X and Y* is defined as

$$d(X, Y) = \min\{d(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Corollary 2.3 *For each positive integer N, there is an infinite class of connected graphs G such that $d(W, S) \geq N$ for every basis W of G and every cr-set S of G.*

Proof. Let G be the graph constructed in the proof of Theorem 2.2 for $p \geq q \geq \max\{3, N+1\}$. Since each basis W of G is a subset of $\{u_p, u_p', v_p, v_p'\}$ and each cr-set S is a subset of $\{u_1, u_1', v_1, v_1', x, y\}$, it follows that $d(W, S) \geq p - 1 \geq N$, as desired. ■

The following three results give the relationship between cr-sets and bases in some well-known classes of graphs, namely complete graphs, complete bipartite graphs, cycles, and trees.

Proposition 2.4 *If G is a complete graph of order at least 3 or a complete bipartite graph that is not a star, then a set W of vertices of G is a basis of G if and only if W is a cr-set of G.*

Proof. If $G = K_n$ for $n \geq 3$, every set of $n-1$ vertices in G form a basis and a cr-set of G. Let $G = K_{r,s}$ with partite sets V_1 and V_2, where $|V_1| = r$

and $|V_2| = s$ and $2 \leq r \leq s$. Then every basis W of G contains exactly $r-1$ vertices from V_1 and exactly $s-1$ vertices from V_2. Since $\langle W \rangle$ is connected, W is a cr-set. By Observation 1.1, we have $\dim(G) = cr(G)$, which implies that every cr-set of G is a basis. ∎

Proposition 2.5 *For a cycle C_n of order $n \geq 4$, every cr-set of C_n is a basis of C_n.*

Proof. Since $\dim(C_n) = cr(C_n) = 2$ for $n \geq 4$ and every cr-set of C_n consists of two adjacent vertices, every cr-set is also a basis. ∎

The converse of Proposition 2.5 is not true for $n \geq 5$ since some basis of C_n consists of two nonadjacent vertices of C_n and therefore, it is not a cr-set of C_n.

A vertex of degree at least 3 in a tree T is called a *major vertex*. An end-vertex u of T is said to be a *terminal vertex of a major vertex* v of T if $d(u,v) < d(u,w)$ for every other major vertex w of T. The *terminal degree* $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of T and let $ex(T)$ denote the number of exterior major vertices of T. Then $\sigma(T)$ is the number of end-vertices of T. The following lemma is useful (see [2]).

Lemma 2.6 *Let T be a nonpath tree of order $n \geq 4$ having p exterior major vertices $v_1, v_2, \cdots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \cdots, u_{ik_i}$ be the terminal vertices of v_i, and let P_{ij} be the $v_i - u_{ij}$ path ($1 \leq j \leq k_i$). Suppose that W is a set of vertices of T. Then W is a resolving set of T if and only if W contains at least one vertex from each of the paths $P_{ij} - v_i$ ($1 \leq j \leq k_i$ and $1 \leq i \leq p$) with at most one exception for each i with $1 \leq i \leq p$. Moreover, W is a basis of T if and only if W contains exactly one vertex from each of the paths $P_{ij} - v_i$ ($1 \leq j \leq k_i$ and $1 \leq i \leq p$) with exactly one exception for each i with $1 \leq i \leq p$.*

By Lemma 2.6, we have the following result whose proof is straightforward and is therefore omitted.

Proposition 2.7 *If T is a tree that is not a path, then every cr-set of T contains a basis of T as a proper subset.*

3 Minimal Connected Resolving Sets in Graphs

A resolving set W of G is a *minimal resolving set* if no proper subset of W is a resolving set. The maximum cardinality of a minimal resolving set is the *upper dimension* $\dim^+(G)$ and a minimal resolving set of cardinality $\dim^+(G)$ is an *upper basis* for G. If G is a nontrivial connected graph, then $\dim(G) \le \dim^+(G)$. These concepts were introduced and studied in [2]. Similarly, a connected resolving set W of G is a *minimal connected resolving set* if no proper subset of W is a connected resolving set. The maximum cardinality of a minimal connected resolving set is the *upper connected resolving number* $cr^+(G)$ and a minimal connected resolving set of cardinality $cr^+(G)$ is called an *upper cr-set* for G. Certainly, every minimum connected resolving set of a graph is a minimal connected resolving set, but the converse is not true.

To illustrate these concepts, consider the graph $G = P_3 \times P_4$ of Figure 4. Since $W = \{u_1, v_1, w_1\}$ is a cr-set for G, it follows that $cr(G) = 3$. Now let $W' = \{v_1, v_2, v_3, w_3, w_4\}$. The representations for the vertices of $V(G) - W'$ with respect to W' are

$$\begin{array}{ll} r(u_1|W') = (1,2,3,4,5) & r(u_2|W') = (2,1,2,3,4) \\ r(u_3|W') = (3,2,1,2,3) & r(u_4|W') = (4,3,2,3,2) \\ r(v_4|W') = (3,2,1,2,1) & r(w_1|W') = (1,2,3,2,3) \\ r(w_2|W') = (2,1,2,1,2) & \end{array}$$

Thus W' is a connected resolving set as well. Let $W_1 = W' - \{v_1\}$, $W_2 = W' - \{v_2\}$, $W_3 = W' - \{v_3\}$, $W_4 = W' - \{w_3\}$, and $W_5 = W' - \{w_4\}$. Since $r(u_2|W_1) = r(v_1|W_1)$, $r(u_3|W_5) = r(v_4|W_5)$, and $\langle W_i \rangle$ is not connected in G for $i = 2, 3, 4$, the sets W_i is not a connected resolving set for all $1 \le i \le 5$. Hence W' is a minimal connected resolving set and so $cr^+(G) \ge 5$. By a case-by-case analysis, one can show that there is no minimal connected resolving sets of cardinality 6. Hence W' is an upper cr-set and $cr^+(G) = 5$.

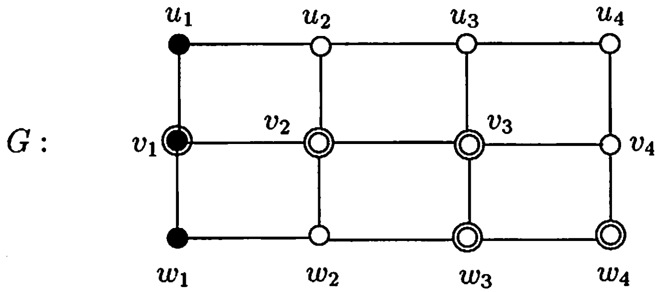

Figure 4: A cr-set and an upper cr-set in G

Thus if G is a nontrivial connected graph of order n, then

$$1 \le cr(G) \le cr^+(G) \le n - 1.$$

The connected resolving numbers of some well-known graphs have been determined in [5]. Next we show that $cr^+(G) = cr(G)$ for these graphs.

Theorem 3.1 *Let G be a nontrivial connected graph. If G is a complete graph, a cycle, a complete k-partite graph ($k \geq 2$), or a tree, then $cr^+(G) = cr(G)$.*

Proof. We will only verify that $cr^+(T) = cr(T)$ for all nontrivial trees T since the proofs for others are routine. If T is a path, then $cr^+(T) = cr(T) = 1$. Thus we may assume that T is not a path. Assume, to the contrary, that there is a nontrivial tree T that is not a path such that $cr^+(T) > cr(T)$. Let T have order $n \geq 4$ and p exterior major vertices $v_1, v_2, \cdots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \cdots, u_{ik_i}$ be the terminal vertices of v_i, and let P_{ij} be the $v_i - u_{ij}$ path $(1 \leq j \leq k_i)$. Let S be an upper cr-set with $|S| = cr^+(T)$. Since S is a resolving set, it follows by Lemma 2.6 that S contains at least one vertex from each of the paths $P_{ij} - v_i$ $(1 \leq j \leq k_i$ and $1 \leq i \leq p)$ with at most one exception for each i with $1 \leq i \leq p$. Let S' be a subset S such that S' consists of all those vertices required by Lemma 2.6. Since $\langle S \rangle$ is connected, S must contain all vertices of T belonging to each $x - y$ path for all $x, y \in S'$. On the other hand, it was shown in [5] that for a set W of vertices of T, W is a cr-set of T if and only if (a) W contains exactly one vertex from each of the paths $P_{ij} - v_i$, $1 \leq j \leq k_i$ and $1 \leq i \leq p$, with exactly one exception for each i with $1 \leq i \leq p$, (b) for each pair i, j with $1 \leq j \leq k_i$ and $1 \leq i \leq p$, if $x_{ij} \in W$, then x_{ij} is adjacent to v_i in the path P_{ij}, and (c) W contains all vertices in the paths between any two vertices described in (b). This implies that if $|S| \geq cr(G)$, then S contains a cr-set as a proper subset, which is a contradiction. ■

By Theorem 3.1 and some known results [5] for the connected resolving numbers of these graphs, we have the following.

(a) If $G = K_n$ for $n \geq 3$ or $G = K_{1,n-1}$ for $n \geq 4$, then $cr^+(G) = cr(G) = n - 1$.

(b) If $G = P_n$ for $n \geq 2$, then $cr^+(G) = cr(G) = 1$.

(c) If $G = C_n$ for $n \geq 4$, then $cr^+(G) = cr(G) = 2$.

(d) For $k \geq 2$, let $G = K_{n_1, n_2, \cdots, n_k}$ be a complete k-partite graph that is not a star. Let $n = n_1 + n_2 + \cdots + n_k$ and ℓ be the number of 1's in $\{n_i : 1 \leq i \leq k\}$. Then

$$cr^+(G) = cr(G) = \begin{cases} n - k & \text{if } \ell = 0 \\ n - k + \ell - 1 & \text{if } \ell \geq 1. \end{cases}$$

(e) Let T be a tree that is not a path, having order $n \geq 4$ and p exterior major vertices $v_1, v_2, \cdots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \cdots, u_{ik_i}$ be the terminal vertices of v_i and let P_{ij} be the $v_i - u_{ij}$ path of length ℓ_{ij} $(1 \leq j \leq k_i)$. Then

$$cr^+(T) = cr(T) = n + \sigma(T) - ex(T) - \sum_{i,j} \ell_{ij}.$$

It was shown in [5] that the graphs K_n and $K_{1,n-1}$ are the only connected graphs of order $n \geq 4$ with connected resolving number $n-1$. In fact, this is also true for the upper connected numbers of graphs, as we show next.

Theorem 3.2 *Let G be a connected graph of order $n \geq 4$. Then $cr^+(G) = n-1$ if and only if $G = K_n$ or $G = K_{1,n-1}$.*

Proof. We have seen that $cr^+(G) = n-1$ if $G = K_n$ or $G = K_{1,n-1}$. It remains to verify the converse. We show that if G is a connected graph of order $n \geq 4$ that is neither a complete graph nor a star, then $cr^+(G) \leq n-2$. To do this, it suffices to show the following stronger statement: If G is a connected graph of order $n \geq 4$ that is neither a complete graph nor a star, then, for each $u \in V(G)$ for which $V(G) - u$ is a connected resolving set, there exist two distinct vertices v and w in $G-u$ such that (1) $V(G)-\{u, v\}$ is a connected resolving set for G and (2) w is adjacent to exactly one of u and v in G. We proceed by induction on the order n of G. For $n = 4$, the graphs G_i $(1 \leq i \leq 4)$ of Figure 5 are the only connected graphs order 4 that are different from K_4 or $K_{1,3}$. For each i $(1 \leq i \leq 4)$, the vertices u, v, w are shown in Figure 5 and $W = V(G_i) - \{u, v\}$ is a connected resolving set in G_i. Note that for G_3 and G_4 there are two possible choices (see Figure 5) for u such that $V(G_3) - \{u\}$ and $V(G_4) - \{u\}$ are connected resolving sets in G_3 and G_4, respectively. Moreover, w is adjacent to exactly one of u and v. Thus the statement is true for $n = 4$. Assume that the statement is true for $n - 1 \geq 4$.

Let G be a connected graph of order $n \geq 5$ that is not K_n or $K_{1,n-1}$ and let x be vertex of G such that $V(G) - x$ is a connected resolving set of G. Let $G' = G - x$. We consider three cases.

Case 1. $G' = K_{n-1}$. Since $G \neq K_n$, there are distinct vertices v, w, y in G' such that x is adjacent to y but not to w. Then $\langle V(G) - \{v, x\}\rangle = K_{n-2}$, $d_G(x, w) = 2$, and $d_G(v, w) = 1$. This implies that $V(G) - \{v, x\}$ is a connected resolving set of G. Moreover, w is adjacent to exactly one of v and x in G.

Case 2. $G' = K_{1,n-2}$. Since $G \neq K_{1,n-1}$, there exist two end-vertices v and w of G' such that x is adjacent to at least one of v and w, say

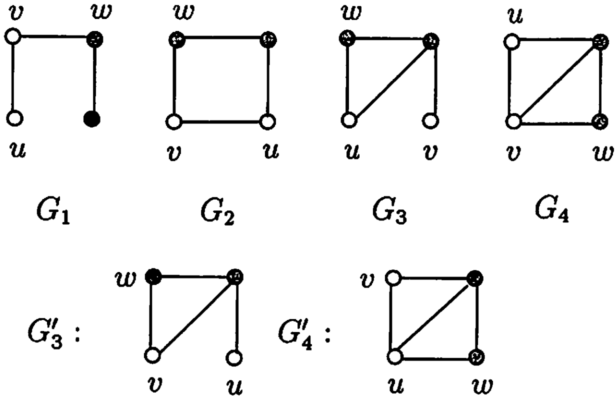

Figure 5: Graphs G_i $(1 \leq i \leq 4)$

x is adjacent to w. Then $\langle V(G) - \{v, x\}\rangle = K_{1,n-2}$, $d_G(x, w) = 1$, and $d_G(v, w) = 2$. This implies that $V(G) - \{v, x\}$ is a connected resolving set of G. Moreover, w is adjacent to exactly one of v and x in G.

Case 3. *$G' \neq K_{n-1}$ and $G' \neq K_{1,n-2}$.* Let $u \in V(G')$ such that $G' - u$ is connected. Therefore, $V(G') - \{u\}$ is a connected resolving set for G'. By the induction hypothesis, there exist two distinct vertices v and w in $G' - u$ such that (1) $V(G') - \{u, v\}$ is a connected resolving set for G' and (2) w is adjacent to exactly one of u and v in G'. There are two subcases.

Subcase 3.1. *w is adjacent to u.* Then w is not adjacent to v. If x is adjacent to w in G, then $d_G(x, w) = 1$ and $d_G(v, w) \geq 2$, implying that $V(G) - \{v, x\}$ is a resolving set of G. Since $V(G') - \{u, v\}$ is a connected resolving set for G', it follows that $\langle V(G') - \{u, v\}\rangle$ is connected. Moreover, u is adjacent to w. Hence $\langle V(G) - \{v, x\}\rangle$ is connected. Therefore, $V(G) - \{v, x\}$ is a connected resolving set of G. Also, w is adjacent to exactly one of v and x. If x is not adjacent to w in G, then $d_G(x, w) \geq 2$ and $d_G(u, w) = 1$, $V(G) - \{u, x\}$ is a resolving set of G. Since $G - u - x = G' - u$ is connected, it follows that $V(G) - \{u, x\}$ is a connected resolving set of G. Moreover, w is adjacent to exactly one of u and x.

Subcase 3.2. *w is adjacent to v.* Then w is not adjacent to u. If x is adjacent to w in G, then w is adjacent to exactly one of v and u. Moreover, an argument similar to one used in Subcase 3.1 shows that $V(G) - \{u, x\}$ is a connected resolving set of G. So we may assume that x is not adjacent to w in G. If there is $z \in V(G) - \{u, v, w, x\}$ such that $uz \in E(G)$, then $V(G) - \{v, x\}$ is a connected resolving set of G and w is adjacent to exactly one of

v and x. Thus we may assume that there is no vertex in $V(G)-\{u,v,w,x\}$ that is adjacent to u. Since G' is connected, it follows that u must be adjacent v. If there is $w' \in V(G)-\{u,v,x\}$ such that $xw' \in E(G)$, then $V(G)-\{u,x\}$ is a connected resolving set of G. Moreover, w' is adjacent to exactly one of u and x. Hence we may assume that there is no edge between $\{u,x\}$ and $V(G)-\{u,v,x\}$. If $|V(G)-\{u,v,x\}|=1$, then $G=K_{1,4}$, which is impossible. If $|V(G)-\{u,v,x\}|=2$, or $|V(G)-\{u,v,x\}|=3$, then there exists $w'' \in V(G)-\{u,v,x\}$ such that w'' is adjacent to w. Since $d_G(w'',w)=1$ and $d_G(x,w)\geq 2$, it follows that $V(G)-\{w'',x\}$ is a connected resolving set of G. Moreover, w is adjacent to exactly one of w'' and x in G. Thus, we may assume that $|V(G)-\{u,v,x\}|\geq 4$. By the induction hypothesis, $\langle V(G')-\{u,v\}\rangle = \langle V(G)-\{u,v,x\}\rangle$ is connected in G'. Then there is $u'v' \in V(G)-\{u,v,x\}$ such that $\langle V(G)-\{u,v,x,u'\}\rangle$ is connected and u' is adjacent to v'. Thus $\langle V(G)-\{x,u'\}\rangle$ is connected. Since $d_G(x,v')\geq 2$ and $d_G(u',v')=1$, it follows that $V(G)-\{x,u'\}$ is a connected resolving set of G. Moreover, v' is adjacent to exactly one of u' and x.

Therefore, in all cases, $cr^+(G)\leq n-2$, as desired. ∎

Note that every graph G encountered thus far has the property that either $cr^+(G)=cr(G)$ or $cr^+(G)-cr(G)\leq 2$. This might lead one to believe that $cr^+(G)$ and $cr(G)$ are close for every connected graph G. However, this is not the case. Indeed, as we next show, every pair a,b of integers with $2\leq a\leq b$ is realizable as the connected resolving number and upper connected resolving number of some graph.

Theorem 3.3 *For every pair a,b of integers with $2\leq a\leq b$, there exists a connected graph G with $cr(G)=a$ and $cr^+(G)=b$.*

Proof. We consider two cases, according to whether $a=2$ or $a\geq 3$.

Case 1. $a=2$. Let $G=P_b\times P_2$, where $u_1,u_2,\cdots,u_b$ and $v_1,v_2,\cdots,v_b$ are two copies of P_b in G. Since $\{u_1,v_1\}$ is a cr-set of G, it follows that $cr(G)=2$.

We now show that $cr^+(G)=b$. If $b=2$, then $G=C_4$ and $cr^+(C_4)=2$. Thus we may assume that $b\geq 3$. Let $U=\{u_1,u_2,\cdots,u_b\}$. We show that U is a minimal connected resolving set of G. Since U is a resolving set and $\langle U\rangle = P_b$, it follows that U is a connected resolving set of G. It remains to show that U is minimal. Assume, to the contrary, that U is not minimal. Then there exists a proper subset S in U such that S is a connected resolving set of G. Since S is connected, $\langle S\rangle = P_s$ is a subpath in $\langle U\rangle$, where $s=|S|$. This implies that at most one of u_1 and u_b belongs to S, say $u_1\notin S$ and $u_b\in S$. Then $r(u_1\mid S)=r(v_2\mid S)$, which is a contradiction. Therefore, U is a minimal connected resolving set of G and so $cr^+(G)\geq |U|=b$.

Next we show that $cr^+(G) \leq b$. Assume, to the contrary, that $cr^+(G) \geq b+1$. Let W be an upper cr-set of G with $|W| = cr^+(G) \geq b+1$. Then there exists i with $1 \leq i \leq b$ such that $\{u_i, v_i\} \subseteq W$. Since $\{u_1, v_1\}$ and $\{u_b, v_b\}$ are cr-sets of G, it follows that $2 \leq i \leq b-1$. Since W is connected, at least one vertex in $\{u_{i-1}, u_{i+1}, v_{i-1}, v_{i+1}\}$ belong to W, say $u_{i-1} \in W$. However, the proper subset $\{u_i, u_{i-1}, v_i\}$ of W is a connected resolving set of G, which is a contradiction. Therefore, $cr^+(G) = b$, as claimed.

Case 2. $a \geq 3$. Let $u_1, u_2, \cdots, u_{b-a+2}$ and $v_1, v_2, \cdots, v_{b-a+2}$ be two copies of P_{b-a+2} in $P_{b-a+2} \times P_2$ and K_a a complete graph with $V(K_a) = \{w_1, w_2, \cdots, w_a\}$. Let G be the graph obtained from the graphs $P_b \times P_2$ and K_a by identifying the vertices u_1 and w_1 and denoting the identified vertex by u_1. Since $\{u_1, v_1, w_3, w_4, \cdots, w_a\}$ is a cr-set of G, it follows that $cr(G) = a$.

Next we show that $cr^+(G) = b$. Let

$$W = \{u_1, u_2, \cdots, u_{b-a+2}, w_3, w_4, \cdots, w_a\}.$$

Then W is a connected resolving set of G. We show, in fact, that W is minimal. By Observation 1.2, every cr-set contains at least $a-2$ vertices from $\{w_2, w_3, w_4, \cdots, w_a\}$. Thus $W - \{w_i\}$ is not a resolving set for all $3 \leq i \leq a$. For each j with $1 \leq j \leq b-a+1$, since $\langle W - \{u_j\} \rangle$ is not connected, $W - \{u_j\}$ is not a connected resolving set. Moreover, $r(u_{b-a+2} \mid W - \{u_{b-a+2}\}) = r(v_{b-a+1} \mid W - \{u_{b-a+2}\})$, implying that $W - \{u_{b-a+2}\}$ is not a resolving set. Thus no proper subset of W is a connected resolving set and so W is minimal. Therefore, $cr^+(G) \geq |W| = b$.

We now show that $cr^+(G) \leq b$. Assume, to the contrary, that $cr^+(G) \geq b+1$. Let W' be an upper cr-set of G with $|W'| = cr^+(G)$. Since W' is a resolving set, W' contains at least $a-2$ vertices from $\{w_2, w_3, w_4, \cdots, w_a\}$. Assume, without loss of generality, that $\{w_3, w_4, \cdots, w_a\} \subseteq W'$. If $W' \subseteq V(K_a)$, then $r(u_2 \mid W') = r(v_1 \mid W')$, which is impossible. Thus W' contains at least one vertex from $V(P_{b-a+2} \times P_2) - \{u_1\}$. Since W' is connected, it follows that $u_1 \in W'$. If $v_1 \in W'$, then the cr-set $\{u_1, v_1, w_3, w_4, \cdots, w_a\}$ is a proper subset of W', which is a contradiction. Thus $v_1 \notin W'$. Since $|W'| \geq b+1$, it follows that W' contains $\{u_i, v_i\}$ for some i with $2 \leq i \leq a-b+2$. Let i_0 be the smallest integer such that $\{u_{i_0}, v_{i_0}\} \subseteq W'$ and let

$$S = \{u_1, u_2, \cdots, u_{i_0}, v_{i_0}, w_3, w_4, \cdots, w_a\}.$$

Then S is a connected resolving set. If $2 \leq i_0 \leq b-a+1$, then $|S| = (i_0+1) + (a-2) \leq (b-a+2) + (a-2) = b$. Thus S is a proper subset of W', which is a contradiction. If $i_0 = b-a+2$, then $S' = S - \{v_{i_0}\} \subseteq W'$ is a connected resolving set G. Since S' is a proper subset of W', a contradiction is produced. Therefore, $cr^+(G) \leq b$. ■

Note that nontrivial paths are the only nontrivial connected graphs with connected resolving number 1. Since the upper connected resolving number of all nontrivial paths is also 1, the following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4 *Let a, b be integers with $1 \leq a \leq b$. Then there exists a connected graph G with $cr(G) = a$ and $cr^+(G) = b$ if and only if $(a, b) \neq (1, i)$ for all $i \geq 2$.*

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