Some results in step domination of graphs

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Abstract

The step domination number of all graphs of diameter two, is determined.

1 Introduction: Definitions and Notation

In this paper we shall refer to graphs as connected graphs. We follow the notation and terminology of [2] and [4]. However, in order to simplify the reading of the paper we shall introduce some of the necessary definitions and notation we are using throughout the paper.

The *distance* between two vertices u, v in a graph G, denoted $d(u,v)$, is the length of a shortest simple path $u - v$ in G. When $d(u,v) = 1$ we say that u and v are *adjacent*. The *eccentricity* of a vertex u, denoted $ecc(u)$, is the distance of the furthest vertex from u, i.e.,

$$ecc(u) = \max\{d(u,x) | x \in V(G)\}.$$

The *diameter* of G ,$d(G)$, is the maximum eccentricity.

The set of vertices at distance k from a vertex v in G is called the *k-neighborhood* of v and is denoted by $N_k(v)$. That is,

$$N_k(v) = \{u \in V(G) | d(v,u) = k\}.$$

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In case $k = 1$ we shall refer to it as the neighborhood of v or *open neighborhood.* In this case we shall denote it, as usual, $N(v)$, while $N[v] = N(v) \cup \{v\}$.

A vertex v in G is said to *dominate* itself and each of its neighbors. A set $S \subseteq V(G)$ is a *domination set* if every vertex of G is dominated by some vertex of S.

The notion of step domination and results along this line are given in [1] and [3].

A set $S = \{v_1, v_2, \cdots, v_t\}$ of vertices in a graph G is defined as a *step domination set* for G if there exist nonnegative integers $k_1, k_2, \cdots, k_t$ such that the set $\{N_{k_i}(v_i)\}$ forms a partition of $V(G)$. This partition is called *the step domination partition associated with S.* The sequence $k_1, k_2, \cdots, k_t$, $(k_1 \leq k_2 \leq \cdots \leq k_t)$ is called a *distance domination sequence associated with S*, while k_i is called the *step* of v_i and denoted $st(v_i) = k_i$. Each vertex u in $N_{k_i}(v_i)$ is said to be *step dominated by* v_i, and v_i *step dominates* u. We assume that in the above definitions $N_{k_i}(v_i)$ is nonempty. Thus, $0 \leq k_i \leq ecc(v_i)$ for each integer k_i in a distance domination sequence associated with S. Since a vertex in a step domination set S cannot step dominate both itself and other vertices, the cardinality of a step domination set for G is at least 2 unless $G = K_1$. On the other hand, $|S| \leq |V(G)|$.

Let G be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Then the set $\{N_0(v_i)\}_{i=1}^{n}$ is obviously a step domination partition of $V(G)$ corresponding to the step domination set $S = V(G)$. Thus, every graph has some step domination set. This leads us to the *step domination number* $\gamma_S(G)$ of a graph G (defined in [1]) to be the minimum cardinality of a step domination set for G. As a consequence of the above, $\gamma_S(G)$ is well defined and satisfies,

$$2 \leq \gamma_S(G) \leq |V(G)|, \tag{1}$$

with $\gamma_S(K_1) = 1$.

However, that concept can be extended to a *sequential step domination number* of a graph G, denoted $\gamma_S(G; k_1, k_2, \cdots, k_t)$, to be the minimum cardinality of a step domination set for G using all values of the sequence $k_1, k_2, \cdots, k_t$. As a consequence of the above, $\gamma_S(G) \leq \gamma_S(G; k_1, k_2, \cdots, k_t)$.

The value ∞ will be given in case there is not a partition of $V(G)$ associated to a particular sequence.

In this paper we deal with graphs having $d(G) = 2$. We determine the values of $\gamma_S(G; 1), \gamma_S(G; 0, 1), \gamma_S(G; 0, 1, 2)$ and $\gamma_S(G; 1, 2)$. Bounds on the remaining numbers, namely, $\gamma_S(G; 0, 2), \gamma_S(G; 2)$, are given, with exact values of some particular graphs defined in the sequel. We end our paper with a slightly more general result concerning k-regular graphs $k \geq 3$, whose girth is at least four, but their diameter is not bounded.

2 Results

Let G be a simple graph. Denote $t(G) = max\{d(u) + d(v) | (u, v) \in E(G)\}$. The girth of a graph G is denoted $g(G)$. We say that G has a *strong* spanning double star if G has a spanning double star with centers, say, at u, v such that $N(u) \cap N(v) = \emptyset$.

For other Graph Theoretical concepts we use [2] and [4].

For convenience we define the following sets.

Definition 2.1 *Let G be a graph with $d(G) = 2$ and $g(G) \geq 4$, where $V(G) = N[u] \cup N[v]$, $(u, v) \notin E(G)$, and u, v are certain vertices.*

We shall denote in this case:

$$A = N(u) \cap N(v)\ ,\ B = N(u) \setminus A\ ,\ C = N(v) \setminus A.$$

So that,

$$|V(G)| = |A| + |B| + |C| + 2.$$

Observe that since $d(G) = 2$ and $g(G) \geq 4$ the induced subgraphs $< A >, < B >, < C >$ are empty graphs, and each vertex of B is adjacent to at least one vertex of C, and vise-versa.

Theorem 2.2 *Let G be a connected graph such that $d(G) = 2$ and $g(G) \geq 4$.*

1. *If G has a spanning double star then, $\gamma_S(G) = \gamma_S(G; 1) = 2$.*
2. *If G does not have a spanning double star then,*
   (a) *either $\gamma_S(G) = \gamma_S(G; 0, 1) = n - t(G) + 2$,*
   (b) *or, $\gamma_S(G) = \gamma_S(G; 0, 2) = 2 + |A|$, where G is the graph defined in definition 2.1.*
3. $\gamma_S(G; 2) = \gamma_S(G; 0, 1, 2) = \infty$

Proof: Observe first that since $g(G) \geq 4$ then, if G has a spanning double star it should be a strong spanning double star. So, suppose G has a spanning double star with centers at u, v. Then, the label $st(u) = st(v) = 1$ yields a domination set of size 2, so that item 1 of the theorem is proved. Thus, to the end of the proof of the theorem we assume that G has no strong spanning double star (in particular $G \neq K_{m,n}$).

Let S be a minimal dominating set in G.

Case 1: $\exists u \in S\ ,\ st(u) = 2$.

Now in order to dominate u we can do it by either one of its neighbors or by a vertex at distance two from it. Let $v \in N(u)$ such that $st(v) = 1$. Since $g(G) \geq 4$ it follows that $N(v) \cap N(u) = \emptyset$ (since otherwise a triangle is created). On the other hand, it follows that $N(v) \subseteq N_2(u)$, which is also impossible since then the vertices in $N(v) \setminus \{u\}$ are dominated by both u and v. Hence, $N(v) \setminus \{u\} = \emptyset$, which is $d(v) = 1$. But this situation is also impossible since it yields $d(v, x) \geq 3$ for all $x \in N_2(u)$ (where $N_2(u) \neq \emptyset$ since $st(u) = 2$). A contradiction.

So we may assume that there exists $v \in N_2(u)$ with $st(v) = 2$. In that case we shall see that the graph G is the graph defined in definition 2.1. Indeed, suppose that the set

$$Y = V \setminus (N[u] \cup N[v]), \tag{2}$$

is not empty. Then each vertex of Y is dominated by both u and v, which is impossible. Hence $Y = \emptyset$ and G is indeed the graph defined in definition 2.1. In that case it is easily observed that a label 1 to any vertex of A leads a domination of both u and v. Suppose $st(x) = 2$ for some vertex $x \in A$. Then x dominates all vertices of $B \cup C$, which are already dominated by u and v, unless, $B = C = \emptyset$. But then, since $< A >$ is the empty graph, it follows that $G = K_{2,|A|}$, which has a spanning double star, and this is impossible by our assumption. Hence, since $|A| \neq 0$ (indeed if $|A| = 0$ it would imply that $d(u, v) \geq 3$) it must be that for all $a \in A$, $st(a) = 0$ so that $\gamma_S(G; 0, 2) \geq 2 + |A|$.

To obtain the upper bound, just label u and v by 2 and the vertices of A by 0.

Hence, we may assume that there are no vertices in S with label 2.

Case 2: $\exists u \in S$, $st(u) = 1$.

In this case the domination of u must be done by one of its neighbors, say, v. Namely, $st(v) = 1$. Hence, $N(u) \cap N(v) = \emptyset$ and let Y be as in (2).

Assume first that $Y \neq \emptyset$. By the diameter of G every vertex of Y must be adjacent to at least one vertex of $N(u)$ and one of $N(v)$. This yields that for all $y \in Y$, $st(y) = 0$. Indeed, a label 1 given to some vertex of Y yields a domination of vertices in $N(u) \cup N(v)$.

Hence, $\gamma_S(G) = \gamma_S(G; 0, 1) = n - t(G) + 2$.

The case $Y = \emptyset$ is impossible since then G will have a strong spanning double star.

Following the previous cases we obtain , $\gamma_S(G; 2) = \gamma_S(G; 0, 1, 2) = \infty$. ∎

Corollary 2.3 *(Theorem 2 in [3])*
If G is a $k-$regular $(k \geq 2)$ graph with $d(G) = 2$ and $g(G) \geq 5$, then,

$$\gamma_S(G) = \gamma_S(G; 0, 1) = 2 + (k - 1)^2.$$

Proof: In [3] it was proved that $\gamma_S(G) = 2 + (k - 1)^2$. On the other hand, the properties required for the graph in the corollary, meet with those of Theorem 2.2 (case 2(a)), with $t(G) = 2k$ and $n = k^2+1$, so that $\gamma_S(G; 0, 1) = 2+(k-1)^2$ and the result follows immediately.
∎

In the next theorem we extend our treatment to graphs having $d(G) = 2$ by extending the girth condition to be $g(G) \geq 3$.

For convenience we shall denote $E_\emptyset = \{(u, v) \in E(G) | N(u) \cap N(v) = \emptyset\}$. If $E_\emptyset \neq \emptyset$ we define,

$$t_1(G) = max\{d(u) + d(v) | (u, v) \in E_\emptyset\}.$$

Theorem 2.4 *Let G be a connected graph such that $d(G) = 2$ and $g(G) \geq 3$. then,* *1.*

$$\gamma_S(G;0,1) = \begin{cases} n - t_1(G) + 2 & , E_\emptyset \neq \emptyset \\ \infty & , E_\emptyset = \emptyset \end{cases} .$$

2. $$\gamma_S(G;0,1,2) = \gamma_S(G;1,2) = \infty.$$

3.

$$\gamma_S(G;1) = \begin{cases} 2 = \gamma_S(G) & , If\ G\ has\ a\ strong\ spanning\ double-star \\ \infty & , otherwise. \end{cases}$$

Proof: Let S be a minimal step-domination set. Obviously if $E_\emptyset = \emptyset$, then $\gamma_S(G;0,1) = \infty$ (and also $\gamma_S(G;1) = \infty$), since there is a triangle for each adjacent vertices, so that a label 1 is not possible. Indeed, if $st(u) = 1$ then there exists $v \in N(u)$ such that $st(v) = 1$. But then u and v dominate a mutual vertex of $N(u) \cap N(v)$.

Hence we may assume $E_\emptyset \neq \emptyset$. This yields $u, v \in V(G)$, $(u,v) \in E(G)$ and $N(u) \cap N(v) = \emptyset$.

Assume, then, that $u, v \in S$ and $st(u) = st(v) = 1$. Let Y be as in (2). We claim that if $Y \neq \emptyset$, then for all $y \in Y$, $st(y) = 0$. Indeed, since $d(G) = 2$, it follows that each $y \in Y$ should be adjacent to at least one vertex of $N(u)$ and $N(v)$. So that any label rather than 0 yields a domination of u or v or a member from $N(u) \cup N(v)$. Hence, $\gamma_S(G;0,1) \geq |Y| + 2 = n - t_1(G) + 2$. To obtain the upper bound just use the same labeling mentioned above.

If $Y = \emptyset$, then G has a strong spanning double star and thus, $\gamma_S(G;1) = 2$. If G has no strong spanning double star then, $Y \neq \emptyset$ and by the same arguments mentioned in the previous paragraph $st(y) = 0$ for all $y \in Y$, which yields that $\gamma_S(G;1) = \infty$, in this case.

Next we determine the values of $\gamma_S(G;1,2)$ and $\gamma_S(G;0,1,2)$. Since the label 2 must occur, let $u \in S$ such that $st(u) = 2$. In order to dominate u (by a vertex v), we can do it in one of the following ways:

Case 1: $st(v) = 2$.

Let, $A = N(u) \cap N(v), B = N(u) \setminus A, C = N(v) \setminus A$. Since $d(G) = 2, A \neq \emptyset$. But then none of the vertices of A can have the label 1 since it will dominate both u and v. On the other hand, $Y = \emptyset$(Y defined in (2)), since otherwise the vertices of Y will be dominated by both u and v. This yields that $V(G) = N[u] \cup N[v]$ and none of the vertices of G has a label 1. Thus, $\gamma_S(G;1,2) = \gamma_S(G;0,1,2) = \infty$, in this case.

Case 2: $st(v) = 1$.

In that case $N(v) \subseteq N(u)$, since otherwise v will dominate vertices which are dominated by u. Now, in order to dominate v, one can easily observe that no vertex with a label 1 can do it, since such a vertex will dominate u, as well. So that there exists $w \in S, st(w) = 2$. It follows than that $N(v) \subseteq N(w)$, for otherwise vertices of $N(v)$ will be dominated by w, as well. In order to dominate w we can do it with a vertex $x \in N(w), st(x) = 1$. This yields that $x \notin N(u)$, otherwise x dominates u, as well. Since $N(v) \subseteq N(u)$, $d(x,v) = 2$ and then there exists $r \in N(x) \cap N(v)$. But than r is dominated by both x and v. Hence, w should be dominated by w_1 such that $st(w_1) = 2$. But this case is exactly the above case 1 (with w and w_1), which is impossible. Thus, $\gamma_S(G;1,2) = \gamma_S(G;0,1,2) = \infty$, in this case.

This completes the proof of the theorem. ∎

Remark 2.5 *There are families of infinite graphs G for which $\gamma_S(G;1) = \gamma_S(G;0,1) = \infty$. The first example is the $n-$spoke wheel ($n \geq 3$) (obtained by taking a simple cycle $C_n, n \geq 3$ with a vertex u inside it adjacent to all vertices of C_n). The second family is obtained from the $n-$spoke wheel ($n \geq 3$) by adding a new vertex, say, v which is adjacent to all vertices of C_n only.*

It is easily checked that $\gamma_S(G;1) = \gamma_S(G;0,1) = \infty$.

For some more accurate values of the size of a minimum step-domination set, in graphs with $d(G) = 2$, we present a straightforward corollary concerning $k-$regular graphs. The only thing we point out is that $t_1(G) = 2k$ in that case.

Corollary 2.6 *Let G be a $k-$regular graph with $d(G) = 2$. Then,*

1.
$$\gamma_S(G;0,1) = \begin{cases} n-2k+2 & ,E_\emptyset \neq \emptyset \\ \infty & ,E_\emptyset = \emptyset \end{cases} .$$

2.
$$\gamma_S(G;0,1,2) = \gamma_S(G;1,2) = \infty.$$

3.
$$\gamma_S(G;1) = \begin{cases} 2 = \gamma_S(G) & ,\textit{If } G \textit{ has a strong spanning double}-\textit{star} \\ \infty & ,\textit{otherwise.} \end{cases}$$

Remark 2.7 *Again, there exist $k-$regular graphs in which $\gamma_S(G;1) = \gamma_S(G;0,1) = \infty$. First K_4 is a $3-$regular such graph. While for $k = 4$ just take the second graph defined in Remark 2.5 with $n = 4$.*

Next we investigate the graphs in which $\gamma_S(G;2), \gamma_S(G;0,2) \neq \infty$. First we define the following set of graphs.

Definition 2.8 sequential *graphs of diameter two are defined as follows:*

$$H_m = H_m(m_0; x_m, y_m, x_{m-1}, y_{m-1}, \cdots, x_1, y_1), \; m_0 \in N \cup \{0\},$$

where,

1. *$H_0 = H_0(m_0)$ is a complete graph K_{m_0}. If $m_0 = 0$ then we put $H_0(m_0) = \emptyset$.*

2. $H_1 = H_1(m_0; x_1, y_1)$ *is defined as follows:*
   $V(H_1) = N[x_1] \cup N[y_1]$, $(x_1, y_1) \notin E(H_1)$, $|N(x_1) \cap N(y_1)| = m_0$, *and* $< A_1 = N[x_1] \cap N[y_1] >$ *is either a graph of type* $H_0(m_0)$ *(which is exactly* K_{m_0}*), called hereafter Type 1, or for each pair of non-adjacent vertices* $u, v \in A_1$ *at least one of them is not connected to at least one member of* $(N[x_1] \setminus A_1) \cup (N[y_1] \setminus A_1)$. *In this case we call it Type 2.*

3. *For* $m \geq 2$ *we define* H_m *as follows:*
   (a) $V(H_m) = N[x_m] \cup N[y_m]$
   (b) *For all* $1 \leq i \leq m$, $(x_i, y_i) \notin E(H_m)$
   (c) $< A_m = N(x_m) \cap N(y_m) >= H_{m-1}(m_0; x_{m-1}, y_{m-1}, \cdots, x_1, y_1)$
   (d) *For all* $1 \leq i \leq m$ *the sets* $N(x_i)$ *and* $N(y_i)$ *contain* $(N[x_j] \cup N[y_j]) \setminus (N[x_j] \cap N[y_j])$ *for all* $i \leq j \leq m$*, as a subset.*
   (e) *For all* $1 \leq i \leq m$ *we define* $B_i = N(x_i) \setminus (N(x_i) \cap N(y_i))$ *and* $C_i = N(y_i) \setminus (N(x_i) \cap N(y_i))$.
   (f) *For all* $1 \leq i \leq m$ *we define* $A_i = (N(x_i) \cap N(y_i)) \setminus \bigcup_{j=i+1}^{m} (B_j \cup C_j \cup \{x_j, y_j\})$.

Remark 2.9 *The value of* m *is calculated according to the fact that* $A_2 = H_1$.

Theorem 2.10 *Let* G *be a graph with* $d(G) = 2$. *Assume,* $\gamma_S(G; 0, 2) \neq \infty$ *or* $\gamma_S(G; 2) \neq \infty$. *Then,*

1. *There exist vertices* x, y , $(x, y) \notin E(G)$, *such that* $V(G) = N[x] \cup N[y]$.
2. *There exist vertices* $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m$ *and* $m_0 \in N \cup \{0\}$ *such that* $G = H_m(m_0; x_m, y_m, x_{m-1}, y_{m-1}, \cdots, x_1, y_1)$, *where* H_m *is the graph defined in definition 2.8.*

Proof: Let x be a vertex such that $st(x) = 2$. Then in order to dominate x we have a vertex y with $st(y) = 2$. Let $Y = V(G) \setminus N[x] \cup N[y]$. Then if $Y \neq \emptyset$ each vertex in Y is dominated by both x and y, which is impossible. Hence, $V(G) = N[x] \cup N[y]$.

Now the vertices of $N[x] \cap N[y]$ are not dominated. If $N[x] \cap N[y] \neq \emptyset$ and G is of Type H_1, then only a label 0 is possible to each of its vertices. Otherwise, we shall have a sequence (of pairs) of non-adjacent vertices $\{x_i, y_i\}_{i=1}^{m}$ (defined, say, by induction) such that $st(x_i) = st(y_i) = 2$, $i = 1, 2, \cdots, m$ where m is determined by the condition $A_2 = H_1$.

Then, we label the vertices of A_1 by 0, and the obtained graph G is the graph H_m defined in definition 2.8. ∎

As a consequence of Theorems 2.4 and 2.10 we have the following result.

Theorem 2.11 *Let* G *be a connected graph such that* $d(G) = 2$. *Then,*

1. *If $\gamma_S(G;2) \neq \infty$ there exist vertices, $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m$ such that, $G = H_m(0; x_m, y_m, \cdots, x_1, y_1)$ and $\gamma_S(G) \leq \gamma_S(G;2) = 2m$.*

2. *If $\gamma_S(G;0,2) \neq \infty$ there exist vertices, $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m$ and $m_0 \in N$ such that, $G = H_m(m_0; x_m, y_m, \cdots, x_1, y_1)$ and $\gamma_S(G) \leq \gamma_S(G;0,2) = 2m + m_0$.*

Proof: Let S be a minimal step domination set. Then one can see that for each $x \in S, st(x) = 2$ there exists $y \in S, st(y) = 2$ such that x dominates y and vise-versa (and both dominate possibly some other vertices of G). Furthermore, we have $V(G) = N[x] \cup N[y]$. Indeed, if there exists $z \in V(G) \setminus N[x] \cup N[y]$, then, z is dominated by both x and y (since $d(G) = 2$). In addition, for each such $x \in S$ there exists a unique such $y \in S$ (by the definition of the step domination). Hence, the vertices of S, with label 2, can be ordered into such pairs $\{x_i, y_i\}_{i=1}^{m}, (m \geq 1)$ which yields $G = H_m$. If $m_0 = 0$ then item 1 of the theorem is achieved. If $m_0 \neq 0$ then the vertices of the set A_1 are labeled by 0 and we get item 2 of the theorem.

This completes the proof of the theorem. ∎

We compute now the values of $\gamma_S(G;0,2)$ and $\gamma_S(G;2)$ for a particular case of k−regular graphs, the almost (k,t)−strongly regular graphs, defined below.

Definition 2.12 *A graph G is called* almost (k,t)-strongly regular *with the parameters k, t, if it is k−regular and every pair of non-adjacent vertices have exactly t common neighbors.*

In the following we assume $k \geq 3$.

Observation 2.13 1. *As a consequence of definition 2.8 for almost (k,t)−regular graphs, it follows that in the graphs H_m, $|A_m| = t$ and if we define $B_m = N(x_m) \setminus A_m, C_m = N(y_m) \setminus A_m$, then, $|B_m| = |C_m| = k - t$ and thus,*

$$\forall m \geq 1 \ , \ |V(H_m)| = n = 2k - t + 2. \qquad (3)$$

2. *If $t = 0$ then $m = 1$ and thus, $\gamma_S(H_1;2) = 2$ (by labeling $st(x_1) = st(y_1) = 2$).*

To the sequel when the graph H_m is mentioned, we mean the graphs defined in definition 2.8 with the additional property of being almost (k,t)−strongly regular graphs, and according to observation 2.13 (2), we assume $t \geq 1$.

Our main result in that case is:

Theorem 2.14 *Let G be a graph which is almost (k,t)−strongly regular. Then,*

1. *either there exists $m \geq 1$ such that $G = H_{m+1}$ and*

$$\gamma_S(H_{m+1};2) = 2m + 2.$$

2. *or there exists $m \geq 1$ such that $G = H_{m+1}$ and*

$$\gamma_S(H_{m+1}; 0, 2) = 2m(t - k - 1) + t + 2.$$

In order to prove Theorem 2.14 we need some preliminaries.

Proposition 2.15 *Let $H_1 = H_1(m_0; x_1, y_1)$ be sequential graph which is almost $(k,t)-$strongly regular. Put $A_1 = H_0$ and $B_1 = N(x_1) \setminus A_1$, $C_1 = N(y_1) \setminus A_1$. Then,*
(a) $|A_1| = m_0 = t$.
(b) $|B_1| = |C_1| = k - t$.
(c) $|V(H_1)| = 2k - t + 2$

Proof: (a) follows from the definition of $H_m, m \geq 1$ as an almost $(k,t)-$regular graph. (b) follows from (a) and the fact that H_1 is $k-$regular. (c) is a particular case noticed in the observation above. ■

In a similar way we prove for $m \geq 2$:

Theorem 2.16 *Let H_{m+1} , $m \geq 1$, be sequential graph which is almost $(k,t)-$strongly regular. Then,*

$$|A_1| = 2m(t - k - 2) + t. \tag{4}$$

Proof: To prove the theorem we have that:

$$t = |N(x_{m+1}) \cap N(y_{m+1})| = |A_1| + \sum_{i=1}^{m}(|B_i| + |C_i|) + 2m, \tag{5}$$

where, $B_i = N(x_i) \setminus (N(x_i) \cap N(y_i))$, $C_i = N(y_i) \setminus (N(x_i) \cap N(y_i))$, Hence, it follows that $|B_i| = |C_i| = k - t$ for all the values of i so that by substituting in (5) together with $t = 2k - n + 2$ (from (3)), we obtain (4). ■

Now we are ready to prove Theorem 2.14.

Proof of Theorem 2.14

The proof of the theorem follows exactly from the proof of Theorem 2.11 with the additional condition upon G to be almost $(k,t)-$strongly regular graph. To obtain the value of $\gamma_S(H_{m+1}; 0, 2)$ we just have to substitute the value of $m_0 = |A_1|$ obtained in (4).

This completes the proof of the theorem. ■

We end our paper with a slightly more general result concerning $k-$ regular graphs where, $k \geq 3$. The cases $k \leq 2$ were dealt in [1].

Theorem 2.17 *Let G be a $k-$ regular connected graph with $g(G) \geq 4$. Then,*

$$\gamma_S(G) \leq \gamma_S(G; 0, 1) \leq n(1 - \frac{1}{2k}) .$$

Proof: Let $(u, v) \in E(G)$. Since $g(G) \geq 4$ we have $N(u) \cap N(v) = \emptyset$. Then there are at most $2k(k-1)+2$ vertices at distance at most three from at least one of the vertices u and v. Since $e(G) = \frac{nk}{2}$ it follows that there are at least $\frac{nk}{2[2(k-1)[k(k-1)+1]]}$ edges whose neighbors are disjoint (since $g(G) \geq 4$). Label $st(u) = st(v) = 1$. Then $2k$ vertices (including u and v) are dominated. Hence, running over all such edges we have that at least $\frac{nk}{2[2(k-1)[k(k-1)+1]]} \bullet 2k$ vertices are dominated. The rest of the vertices are labeled 0.

Hence,

$\gamma s(G;0,1) \leq n - k\frac{nk}{2(k-1)[k(k-1)+1]} + \frac{nk}{2(k-1)[k(k-1)+1]} = n - n\frac{k(k-1)}{2(k-1)[k(k-1)+1]} = n\left(1 - \frac{k}{2[k(k-1)+1]}\right) \leq n(1-\frac{1}{2k})$. ■

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