A few identities involving partitions with a fixed number of parts

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1 Notation and elementary identities

The partition function $P(n)$ gives the number of ways of writing an integer n as a sum of positive integers without regard to the order. Let $p(n,k)$ be the number of solutions of the diophantine equation

$$x_1 + x_2 + \dots + x_k = n, \tag{1}$$

where $0 < x_1 \le x_2 \le \dots \le x_k$ and denote by $p_r(n,k)$ the number of solutions of (1) with $r \le x_1 \le x_2 \le \dots \le x_k$.

By solving the recurrence relationship

$$p(n,m) - p(n-m,m) = p(n-1,m-1)$$

for small values of m, Colman [1] obtained formulas for $p(n,1)$, $p(n,2)$, $p(n,3)$, $p(n,4)$, $p(n,5)$ and $p(n,6)$. With simple combinatorial considerations together with the formulas

$$\left[\frac{n-1}{k}\right] + \left[\frac{n-2}{k}\right] + \dots + \left[\frac{n-k}{k}\right] = n-k, \tag{2}$$

$$\sum_{i=1}^{3} \left\| \frac{(n-i)^2}{12} \right\| = \begin{cases} \left(\frac{n-2}{2}\right)^2 & \text{if } n \text{ is even,} \\ \frac{(n-3)(n-1)}{4} & \text{if } n \text{ is odd,} \end{cases} \tag{3}$$

where $\|x\| = [x + 1/2]$ is the integer closest to x (and $[\ ]$ is the greatest integer function), and the formulas for the sum of the k^{th} powers of the first n integers, we will derive some slick expressions for $p(n,1)$, $p(n,2)$, $p(n,3)$ and $p(n,4)$. Finally, we are going to use these identities together with a technique of Hirschhorn (see [2]) to give a formula for $p(n,5)$. Our approach and our method appear to be elementary and original and lead to formulas equivalent to those of Colman [1].

First, let us remark that

$$p(n,1) = 1, \quad p(n,2) = \left[\frac{n}{2}\right], \quad p_r(n,2) = \left[\frac{n}{2}\right] - (r-1)$$

are easy cases to verify. Let us see now what is going on with $p(n,3)$ and $p(n,4)$.

2 Formulas for $p(n,3)$ and $p(n,4)$

We are studying

$$x_1 + x_2 + x_3 = n.$$

In this equation, $1 \leq x_1 \leq [n/3]$. For each value of x_1, we are looking for the number of solutions of $x_2 + x_3 = n - x_1$. So, for each fixed value of x_1, this number is $p_{x_1}(n - x_1, 2)$. Then we get

$$p(n,3) = \sum_{r=1}^{[n/3]} p_r(n-r,2) = \sum_{r=1}^{[n/3]} \left(\left[\frac{n-r}{2}\right] - (r-1) \right) = \left\| \frac{n^2}{12} \right\|,$$

where we have used (2) to simplify the last sum.

Let us look at $p_r(n,3)$ now. It is a matter of subtracting from $p(n,3)$, the number of solutions to

$$x_1 + x_2 + x_3 = n$$

such that $x_1 < r$, which gives

$$p_r(n,3) = \left\| \frac{n^2}{12} \right\| - \sum_{l=1}^{r-1} p_l(n-l,2) = \left\| \frac{n^2}{12} \right\| - \sum_{l=1}^{r-1} \left(\left[\frac{n-l}{2}\right] - (l-1) \right).$$

The simplification of the sum is going to be done in the process of finding an expression for $p(n,4)$.

By the same argument, we have $p(n,4) = \sum_{r=1}^{[n/4]} p_r(n-r,3)$. So we compute

$$\begin{aligned} p(n,4) &= \sum_{r=1}^{[n/4]} p_r(n-r,3) \\ &= \sum_{r=1}^{[n/4]} \left(\left\| \frac{(n-r)^2}{12} \right\| - \sum_{l=1}^{r-1} \left(\left[\frac{(n-r)-l}{2}\right] - (l-1) \right) \right) \\ &= \sum_{r=1}^{[n/4]} \left\| \frac{(n-r)^2}{12} \right\| - \sum_{r=2}^{[n/4]} \sum_{l=1}^{r-1} \left[\frac{(n-r)-l}{2}\right] + \frac{[n/4]^3}{6} - \frac{[n/4]^2}{2} + \frac{[n/4]}{3} \end{aligned}$$

and it is now a matter of simplifying two sums.

Let us denote $\lfloor n/4 \rfloor$ by j and use (2) to get

$$\sum_{r=2}^{\lfloor n/4 \rfloor}\sum_{l=1}^{r-1}\left[\frac{(n-r)-l}{2}\right] = \begin{cases} -\frac{1}{4}j^3 + \frac{2n-1}{8}j^2 - \frac{n-1}{4}j & \text{if } j \text{ and } n \text{ are even,} \\ -\frac{1}{4}j^3 + \frac{2n-1}{8}j^2 - \frac{n-1}{4}j + \frac{1}{8} & \text{if } j \text{ is odd and } n \text{ is even,} \\ -\frac{1}{4}j^3 + \frac{2n-1}{8}j^2 - \frac{n-2}{4}j & \text{if } j \text{ is even and } n \text{ is odd,} \\ -\frac{1}{4}j^3 + \frac{2n-1}{8}j^2 - \frac{n-2}{4}j - \frac{1}{8} & \text{if } j \text{ and } n \text{ are odd.} \end{cases}$$

Then, with (3), we find (recalling that $j = \lfloor n/4 \rfloor$) that for n even,

$$\sum_{r=1}^{j}\left\|\frac{(n-r)^2}{12}\right\| = \begin{cases} \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-1)}{12}j & \text{if } j \equiv 0 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n+7)}{12}j - \frac{(12n-25)}{72} & \text{if } j \equiv 1 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-1)}{12}j + \frac{1}{9} & \text{if } j \equiv 2 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-1)}{12}j + \frac{1}{8} & \text{if } j \equiv 3 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n+31)}{24}j^2 + \frac{(n^2+7n-1)}{12}j - \frac{(3n^2-8)}{36} & \text{if } j \equiv 4 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-9)}{12}j + \frac{17}{72} & \text{if } j \equiv 5 \pmod 6, \end{cases}$$

and that for n odd,

$$\sum_{r=1}^{j}\left\|\frac{(n-r)^2}{12}\right\| = \begin{cases} \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-1)}{12}j & \text{if } j \equiv 0 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-9)}{12}j - \frac{(12n-29)}{72} & \text{if } j \equiv 1 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-9)}{12}j + \frac{(3n-7)}{18} & \text{if } j \equiv 2 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-1)}{12}j - \frac{1}{8} & \text{if } j \equiv 3 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n+31)}{24}j^2 + \frac{(n^2+7n-25)}{12}j + \frac{(3n^2-18n+19)}{36} & \text{if } j \equiv 4 \pmod 6, \\ \frac{1}{36}j^3 - \frac{(2n-1)}{24}j^2 + \frac{(n^2-n-9)}{12}j - \frac{(12n-37)}{72} & \text{if } j \equiv 5 \pmod 6. \end{cases}$$

This is summarized with

$$p(n,4)=\begin{cases}
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n)}{12}j & \text{if } j\equiv 0 \pmod 6 \text{ and } n \text{ is even},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n+8)}{12}j-\frac{(6n-8)}{36} & \text{if } j\equiv 1 \pmod 6 \text{ and } n \text{ is even},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n)}{12}j+\frac{1}{9} & \text{if } j\equiv 2 \pmod 6 \text{ and } n \text{ is even},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n)}{12}j & \text{if } j\equiv 3 \pmod 6 \text{ and } n \text{ is even},\\
\frac{4}{9}j^3-\frac{(n+5)}{3}j^2+\frac{(n^2+10n)}{12}j-\frac{(3n^2-8)}{36} & \text{if } j\equiv 4 \pmod 6 \text{ and } n \text{ is even},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n)}{12}j+\frac{1}{9} & \text{if } j\equiv 5 \pmod 6 \text{ and } n \text{ is even},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2-2n-3)}{12}j & \text{if } j\equiv 0 \pmod 6 \text{ and } n \text{ is odd},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n-11)}{12}j+\frac{(3n-5)}{18} & \text{if } j\equiv 1 \pmod 6 \text{ and } n \text{ is odd},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n-11)}{12}j+\frac{(3n-7)}{18} & \text{if } j\equiv 2 \pmod 6 \text{ and } n \text{ is odd},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n-3)}{12}j & \text{if } j\equiv 3 \pmod 6 \text{ and } n \text{ is odd},\\
\frac{4}{9}j^3-\frac{(n+5)}{3}j^2+\frac{(n^2+10n-27)}{12}j-\frac{3n^2-18n+19}{36} & \text{if } j\equiv 4 \pmod 6 \text{ and } n \text{ is odd},\\
\frac{4}{9}j^3-\frac{(n+1)}{3}j^2+\frac{(n^2+2n-11)}{12}j-\frac{(24n-65)}{72} & \text{if } j\equiv 5 \pmod 6 \text{ and } n \text{ is odd}.
\end{cases}$$

which seems to be inelegant but, by considering all cases one by one, one can find

$$p(n,4)=\begin{cases}
\left\|\frac{n^3+3n^2}{144}\right\| & \text{if } n \text{ is even},\\
\left\|\frac{n^3+3n^2-9n}{144}\right\| & \text{if } n \text{ is odd}.
\end{cases}$$

To represent $p(n,4)$ with a unified formula, one can write

$$p(n,4) \;=\; \left\|\frac{n^3+3n^2-9n\left(\frac{1+(-1)^{n-1}}{2}\right)}{144}\right\|.$$

3 What about $p(n,k)$ when $k>4$?

As we justified before, we have

$$p(n,k)=\sum_{r=1}^{[n/k]} p_r(n-r,k-1).$$

To calculate this last sum for $k=3$ (resp. $k=4$), we used (2) (resp. (3)), but there is no guarantee that identities like these are available for $k\geq 5$. Also, upon thinking about the large number of cases to deal with, one may wonder whether there is another way of obtaining a close formula for $p(n,5)$. Actually, we get the answer in the following application.

How many triangles are there with integral sides are there with perimeter equal to n? Hirschhorn [2] showed that the answer is

$$\begin{cases} p\left(\frac{n-6}{2},1\right)+p\left(\frac{n-6}{2},2\right)+p\left(\frac{n-6}{2},3\right) & \text{if } n \text{ is even,} \\ p\left(\frac{n-3}{2},1\right)+p\left(\frac{n-3}{2},2\right)+p\left(\frac{n-3}{2},3\right) & \text{if } n \text{ is odd.} \end{cases}$$

By the use of the generating functions, one can find an expression for the number of partitions of n in at most three parts, which is useful for the two preceding expressions. Hirschhorn proposes an original method in relation with Taylor's series to avoid boring work with partial fractions. He gets

$$p(n,1)+p(n,2)+p(n,3) = \left\|\frac{(n+3)^2}{12}\right\|,$$

i.e., he obtains, with the notation of this note,

$$1+\left[\frac{n}{2}\right]+\left\|\frac{n^2}{12}\right\| = \left\|\frac{(n+3)^2}{12}\right\|.$$

With his method, we find

$$p(n,1)+p(n,2)+p(n,3)+p(n,4) = \left\|\frac{2n^3+30n^2+135n+166+9(-1)^n(n+1)}{288}\right\|$$

and

$$p(n,1)+p(n,2)+p(n,3)+p(n,4)+p(n,5) =$$
$$\left\|\frac{n^4+30n^3+310n^2+1275n+1806+45(-1)^n(n+1)}{2880}\right\|.$$

Then, one can now isolate $p(n,5)$ to obtain

$$p(n,5) = \left\|\frac{n^4+10n^3+10n^2-75n-45n(-1)^n+185}{2880}\right\|.$$

ACKNOWLEDGEMENT: The author is grateful to the referee for his (her) judicious remarks.

References

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