

# Chromatic Equivalence Classes of Certain Generalized Polygon Trees, II

BEHNAZ OMOOMI <sup>a</sup> and YEE-HOCK PENG <sup>b1</sup>

<sup>a</sup>*Department of Mathematical Sciences  
Isfahan University of Technology  
84154, Isfahan, Iran*

<sup>b</sup>*Department of Mathematics, and  
Institute for Mathematical research  
University Putra Malaysia  
43400UPM Serdang, Malaysia*

## ABSTRACT

Let  $P(G)$  denote the chromatic polynomial of a graph  $G$ . Two graphs  $G$  and  $H$  are chromatically equivalent, written  $G \sim H$ , if  $P(G) = P(H)$ . A graph  $G$  is chromatically unique if for any graph  $H$ ,  $G \sim H$  implies that  $G$  is isomorphic with  $H$ . In "Chromatic Equivalence Classes of Certain Generalized Polygon Trees", *Discrete Mathematics Vol.172, 103 - 114 (1997)*, Peng et al. studied the chromaticity of certain generalized polygon trees. In this paper, we present a chromaticity characterization of another big family of such graphs.

1980 Mathematical Subject Classification. *Primary* 05C15.

## 1. Introduction

The graphs that we consider are finite, undirected and simple. Let  $P(G)$  denote the chromatic polynomial of a graph  $G$ . Two graphs  $G$  and  $H$  are said to be *chromatically equivalent*, and we write  $G \sim H$ , if  $P(G) = P(H)$ . A graph  $G$  is *chromatically unique* if  $G \sim H$  implies that  $H$  is isomorphic to  $G$ . A set of graphs  $S$  is called a *chromatic equivalence class* if any two element of  $S$  are

---

<sup>1</sup>Corresponding author. E-mail: yhpeng@fsas.upm.edu.my

This work was partly supported by the University Putra Malaysia research grant 2000

chromatically equivalent, and if any graph which is chromatically equivalent with a graph  $G$  in  $\mathcal{S}$  is also isomorphic to some element of  $\mathcal{S}$ . Although chromatically unique graphs have been the subject of many recent papers (see [2] and [3]), relatively few results concerning the chromatic equivalence classes of graphs are known.

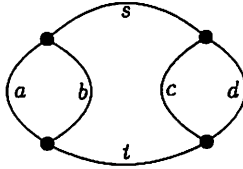


Figure 1.  $G_t^s(a, b; c, d)$

A path in  $G$  is called *simple* if the degree of each interior vertex is two in  $G$ . A *generalized polygon tree* is a graph defined recursively as follows. A cycle  $C_p$  ( $p \geq 3$ ) is a generalized polygon tree. Next, suppose  $H$  is a generalized polygon tree containing a simple path  $P_k$ , where  $k \geq 1$ . If  $G$  is a graph obtained from the union of  $H$  and a cycle  $C_r$ , where  $r > k$ , by identifying  $P_k$  in  $H$  with a path of length  $k$  in  $C_r$ , then  $G$  is also a generalized polygon tree. Consider the generalized polygon tree  $G_t^s(a, b; c, d)$  shown in Figure 1. The integers  $a, b, c, d, s$  and  $t$  represent the lengths of the respective paths between the vertices of degree  $> 2$ , where  $s \geq 0, t \geq 0$ . Without loss of generality, assume that  $a \leq b, a \leq c \leq d$  and if  $a = c$ , then  $b \leq d$ . Thus,  $\min\{a, b, c, d\} = a$ . Let  $r = s + t$ . We now form a family  $\mathcal{C}_r(a, b; c, d)$  of the graphs  $G_t^s(a, b; c, d)$  where the values of  $a, b, c, d$  and  $r$  are fixed but the values of  $s$  and  $t$  vary; that is

$$\mathcal{C}_r(a, b; c, d) = \{ G_t^s(a, b; c, d) \mid r = s + t, s \geq 0, t \geq 0 \}.$$

It is clear that the families  $\mathcal{C}_0(a, b; c, d)$  and  $\mathcal{C}_1(a, b; c, d)$  are singletons.

Note that  $G_t^s(a, b; c, d)$  is a connected  $(n, n+2)$ -graph, whose chromatic polynomials were computed by Chao and Zhao (see [1]), who also determined several chromatic equivalence classes, excluding among others the graph  $G_t^s(a, b; c, d)$ .

In [6], Peng et al. showed that  $\mathcal{C}_r(a, b; c, d)$  is a chromatic equivalence class for  $a, b, c, d$  at least  $r + 3$ . As a corollary, the graph  $G_t^s(a, b; c, d)$  is chromatically unique for  $a, b, c, d$  at least four (see also Peng [5]). In [4], Omoomi and Peng characterized the chromaticity of  $\mathcal{C}_1(a, b; c, d)$  for the minimum  $a, b, c$ , and  $d$

less than four. In this paper, we characterize the chromaticity of  $C_r(a, b; c, d)$  for the minimum of  $a, b, c$  and  $d$  equal to  $r + 2$ , and  $r \geq 2$ . Also we discover that the following conjecture is not true for each  $r \geq 2$ .

**Conjecture [6].** *The family of graphs  $C_r(a, b; c, d)$  is a chromatic equivalence class whenever  $a, b, c,$  and  $d$  are each at least four.*

In the remaining of this section, we give some known results that will be used to prove our main theorem. The girth of  $G$ , denoted by  $g(G)$ , is the length of a shortest cycle of  $G$ .

**Theorem A (Whitney [7]).** *Let  $G$  and  $H$  be chromatically equivalent graphs. Then*

- (a)  $|V(G)| = |V(H)|$ .
- (b)  $|E(G)| = |E(H)|$ .
- (c)  $g(G) = g(H)$ .
- (d)  $G$  and  $H$  have the same number of shortest cycles.

**Theorem B (Chao and Zhao [1], Peng et al. [6]).** *All the graphs in  $C_r(a, b; c, d)$  are chromatically equivalent.*

By this theorem we only need to compute  $P(G_r^0(a, b; c, d))$  for computing the chromatic polynomial of  $G_r^i(a, b; c, d)$ .

The next result is Case 1 in the proof of Theorem 6 in [6].

**Theorem C (Peng et al. [6]).** *If  $G_r^s(a, b; c, d)$  and  $G_r^{s'}(a', b'; c', d')$  are chromatically equivalent and  $s+t = s'+t'$ , then  $G_r^{s'}(a', b'; c', d') \in C_r(a, b; c, d)$ , where  $r = s + t$ .*

The next result gives the chromatic polynomial of  $G_r^s(a, b; c, d)$ . In [1], Chao and Zhao also determined the chromatic polynomial of this graph, but we shall use the computed chromatic polynomial of  $G_r^i(a, b; c, d)$  in [6] to prove our main results.

**Theorem D (Peng et al. [6]).** *Let the order of  $G_r^i(a, b; c, d)$  be  $n$  ( $n = a + b + c + d + r - 2$ ), and  $x = 1 - \lambda$ . Then we have*

$$P(G_r^i(a, b; c, d)) = \frac{(-1)^n x}{(x-1)^2} \cdot Q(G_r^i(a, b; c, d)),$$

where

$$Q(G_i^s(a, b; c, d)) = (x^{n+1} - x^{a+b+r} - x^{c+d+r} + x^{r+1} - x) - \\ (1 + x + x^2) + (x+1)(x^a + x^b + x^c + x^d) - \\ (x^{a+c} + x^{a+d} + x^{b+c} + x^{b+d}).$$

## 2. Main Theorem

In this section, we shall characterize the chromaticity of the family  $C_r(a, b; c, d)$  when  $\min\{a, b, c, d\} = r + 2$ , which gives us two counterexamples for the Conjecture.

**Theorem 1.** *The family of graphs in  $C_r(a, b; c, d)$  is a chromatic equivalence class if  $r \geq 2$  and  $\min\{a, b, c, d\} = r + 2$ , except the two families  $C_r(r + 2, b; b + 1, b + r + 2)$  and  $C_r(r + 2, c + r + 2; c, c + 1)$ .*

**Proof.** Let  $G = G_i^s(a, b; c, d) \in C_r(a, b; c, d)$  and  $H \sim G$ . By Lemma 4 and Theorem 2 in [1],  $H \cong G_i^{s'}(a', b'; c', d')$ , where  $a', b', c', d' \geq 1$ . Let  $r' = s' + t'$ . If  $r' = r$ , then by Theorem C,  $H \in C_r(a, b; c, d)$ . Now assume  $r' \neq r$ . We solve the equation  $Q(G) = Q(H)$ . After cancelling the terms  $x^{n+1}$ ,  $-x$  and  $-(1 + x + x^2)$ , we have  $Q_1(G) = Q_1(H)$  where

$$Q_1(G) = x^{r+1} + (x+1)(x^a + x^b + x^c + x^d) - x^{r+a+b} - \\ x^{r+c+d} - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_1(H) = x^{r'+1} + (x+1)(x^{a'} + x^{b'} + x^{c'} + x^{d'}) - x^{r'+a'+b'} - \\ x^{r'+c'+d'} - x^{a'+c'} - x^{a'+d'} - x^{b'+c'} - x^{b'+d'};$$

and  $a + b + c + d + r = a' + b' + c' + d' + r'$ ;  $a \leq b, a \leq c \leq d; a' \leq b', a' \leq c' \leq d'$ .

Since by assumption  $\min\{a, b, c, d\} = r + 2$ , the term  $x^{r+1}$  in  $Q_1(G)$  cannot be cancelled. Hence  $x^{r+1}$  is in  $Q_1(H)$  and this implies  $r + 1 = \min\{r' + 1, a', b', c', d'\}$ . Thus  $r' + 1 = r + 1$  or  $a' = r + 1$ . By our assumption, we must have  $a' = r + 1$ . Since  $a = \min\{a, b, c, d\} = r + 2$ , we have  $Q_2(G) = Q_2(H)$  where

$$Q_2(G) = x^{r+3} + (x+1)(x^b + x^c + x^d) - x^{2r+b+2} - \\ x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$

$$Q_2(H) = x^{r'+1} + (x+1)(x^{b'} + x^{c'} + x^{d'}) - x^{r'+r+b'+1} - \\ x^{r'+c'+d'} - x^{r+c'+1} - x^{r+d'+1} - x^{b'+c'} - x^{b'+d'};$$

$$\text{and } b+c+d+r+1 = b' + c' + d' + r';$$

$$r+2 = a \leq b, r+2 \leq c \leq d, r+1 \leq b', r+1 \leq c' \leq d'.$$

The lowest power positive term in  $Q_2(G)$  cannot be cancelled, hence we have  $\min\{b', c', d', r'+1\} \geq r+2$ . The term  $x^{r+3}$  in  $Q_2(G)$  cannot be cancelled. Hence  $x^{r+3}$  is a term in  $Q_2(H)$ , and thus we have  $\underline{r'+1 = r+3}$  or  $\underline{b' = r+3}$  or  $\underline{c' = r+3}$  or  $\underline{c'+1 = r+3}$  or  $d' = r+3$  or  $d'+1 = r+3$  or  $b'+1 = r+3$ . Since  $b, c, d \geq r+2$ , we have  $g(H) = g(G) \geq 2r+4$ . So  $b' \geq r+3$  because  $a' = r+1$ . Thus  $b' = r+2$  is impossible. Also  $d' = r+3$  or  $d' = r+2$  imply that  $c' = r+2$  or  $c' = r+3$ . Therefore we need to consider only the first four cases (underline).

**Case 1** Suppose  $r'+1 = r+3$  (or  $r' = r+2$ ). Then we have  $Q_3(G) = Q_3(H)$  where

$$Q_3(G) = (x+1)(x^b + x^c + x^d) - x^{2r+b+2} - \\ x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$

$$Q_3(H) = (x+1)(x^{b'} + x^{c'} + x^{d'}) - x^{2r+b'+3} - \\ x^{r+c'+d'+2} - x^{r+c'+1} - x^{r+d'+1} - x^{b'+c'} - x^{b'+d'};$$

$$\text{and } b+c+d = b' + c' + d' + 1;$$

$$r+2 \leq b, r+2 \leq c \leq d, r+3 \leq b', r+2 \leq c' \leq d'.$$

It is easy to see that  $\min\{b, c, d\} = \min\{b', c', d'\}$ . We consider two subcases:  $b \leq c$  and  $b > c$ .

**Subcase 1.1** Suppose  $b \leq c$ . Then we have  $\min\{b, c, d\} = b$  and  $g(G) = a+b$ . Also we have  $b = b'$  (if  $b' \leq c'$ ) or  $b = c'$  (if  $b' > c'$ ). If  $b = b'$ , then  $g(H) = a' + b' = g(G) = a+b$ , and we have  $a = a'$ , a contradiction (since  $a = r+2$  and  $a' = r+1$ ). Hence we have  $b = c'$  and  $g(G) = a+b = a+c' = r+2+c'$ .

Then  $g(H)$  is equal to either  $a' + b'$  or  $c' + d'$  or  $a' + r' + c' = 2r+3+c'$ . Since  $g(H) = g(G)$ , the last possibility is impossible. We now look at the other two possibilities.

**Subcase 1.1.1** Suppose  $g(H) = a' + b' = r+1+b'$  ( $b = c'$ ). Then  $g(G) = r+2+c' = r+1+b'$  and we have  $b' = c'+1 = b+1$ . Moreover, we have  $Q_4(G) = Q_4(H)$  where

$$Q_4(G) = (x+1)(x^c + x^d) - x^{2r+b+2} - \\ x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$

$$Q_4(H) = (x+1)(x^{b'} + x^{d'}) - x^{2r+b+4} - \\ x^{r+b+d'+2} - x^{r+b+1} - x^{r+d'+1} - x^{2b+1} - x^{b+d'+1};$$

$$\text{and } c+d = b' + d' + 1; r+2 \leq b \leq c \leq d, r+3 \leq b' = c'+1, b = c' \leq d'.$$

It can be seen that the term  $x^c$  in  $Q_4(G)$  cannot be cancelled since if  $c = 2r + b + 2$ , then at least one of the terms  $x^{b'}$  and  $x^{b'+1}$  is neither cancelled in  $Q_4(H)$  nor is a term of  $Q_4(G)$ . Thus we must have  $x^c$  in  $Q_4(H)$ . So we have  $c = b'$  or  $c = d'$ .

Subcase 1.1.1.1 Suppose  $c = b'$  (or  $c = b + 1$ ). Then we have  $d = d' + 1$  and from  $Q_4(G) = Q_4(H)$ , after cancelling equal terms, we have  $Q_5(G) = Q_5(H)$  where

$$Q_5(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2},$$

$$Q_5(H) = x^{d-1} - x^{2r+b+4} - x^{r+b+d+1} - x^{r+b+1} - x^{r+d},$$

The terms  $x^{d+1}$  and  $x^{d-1}$  must be cancelled in  $Q_5(G)$  and  $Q_5(H)$ , respectively; otherwise, there is no solution. The term  $x^{d+1}$  can be cancelled with  $-x^{2r+b+2}$  or  $-x^{r+c+2}$ , and the term  $x^{d-1}$  can be cancelled with  $-x^{2r+b+4}$  or  $-x^{r+b+1}$ . If  $d + 1 = 2r + b + 2$  then  $d - 1 = 2r + b$  and  $Q(G) = Q(H)$  has no solution. If  $d + 1 = r + c + 2$ , then  $d - 1 = r + c = r + b + 1$  and we have many solutions:  $a = r + 2, c = b + 1, d = b + r + 2$ ; and  $a' = r + 1, b' = b + 1, c' = b, d' = b + r + 1$ , and  $r' = r + 2$ . In other words, we have  $G_r^0(r + 2, b; b + 1, b + r + 2) \sim G_{r+2}^0(r + 1, b + 1; b, b + r + 1)$ , but  $G_{r+2}^0(r + 1, b + 1; b, b + r + 1) \notin C_r(r + 2, b; b + 1, b + r + 2)$ . Note that  $G_r^0(r + 2, b; b + 1, b + r + 2) \not\cong G_{r+2}^0(r + 1, b + 1; b, b + r + 1)$  where  $b \geq r + 2$ .

Subcase 1.1.1.2 Suppose  $c = d'$ . Recall that we also have  $b = c' = b' - 1$ . Then  $d = b' + 1 = b + 2$  and from  $Q_4(G) = Q_4(H)$ , after cancelling equal terms, we have  $Q_6(G) = Q_6(H)$  where

$$Q_6(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$

$$Q_6(H) = x^{d-1} - x^{2r+b+4} - x^{r+b+c+2} - x^{r+b+1} - x^{r+c+1} - x^{2b+1} - x^{b+c+1},$$

Now  $x^{d-1} = x^{b+1}$  in  $Q_6(H)$  cannot be cancelled because  $b \leq c$  but  $x^{b+1}$  is not a term in  $Q_6(G)$ . This is a contradiction.

Subcase 1.1.2 Suppose  $g(H) = c' + d'$ . Then  $g(G) = r + 2 + c' = c' + d'$ , thus  $d' = r + 2$ . Since  $r + 2 \leq b = c' \leq d' = r + 2$ , we have  $c' = r + 2 = b$ . From  $Q_3(G) = Q_3(H)$ , we have  $Q_7(G) = Q_7(H)$  where

$$Q_7(G) = (x + 1)(x^c + x^d) - x^{3r+4} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{r+2+c} - x^{r+2+d},$$

$$Q_7(H) = (x + 1)(x^{b'} + x^{r+2}) - x^{2r+b'+3} - x^{3r+6} - 2x^{2r+3} - 2x^{b'+r+2} - x^{b'+r+2},$$

and  $c + d = b' + r + 3$ ;

$$a = b = r + 2 \leq c \leq d, a' = r + 1, b' \geq r + 3, c' = r + 2 = d'.$$

The term  $x^{r+2}$  in  $Q_7(H)$  cannot be cancelled, therefore  $x^{r+2}$  is a term of  $Q_7(G)$  and  $c = r + 2$ . Thus  $d = b' + 1$  and after cancelling equal terms, we have  $Q_8(G) = Q_8(H)$  where

$$\begin{aligned} Q_8(G) &= x^{d+1} - x^{3r+4} - x^{2r+2+d} - 2x^{2r+4} - 2x^{r+d+2}; \\ Q_8(H) &= x^{d-1} - x^{2r+d+2} - x^{3r+6} - 2x^{2r+3} - 2x^{d+r+1}. \end{aligned}$$

Note that  $x^{d+1}$  and  $x^{d-1}$  must be cancelled in  $Q_8(G)$  and  $Q_8(H)$  respectively, but this is impossible. Therefore there is no solution.

Subcase 1.2 Suppose  $b > c$ . Thus  $\min\{b, c, d\} = c$ . So we have  $c = b'$  (that is if  $\min\{b', c', d'\} = b'$ ) or  $c = c'$  (that is if  $\min\{b', c', d'\} = c'$ ).

Subcase 1.2.1 Suppose  $c = b'$ . Then  $g(H) = a' + b' = r + 1 + b' = g(G)$ . If  $g(G) = a + b = r + 2 + b$ , then  $r + 2 + b = r + 1 + b' = r + 1 + c$  or  $c = b + 1$ . This contradicting our assumption that  $b > c$ . If  $g(G) = c + d$ , then  $c + d = r + 1 + c$  but  $d \geq r + 2$ , a contradiction. If  $g(G) = a + c + r = 2r + c + 2$ , then  $2r + c + 2 = r + 1 + c$  and this is a contradiction. Therefore  $c = b'$  is impossible.

Subcase 1.2.2 Suppose  $c = c'$ . Then from  $Q_3(G) = Q_3(H)$  we have  $Q_9(G) = Q_9(H)$  where

$$\begin{aligned} Q_9(G) &= (x + 1)(x^b + x^d) - x^{2r+b+2} - \\ &\quad x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d}, \\ Q_9(H) &= (x + 1)(x^{b'} + x^{d'}) - x^{2r+b'+3} - \\ &\quad x^{r+c+d'+2} - x^{r+c+1} - x^{r+d'+1} - x^{b'+c} - x^{b'+d'}, \end{aligned}$$

and  $b + d = b' + d' + 1$ ,  $a = r + 2 \leq b$ ,  $c \leq d$ ,  $c < b$ ;

$$a' = r + 1 \leq b', c' \leq d', c' < b', c \leq d', c \leq b', b' \geq r + 3.$$

Note that there is at least one positive term in  $Q_9(G)$  that cannot be cancelled by a negative term. This can be seen as follows. For the case of  $b < d$ , at least one of the terms  $x^b$  or  $x^{b+1}$  cannot be cancelled. Also for the case of  $b \geq d$ , at least one of the terms  $x^d$  or  $x^{d+1}$  cannot be cancelled. Now consider the positive terms in  $Q_9(G)$  and  $Q_9(H)$ . Since  $Q_9(G) = Q_9(H)$ , we have eight possibilities:  $b = b'$ ,  $b = b' + 1$ ,  $b = d'$ ,  $b = d' + 1$ ,  $b + 1 = b'$ ,  $b + 1 = d'$ ,  $d + 1 = b'$ , or  $d + 1 = d'$ .

Subcase 1.2.2.1 Suppose  $b = b'$ . Then  $d = d' + 1$ , and from  $Q_9(G) = Q_9(H)$ , after cancelling equal terms, we have  $Q_{10}(G) = Q_{10}(H)$  where

$$Q_{10}(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+d},$$

$$Q_{10}(H) = x^{d-1} - x^{2r+b+3} - x^{r+c+d+1} - x^{r+c+1} - x^{r+d} - x^{b+d-1}.$$

The term  $-x^{r+c+1}$  is in  $Q_{10}(H)$ , but  $-x^{r+c+1}$  is not in  $Q_{10}(G)$  (since  $c < b$  and  $c \leq d$ ). Therefore it must be cancelled by a positive term in  $Q_{10}(H)$ . Thus  $r+c+1 = d-1$ , hence  $d+1 = r+c+3$  and the term  $x^{d+1}$  cannot be cancelled in  $Q_{10}(G)$ , which is a contradiction.

Subcase 1.2.2.2 Suppose  $b = b' + 1$ . Then  $d = d'$ , and from  $Q_9(G) = Q_9(H)$ , after cancelling equal terms, we have  $Q_{11}(G) = Q_{11}(H)$  where

$$Q_{11}(G) = x^{b+1} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$

$$Q_{11}(H) = x^{b-1} - x^{r+c+d+2} - x^{r+c+1} - x^{r+d+1} - x^{b+c-1} - x^{b+d-1}.$$

The term  $-x^{r+c+1}$  is in  $Q_{11}(H)$ , but  $-x^{r+c+1}$  is not in  $Q_{11}(G)$  (since  $c < b$  and  $c \leq d$ ). Therefore it must be cancelled by a positive term in  $Q_{11}(H)$ . Thus  $r+c+1 = b-1$ , hence  $b+1 = r+c+3$  and the term  $x^{b+1}$  can be cancelled in  $Q_{11}(G)$  if and only if  $b+1 = r+c+3 = r+d+2$ , or  $d = c+1$ . Thus, we have many solutions:  $a = r+2$ ,  $b = r+c+2$ ,  $d = c+1$ ;  $a' = r+1$ ,  $b' = b-1 = r+c+1$ ,  $c' = c$ ,  $d' = d = c+1$ ,  $r' = r+2$ . In other words, we have  $G_r^0(r+2, r+c+2; c, c+1) \sim G_{r+2}^0(r+1, r+c+1; c, c+1)$  but  $G_{r+2}^0(r+1, r+c+1; c, c+1) \notin \mathcal{C}_r(r+2, r+c+2; c, c+1)$  for  $c \geq r+2 \geq 4$ . Note that  $G_r^0(r+2, r+c+2; c, c+1) \not\cong G_{r+2}^0(r+1, r+c+1; c, c+1)$  for  $c \geq r+2 \geq 4$ .

Subcase 1.2.2.3 Suppose  $b = d'$ . Then  $d = b' + 1$ , and from  $Q_9(G) = Q_9(H)$ , after cancelling equal terms, we have  $Q_{12}(G) = Q_{12}(H)$  where

$$Q_{12}(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$

$$Q_{12}(H) = x^{d-1} - x^{2r+d+2} - x^{r+c+b+2} - x^{r+c+1} - x^{r+b+1} - x^{d+c-1} - x^{b+d-1}.$$

The term  $-x^{r+c+1}$  is in  $Q_{12}(H)$ , but  $-x^{r+c+1}$  is not in  $Q_{12}(G)$  (since  $c < b$  and  $c \leq d$ ). Therefore it must be cancelled by a positive term in  $Q_{12}(H)$ .

Thus  $r+c+1 = d-1$ , hence  $d+1 = r+c+3$  and the term  $x^{d+1}$  can be cancelled in  $Q_{12}(G)$  if and only if  $d+1 = r+c+3 = b+c$ , or  $b = r+3$ . Since  $r+2 \leq c < b = r+3$ , we have  $c = r+2$ , and hence  $d = 2r+4$ . So we have a solution for  $Q(G) = Q(H)$ :  $a = r+2$ ,  $b = r+3$ ,  $c = r+2$ ,  $d = 2r+4$ ;  $a' = r+1$ ,  $b' = d-1 = 2r+3$ ,  $c' = c = r+2$ ,  $d' = b = r+3$ ,  $r' = r+2$ . In other words, we have  $G_r^0(r+2, r+3; r+2, 2r+4) \sim G_{r+2}^0(r+1, 2r+3; r+2, r+3)$  but  $G_{r+2}^0(r+1, 2r+3; r+2, r+3) \notin \mathcal{C}_r(r+2, r+3; r+2, 2r+4)$  for  $r \geq 2$ . Note that  $G_r^0(r+2, r+3; r+2, 2r+4) \not\cong G_{r+2}^0(r+1, 2r+3; r+2, r+3)$ . This solution is a special case of solution in Subcase 1.2.2.2, where  $c = r+2$ .



Subcase 1.2.2.4 Suppose  $b = d' + 1$ . Then  $d = b'$ , and from  $Q_9(G) = Q_9(H)$ , after cancelling equal terms, we have  $Q_{13}(G) = Q_{13}(H)$  where

$$Q_{13}(G) = x^{b+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$

$$Q_{13}(H) = x^{b-1} - x^{2r+d+3} - x^{r+c+b+1} - x^{r+c+1} - x^{r+b} - x^{d+c} - x^{b+d-1}.$$

The term  $-x^{r+c+1}$  is in  $Q_{13}(H)$ , but  $-x^{r+c+1}$  is not in  $Q_{13}(G)$  (since  $c < b$  and  $c \leq d$ ). Therefore it must be cancelled by a positive term in  $Q_{13}(H)$ . Thus  $r + c + 1 = b - 1$ , hence  $b + 1 = r + c + 3$  and the term  $x^{b+1}$  can be cancelled in  $Q_{13}(G)$ , only if  $b + 1 = r + c + 3 = r + d + 2$  or  $d = c + 1$ . Thus the term  $-x^{r+c+2}$  cannot be cancelled in  $Q_{13}(G)$ , and  $-x^{r+c+2}$  is not a term of  $Q_{13}(H)$ , which is a contradiction.

Subcase 1.2.2.5 Suppose  $b+1 = b'$ . Then  $d = d' + 2$ , and from  $Q_9(G) = Q_9(H)$ , after cancelling equal terms, we have  $Q_{14}(G) = Q_{14}(H)$  where

$$Q_{14}(G) = x^b + (x+1)x^d - x^{2r+b+2} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$

$$Q_{14}(H) = x^{b+2} + (x+1)x^{d-2} - x^{2r+b+4} - x^{r+c+1} - x^{r+d-1} - x^{b+c+1} - x^{b+d-1}.$$

The term  $-x^{r+c+1}$  is in  $Q_{14}(H)$  but it is not in  $Q_{14}(G)$  (since  $c < b$  and  $c \leq d$ ). Hence  $-x^{r+c+1}$  must be cancelled by a positive term in  $Q_{14}(H)$ . We have either  $r + c + 1 = b + 2$  or  $r + c + 1 = d - 2$  or  $r + c + 1 = d - 1$ . If  $r + c + 1 = b + 2$ , then the term  $x^b = x^{r+c-1}$  cannot be cancelled in  $Q_{14}(G)$  which means that  $x^b$  is a term of  $Q_{14}(H)$ . Thus we have either  $b = d - 2$  or  $b = d - 1$ . In each case the term  $x^d$  cannot be cancelled in  $Q_{14}(G)$ , which is a contradiction.

If  $r + c + 1 = d - 2$ , then we have the term  $x^{d-1} = x^{r+c+2}$  in  $Q_{14}(H)$  and this term cannot be cancelled in  $Q_{14}(G)$  because  $r + 2 \leq c < b$ . Since  $-x^{r+c+2}$  occurs in  $Q_{14}(G)$ , we must have the term  $2x^{r+c+2}$  in  $Q_{14}(G)$ , which is impossible.

If  $r + c + 1 = d - 1$ , then  $d - 2 = r + c$  and the term  $x^{d-2} = x^{r+c}$  cannot be cancelled in  $Q_{14}(H)$ . Thus, we must have the term  $x^{r+c} = x^{d-2}$  in  $Q_{14}(G)$ . This implies  $b = r + c$ . Now the term  $x^{b+2} = x^{r+c+2}$  cannot be cancelled in  $Q_{14}(H)$  because  $r + 2 \leq c < b$ . Since  $-x^{r+c+2}$  occurs in  $Q_{14}(G)$ , we must have the term  $2x^{r+c+2}$  in  $Q_{14}(G)$ , which is impossible. Therefore there is no solution. In the remaining three possibilities, that is  $b+1 = d'$ ,  $d+1 = b'$ , and  $d+1 = d'$ , there is no solution for  $Q(G) = Q(H)$ . The proof is similar to that of Subcase 1.2.2.5.

For Case 2 ( $b' = r + 3$ ), Case 3 ( $c' = r + 2$ ), and Case 4 ( $c' = r + 3$ ), there is no new solution for the equation  $Q(G) = Q(H)$ . The proof is similar to that of Case 1. The detailed proof can be obtained by e-mail from the second author or viewed at <http://www.fsas.upm.edu.my/~yhpeng/publish/p2c234.pdf>.  $\square$

**Corollary.** *We discover that the conjecture in [6] is only true for  $r = 1$ . For each  $r \geq 2$ , we provide two counterexamples as follows.*

- $G_r^0(r+2, b; b+1, b+r+2) \sim G_{r+2}^0(r+1, b+1; b, b+r+1)$  for  $b \geq 4$  but  $G_{r+2}^0(r+1, b+1; b, b+r+1) \notin \mathcal{C}_r(r+2, b; b+1, b+r+2)$ .
- $G_r^0(r+2, c+r+2; c, c+1) \sim G_{r+2}^0(r+1, c+r+1; c, c+1)$  for  $c \geq 4$  but  $G_{r+2}^0(r+1, c+r+1; c, c+1) \notin \mathcal{C}_r(r+2, c+r+2; c, c+1)$ .

### Acknowledgment

The authors would like to express their sincere thanks to the referees for their helpful and valuable comments.

### References

1. Chao, C.Y., Zhao, L.C.: Chromatic polynomials of a family of graphs. *Ars Combinatoria*. **15**, 111-129 (1983)
2. Koh, K.M., Teo, K.L.: The search for chromatically unique graphs. *Graphs and Combinatorics*. **6**, 259-285 (1990)
3. Koh, K.M., Teo, K.L.: The search for chromatically unique graphs, II. *Discrete Mathematics*. **172**, 59-78 (1997)
4. Omoomi, B., Peng, Y.H.: Chromatic equivalence classes of certain cycles with edges. *Discrete Mathematics*. **232**, 175-183 (2001)
5. Peng, Y.H.: Another family of chromatically unique graphs. *Graphs and Combinatorics*. **11**, 285-291 (1995)
6. Peng, Y.H., Little, C.H.C., Teo, K.L., and Wang, H.: Chromatic equivalence classes of certain generalized polygon trees. *Discrete Mathematics*. **172**, 103-114 (1997)
7. Whitney, H.: A logical expansion in mathematics. *Bull. Amer. Math. Soc.* **38**, 572-579 (1932)