On Monochromatic-Rainbow Generalizations of Two Ramsey Type Theorems

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1. Introduction

An edge colored graph is called a *rainbow* if no two of its edges have the same color. Let $\mathcal{H}$ and $\mathcal{G}$ be two families of graphs. Denote by $RM(\mathcal{H}, \mathcal{G})$ the smallest integer R, if it exists, having the property that every coloring of the edges of K_R by an arbitrary number of colors implies that either there is a monochromatic subgraph of K_R that is isomorphic to a graph in $\mathcal{H}$, or there is a rainbow subgraph of K_R that is isomorphic to a graph in $\mathcal{G}$. If there is no integer R that satisfies the property above, then we write $RM(\mathcal{H}, \mathcal{G}) = \infty$. If one of the sets $\mathcal{H}$ or $\mathcal{G}$ contains only a single graph H or G, respectively, then we use the simplified notation $RM(H, G)$ or $RM(H, \mathcal{G})$ or $RM(\mathcal{H}, G)$. For a family of graphs $\mathcal{H}$ and an integer s, we denote by $r(\mathcal{H}, s)$ the corresponding Ramsey number, which is defined to be the smallest integer r such that every s-coloring of the edges of K_r implies the existence of a monochromatic subgraph of K_r that is isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H}$ contains only a single graph H, then we denote $r(\mathcal{H}, s)$ by $r(H, s)$.

We use $e(G)$ and $v(G)$ to denote the set of edges and the set of vertices of the graph G, respectively. Furthermore, K_n, $K(A)$, $K_{m,n}$, and $K(A, B)$ denote the complete graph on n vertices, the complete graph on the vertex set A, the complete bipartite graph on $m + n$ vertices, and the complete bipartite graph on the sets A and B, respectively.

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The investigation of $RM(H,G)$ began in [9] with the investigation of $RM(H,K_{1,s})$, which was called the $(s-1)$-local Ramsey number and denoted by $r_{\text{loc}}^{s-1}(H)$. It was proved in [9] that $r_{\text{loc}}^{s-1}(H) < \infty$, for every graph H and every integer $s > 1$. Results concerning local Ramsey numbers, their variations and generalizations appear in [4], [6], [8], [9], [10], [11], [12], [13], and [14]. In [13], it is proved that $RM(H,G) < \infty$, for any graph H if and only if G is a forest. As will be seen in this paper this theorem does not apply if H is replaced by a set $\mathcal{H}$. We begin with the following observation.

Observation: Let $n = \max\{|e(H)| : H \in \mathcal{H}\}$ and let $m = \min\{|e(G)| : G \in \mathcal{G}\}$. If $m > n$, then $r(\mathcal{H},n) \leq RM(\mathcal{H},\mathcal{G})$. Indeed, consider a coloring of $e(K_{r(\mathcal{H},n)-1})$ by n colors that avoids a monochromatic copy of a graph in $\mathcal{H}$. By the definition of $r(\mathcal{H},n)$ such a coloring exists. Moreover, since $m > n$, i.e., every graph in $\mathcal{G}$ has more than n edges, it follows that this coloring avoids a rainbow copy of a graph in $\mathcal{G}$ as well.

In view of this observation it is of interest to discover cases where $r(\mathcal{H},n) = RM(\mathcal{H},\mathcal{G})$. For example, the authors proved in [2] that $r(nK_2, n-1) = RM(nK_2, nK_2)$. In this paper we investigate $RM(\mathcal{H},\mathcal{G})$, where $\mathcal{H}$ is either the set of all trees on n vertices denoted by $\mathcal{T}_n$ or $\mathcal{H}$ consists of the single graph of n matchings denoted by nK_2. Both families have received attention in the context of Ramsey numbers for graphs. We know that $r(\mathcal{T}_n,2) = n$ and $r(nK_2,2) = 3n-1$. Generalizations of these two theorems appear in [1], [2], [3], [5] and [7]. We now investigate further generalizations of these two theorems by determining $RM(\mathcal{T}_n,G)$ and $RM(nK_2,G)$ for all graphs G having three edges.

2. Preliminary Results

The five graphs with three edges are: $K_{1,3}$, $3K_2$, K_3, P_4, and $P_3 \cup P_2$, where P_n denotes a path on n vertices. In view of what follows we see that only the first three graphs need to be investigated.

Theorem 2.1: *Let $n \geq 5$. If $e(K_n)$ is colored by at least three colors, then K_n contains a rainbow copy of P_4 and $P_3 \cup P_2$.*

Proof: The validity of the theorem follows from a simple verification of the various colorings of $e(K_5)$.

Corollary 2.2: *If $n \geq 5$, then*
$r(\mathcal{T}_n, 2) = RM(\mathcal{T}_n, P_4) = RM(\mathcal{T}_n, P_3 \cup P_2) = n$.

Corollary 2.3: *If $n \geq 5$, then*
$r(nK_2, 2) = RM(nK_2, P_4) = RM(nK_2, P_3 \cup P_2) = 3n - 1$.

3. Spanning trees

Theorem 3.1: $RM(\mathcal{T}_n, K_3) = n$.

Proof: Since $r(\mathcal{T}_n, 2) = n$, it follows that $RM(\mathcal{T}_n, K_3) \geq n$. We will prove the reverse inequality. Consider a coloring of $e(K_n)$ by an arbitrary number of colors. Choose a monochromatic tree T with the maximum number of vertices. Suppose that T is colored red. We can assume that T is not a spanning tree and hence there exists a vertex x, where $x \notin v(T)$. Assuming that K_n does not contain a rainbow K_3, we will prove that the star whose center is x and whose end points are the vertices of T is monochromatic. Indeed, consider the color of xu_0, where u_0 is an arbitrary vertex of T. By the maximality of T the color of xu_0 is not red; suppose it is blue. If u_k is any vertex of T distinct from u_0, then there is a path $u_0, u_1, \ldots, u_k$ joining u_0 and u_k. Again, by the maximality of T, the edge xu_1 is not red and since by our assumption K_n does not contain a rainbow K_3, it follows that xu_1 is colored blue. Similarly, we get that all the edges $xu_2, \ldots, xu_k$ are colored blue. Thus we proved that the star whose center is x and whose end points are all the vertices of T is blue. But this star has one more vertex than T, contradicting the maximality of T. □

We wonder whether some stronger results can be proved. Namely, is it true that $RM(\mathcal{U}, K_3) = n$, where $\mathcal{U}$ is any of three subfamilies of $\mathcal{T}_n$. In order to state the problems, we need to intoduce the folowing definitions:

(a) The family $\mathcal{D}_n(k)$ denotes all trees T on n vertices for which diam(T) $\leq k$, where diam(T) is the diameter of T.

(b) A *k-superstar* is a tree that has one vertex of degree k (called the *center* of the superstar) and all other vertices of degree 1 or 2. We denote by $\mathcal{S}_n(t)$ the set of all k-superstars on n vertices, with $t \leq k \leq n-1$.

(c) A *broom* is a combination of a star and a path that have in common only one vertex: the center of the star and an end vertex of the path. We denote by $\mathcal{B}_n$ the set of all brooms on n vertices.

Parts (a) and (b) of the following theorem were proved in [2] and part (c) in [5].

Theorem 3.2:

(a) $r(\mathcal{D}_n(4), 2) = n$

(b) $r(\mathcal{S}_n(\lceil \frac{n-1}{2} \rceil), 2) = n$

(c) $r(\mathcal{B}_n, 2) = n$

Problem 3.3: Is it true that the following hold?

(a) $RM(\mathcal{D}_n(4), K_3) = n$

(b) $RM(\mathcal{S}_n(\lceil \frac{n-1}{2} \rceil), K_3) = n$

(c) $RM(\mathcal{B}_n, K_3) = n$

Next, we consider $RM(\mathcal{T}_n, K_{1,3})$.

Theorem 3.4: $RM(\mathcal{T}_n, K_{1,3}) = f(n)$, *where*

$$f(n) = \begin{cases} 3k-1 & \text{if } n = 2k \\ 3k+1 & \text{if } n = 2k+1 \end{cases}$$

Proof: (a) For every $n \geq 3$ we will describe a 3-coloring of $e(K_{f(n)-1})$ that avoids a monochromatic tree on n vertices as well as a rainbow copy of $K_{1,3}$. Consider the following coloring of $e(K_{f(n)-1})$. First partition $v(K_{f(n)-1})$ into three parts $A, B, \text{and } C$ as follows: if $f(n)-1 = 3k-2$, then $|A| = |B| = k-1$, and $|C| = k$, and if $f(n)-1 = 3k$, then $|A| = |B| = |C| = k$. Next, color the edges of $K(A)$, $K(B)$, and $K(C)$ by three distinct colors and the edges of $K(A,B)$, $K(B,C)$, and $K(C,A)$ by the same three colors, respectively. It is easy to see that this coloring implies that $RM(T_n, K_{1,3}) \leq f(n)$. In order to prove the reverse inequality consider a coloring of $e(K_{f(n)})$ that avoids a rainbow copy of $K_{1,3}$. We consider two cases.

CASE 1: There is a color which induces a subgraph G of $K_{f(n)}$, that has at least n vertices.

We assume that the components of G are nontrivial. If G has a component with at least n vertices, then the proof is complete. Otherwise, there is a component H of G with no more that $\left\lceil \frac{n-1}{2} \right\rceil \leq k$ vertices. Since the coloring avoids a rainbow copy of $K_{1,3}$, it follows that if $x \in v(H)$, then all the edges joining x to a vertex outside of H have the same color. Moreover, since $|v(H)| \leq k$, it follows that the number of vertices outside of H is at least $2k-1$ if $n = 2k$, or at least $2k+1$ if $n = 2k+1$. In both cases we get a monochromatic star with at least n vertices.

CASE 2: The maximum number of vertices of a subgraph G of $K_{f(n)}$ induced by any color is less than n.

Let G be a subgraph of $K_{f(n)}$ induced by one of the colors and assume that $|v(G)| = t < n$. We assume that the components of G are nontrivial. Since the coloring avoids a rainbow copy of $K_{1,3}$, it follows that the edges that join $v(G)$ and $v(K_{f(n)}) \setminus v(G)$ are colored by only two colors; moreover, if $x \in v(G)$, then all the edges that join x to a vertex in $v(K_{f(n)}) \setminus v(G)$ have the same color. Hence there are at least $\left\lceil \frac{t}{2} \right\rceil$ vertices in G that are connected to the vertices in $v(K_{f(n)}) \setminus v(G)$ by the same color which results in a complete bipartite graph with at least $\left\lceil \frac{t}{2} \right\rceil + (f(n) - t) = f(n) - \left\lfloor \frac{t}{2} \right\rfloor$ vertices. Since $t < n$, it follows that $f(n) - \left\lfloor \frac{t}{2} \right\rfloor \geq n$ and the proof is complete. □

Remark: It is worthwhile mentioning that in the proof of the previous theorem, we actually showed that $RM(\mathcal{B}_n, K_{1,3}) = f(n)$.

Lemma 3.5: $RM(\mathcal{T}_3, 3K_2) = RM(\mathcal{T}_4, 3K_2) = 6$ *and* $RM(\mathcal{T}_5, 3K_2) = RM(\mathcal{T}_6, 3K_2) = 7$

Proof: The result follows from a case by case analysis. We omit all the details, but we would like to depict a coloring of K_6 that avoids a monochromatic copy of $\mathcal{T}_5$ as well as a rainbow $3K_2$. Let $v(K_6) = \{a, b, c, d, e, f\}$. Color ab, ac, de and df by color 1, color the edges of the complete graph spanned by $\{a, b, d, e\}$ by color 2, and the remaining edges by color 3.

Lemma 3.6: $RM(\mathcal{T}_7, 3K_2) = 8$

Proof: Consider an arbitrary coloring of $e(K_8)$ and let T be a monochromatic tree with the maximum number of vertices. Furthermore, suppose that the color of T is red. If $|v(T)| = 2$, then since $8 > 6$, a rainbow copy of $3K_2$ is assured. Next, we will consider three cases.

CASE 1: $|v(T)| = 3$

Let T consist of the vertices x, y, and z and the edges xy and yz, and let the remaining vertices be denoted by v_i, where $i = 1, 2, 3, 4, 5$. By the maximality of $|v(T)|$, we can assume that the color of xv_1 is blue. If $K(\{v_2, v_3, v_4, v_5\})$ contains an edge that is neither red nor blue, then we get a rainbow copy of $3K_2$; otherwise it is colored only by red and blue and hence contains a spanning tree on 4 vertices, contradicting the maximality of T.

CASE 2: $|v(T)| = 4$

Let T consist of the vertices x, y, z and t and the edges and let the remaining vertices be denoted by v_i, where $i = 1, 2, 3, 4$. By the maximality of

$|v(T)|$, we can assume that the color of xv_1 and yv_2 are blue. If one of the edges in $K(\{v_1, v_2\}, \{v_3, v_4\})$ is neither red nor blue, then we obtain a a rainbow copy of $3K_2$. Hence it is monochromatic either in red or in blue. If it is blue, then there is a blue tree on the six vertices x, y, v_1, v_2, v_3, v_4, a contradiction. Hence it is red. Finally, consider the graph $K(\{\{v_1, v_2\}, \{z, t\})$. If one of its edges is neither red nor blue, then we obtain a a rainbow copy of $3K_2$. Hence it is monochromatic either in red or in blue. However, each of these cases contradicts $|v(T)| = 4$.

CASE 3: $|v(T)| > 4$

We can assume that $|v(T)| \leq 6$. Let $\{x, y\} = v(K_8) \backslash v(T)$ and let ab be an arbitrary edge of T. We consider the color of the edges xc and yd, where $c, d \in v(T) \backslash \{a, b\}$. If at least one of the edges xc or yd is red, then we have contradicted the maximality of $|v(T)|$. If the two edges have distinct colors, then we can combine them with the edge ab to obtain a rainbow copy of $3K_2$. Thus xc and yd have the same color. Since c and d were arbitrary we can conclude that the graph $K(\{x, y\}, v(T) \backslash \{a, b\})$ is monochromatic and since $|v(T)| > 4$, it follows that it has at least 6 vertices, and hence it contains a monochromatic spanning tree, say, S. Consider the edge xm, where $m \in \{a, b\}$ and m is not the center of a star in case that T is a star . By the maximality of T the color of xm is not the same as the color of T. If it is the color of S, then we get a monochromatic tree on 7 vertices and the proof is complete. Otherwise, the edge xm has a new color and together with yd and an additional edge from T, we obtain a rainbow copy of $3K_2$ and the proof is complete. □

Theorem 3.7: *If $n \geq 7$, then $RM(T_n, 3K_2) = n + 1$.*

Proof: For every $n \geq 7$ we will describe a 3-coloring of $e(K_n)$ that avoids a monochromatic tree on n vertices as well as a rainbow copy of $3K_2$. Consider the following coloring of $e(K_n)$. First partition $v(K_n)$ into three parts $A, B,$ and C, where $|A| = |B| = 1$, and $|C| = n - 2$. Next, color the edges of $K(C)$ and

$K(C,A)$ by red, the edges of $K(C,B)$ by blue and the remaining edge of $K(A,B)$, by green. It is easy to see that this coloring implies that $RM(\mathcal{T}_n, 3K_2) \geq n+1$. In order to prove the reverse inequality we use induction on n. By Lemma 3.6 the theorem holds for $n = 7$, so assume that it holds for some n, where $n \geq 7$, and consider a coloring of $e(K_{n+1})$. Applying the induction hypothesis we get either a rainbow copy of $3K_2$ and the proof is complete, or a monochromatic copy of $T_{n-1} \in \mathcal{T}_{n-1}$. Let $\{x,y\} = v(K_{n+1})\backslash v(T_{n-1})$ and let ab be an arbitrary edge of T_{n-1}. We consider the color of the edges xc and yd, where $c,d \in v(T_{n-1})\backslash\{a,b\}$. If at least one of the edges xc or yd has the color of T_{n-1}, then we obtain a monochromatic tree on n vertices. If the two edges have distinct colors, then we can combine them with the edge ab to get a rainbow copy of $3K_2$. Thus xc and yd have the same color. Since c and d were arbitrary we can conclude that the graph $K(\{x,y\}, v(T_{n-1})\backslash\{a,b\})$ is monochromatic and since it has $n-1$ vertices, it contains a monochromatic spanning tree, say, S_{n-1}. Consider the edge xm, where $m \in \{a,b\}$ and m is not the center of a star in case that T_{n-1} is a star . If its color is the same as the color of T_{n-1} or the color of S_{n-1}, then we have a monochromatic tree on n vertices and the proof is complete. Otherwise, the edge xm has a new color and together with yd and an additional edge from T_{n-1}, we get a rainbow copy of $3K_2$, which completes the proof. □

Remark: It is worth noting that the coloring in the previous proof implies that $RM(\mathcal{T}_n, \{3K_2, K_{1,3}\}) = n+1$.

4. Matchings

Theorem 4.1: $RM(nK_2, K_3) = \infty$.

Proof: The theorem follows from [13], but because of its simplicity we present its proof here. Let the vertices of K_n be $v_1, v_2, \ldots, v_n$, and let the color set be $\{2, 3, \ldots, n\}$. If $i < j$, then color the edge v_iv_j by color j.

Theorem 4.2: $RM(nK_2, K_{1,3}) = 3n - 1$.

Proof: This is Theorem 13 of [9]

Theorem 4.3: $RM(2K_2, 3K_2) = 2$, $RM(3K_2, 3K_2) = 6$, and $RM(nK_2, 3K_2) = 3n - 1$ for $n \geq 3$.

Proof: The first two values of RM can be easily checked. We will prove that $RM(nK_2, 3K_2) = 3n - 1$ for $n \geq 3$. Since $RM(nK_2, 3K_2) \geq r(nK_2, 2)$ and since $r(nK_2, 2) = 3n - 1$ by [7], it suffices to prove that $RM(nK_2, 3K_2) \leq 3n - 1$. We will show that $RM(3K_2, 3K_2) \leq 8$ and then proceed for $n \geq 4$. In any coloring of K_8 consider four independent edges u_1v_1, u_2v_2, u_3v_3 and u_4v_4. W.l.o.g we can assume that u_1v_1 and u_2v_2 are red, and u_3v_3 and $u_4v_4$are blue; otherwise the proof is complete. Any new color joining a vertex from $\{u_1, v_1, u_2, v_2\}$ and a vertex from $\{u_3, v_3, u_4, v_4\}$ will result in a rainbow $3K_2$. Thus we can assume that all the sixteen edges that join $\{u_1, v_1, u_2, v_2\}$ and $\{u_3, v_3, u_4, v_4\}$ are either red or blue. If the 4-cycle $u_1u_3v_1v_3$ has three edges of the same color or two opposite edges of the same color, then we get a monochromatic $3K_2$. Hence w.l.o.g. we can assume that u_1u_3 and u_1v_3 are both red and v_1u_3 and v_1v_3 are both blue. Now examine the edge u_3u_2. A new color forces a rainbow $3K_2$ and a blue forces a blue $3K_2$, so it must be red. Finally, any color choice for the edge v_3v_2 forces either a monochromatic or a rainbow $3K_2$.

Consider a coloring of K_{3n-1} by an arbitrary set of colors, say, C. If $|C| \leq 2$, then by [7] K_{3n-1} contains a monochromatic copy of nK_2. Hence, we can assume $|C| \geq 3$. Choose 3 representative edges from $e(K_{3n-1})$, where each edge is colored by a different color. Denote by H the graph induced by the representing edges and let d denote the number of its vertices. Also denote by M the complete graph induced by the vertices that do not belong to H. We can choose the representative edges such that the number of K_2's that are components in H is maximal. If $d = 6$, then the proof is complete. Hence we can assume that $d \leq 5$ and that there are edges in H that are not components. An important fact is that by the maximality of H, the set of colors used in M is a subset of the set of colors used in H. Moreover, the colors of the edges that are not components in H are not used in M. Since $d \leq 5$, it follows that the number of vertices in M is at least $3n - 6$; furthermore, since $n \geq 4$, it follows that $3n - 6 \geq 2n - 2$. Thus M contains at least $n - 1$ copies of K_2. If these copies are monochromatic, then adding one more matching from H results in a monochromatic nK_2. Otherwise there are two disjoint matchings in M having two different colors, yielding, by the maximality of H, $d \geq 5$. Thus $d = 5$ and $H = K_2 \cup K_{1,2}$; furthermore the only color that appears in M is the color of the K_2 from H. But the fact that M is monochromatic contradicts our assumption that there are two disjoint matchings in M having two different colors. This completes the proof. □

Corollary 4.4: $RM(nK_2, 3K_2) = r(nK_2, 2) = 3n - 1$.

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References

[1] A. Bialostocki and P. Dierker, *On the Erdös-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings.* Discrete Math. **110** (1992), 1-8.

[2] A. Bialostocki, W. Voxman and P. Dierker, *Either a graph or its complement is connected: A continuing saga*, to appear in Mathematics Magazine.

[3] A. Bialostocki and W. Voxman, *Generalization of some Ramsey-Type Theorems for matchings*, to appear in Discrete Math.

[4] B.Bollobás, A. Kostochka, and R.H. Schelp *Local and mean Ramsey numbers for trees*. J. Combin. Theory Ser. B **79** (2000), 100-103

[5] S. Burr, *A graph or its complement contains a spanning broom*, manuscript (1992).

[6] Y. Caro and Zs. Tuza, *On k-local and k-mean colorings of graphs and hypergraphs*, Quart. J. Math. Oxford Ser. (2) **44** (1993), 385-398.

[7] E. J. Cockayne and P. J. Lorimer, *The Ramsey number for stripes*, J. Austral. Math. Soc. (Ser A) **19** (1975), 252-256.

[8] P. Erdös, R. J. Faudree, C. C Rousseau, and R. H Schelp, *A local density condition for triangles*, Graph theory and applications (Hakone, 1990). Discrete Math. **127** (1994), 153-161.

[9] A. Gyárfás, J. Lehel, R. H. Schelp and Zs. Tuza, *Ramsey numbers for local colorings*, Graphs Combin. **3** (1987), 267-277.

[10] A. Gyárfás, A, J. Lehel, J. Nesetril, V. Rödl, R. H. Schelp, Zs. Tuza, *Local k-colorings of graphs and hypergraphs*, J. Combin. Theory Ser. B **43** (1987), 127-139.

[11] A. Rucinski and M. Truszczynski, *A note on local colorings of graphs*, The Second Krakow Conference on Graph Theory (Zgorzelisko, 1994). Discrete Math. **164** (1997), 251-255.

[12] R. H. Schelp, *Local and mean k-Ramsey numbers for complete graphs* J. Graph Theory **24** (1997), 201-203.

[13] M. Truszczynski, *Generalized local colorings of graphs*, J. Combin. Theory Ser. B **54** (1992), 178-188.

[14] M. Truszczynski and Zs. Tuza *Linear upper bounds for local Ramsey numbers*, Graphs Combin. 3 (1987), 67-73.