ABOUT MULTICOLOURED CYCLES IN K_{24n+1}

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ABSTRACT. Colour the edges of a K_{24n+1} by 12 colours so that every vertex in every colour has degree $2n$. Is there a totally multicoloured C_{12} (i.e. every edge gets a different colour)? Here we answer affirmative to this question. In [1] P. Erdos stated the same problem for K_{12n+1} and 6 colours. It was settled in [2].

In this paper we follow the terminology and symbols of [3]. We assume the complete graph K_{24n+1} to have the vertex-set
$V=V(K_{24n+1}) = \{1, 2, \dots, 24n+1\}$, $n>=1$.

MAIN RESULT

If the edges of a K_{24n+1} are coloured by 12 colours so that every vertex is an endvertex of $2n$ edges of every colour then there are at least n totally multicoloured C_{12}.

Proof. I. Let us start with the case $n=1$, i.e. K_{25}. First we decompose K_{25} into 25 subgraphs isomorphic to the graph G in Fig.1 constructing a cyclic G-decomposition [3, pp.57-59]. The base system consists of a single element - the graph G_1 in Fig.1. For K_{25}, with vertex-set $V= \{1, 2, \dots, 25\}$ the cyclic permutation is:

$$\begin{matrix} 1 & 2 & \dots & 24 & 25 \\ 2 & 3 & \dots & 25 & 1 \end{matrix}$$

Since the length of an edge (i, j) of a graph with vertex-set $\{1, 2, \dots, v\}$ is the number

$$\min \{|i-j|,\ v-|i-j|\}$$

the edges of K_{25} have lengths: 1, 2, ... 12. Thus applying 25 cyclic permutations (including the identity) we obtain a G-decomposition $G_1, G_2, \dots, G_{25}$ of K_{25}. Let us indicate the edges' lengths of G_k, $k=1, 2, \dots, 25$ using the corresponding vertices of G in Fig.1.

edge:	*he*	*bf*	*ae*	*cb*	*hg*	*dc*	*ab*	*ef*	*da*	*gc*	*hd*	*gf*
length:	1	2	3	4	5	6	7	8	9	10	11	12

Now we colour the edges of G_k, $k=1, 2, \dots, 25$ by 12 colours, for example A_1, $A2$, $\dots, A_{12}$ giving to each edge of length i the colour A_i. So each vertex has in every colour degree 2.

A multicoloured cycle C_{12} of K_{25} is the following:
C_{12}: 8, 9, 11, 14, 18, 13, 19, 12, 4, 20, 10, 21, 8.
The consecutive colours of the cycle's edges are: $A_1, A_2, \dots, A_{12}$.

II. Now let n>=2.For colouring K_{24n+1} by 12 colours we shall apply composite method. Let separate the vertices of K_{24n+1} into sets:

$$\{1,2,\ldots,24\},\ \{25,26,\ldots,48\},\ldots,\{24n\text{-}23,\ldots,24n\},\ \{24n+1\}.$$

Using these sets of vertices we represent K_{24n+1} as follows:

$$K_{24n+1}=n.K_{25}+\ 1/2n(n\text{-}1)\ K_{24,24}$$

where each $V(K_{25})$ consists of the vertex $24n+1$ and 24 vertices from one of the other sets; each bipartite graph $K_{24,24}$ has as parts two of the above sets except $\{24n+1\}$. Colour now every K_{25} as in I. Each vertex of K_{24n+1} except the vertex $24n+1$ belongs to one K_{25} and has degree 2 in every colour; the vertex $24n+1$ belongs to all graphs K_{25} and has degree $2n$ in every colour.

For colouring the graphs $K_{24,24}$ let us assume a graph $K_{24,24}$ to have parts

$$\{u_1, u_2,\ldots,u_{24}\} \text{ and } \{v_1, v_2,\ldots,v_{24}\};$$

then we represent $V(K_{24,24})=B_1+B_2+B_3+B_4$, where

$$B_1=\{u_1,u_2,\ldots,u_{12}\};\ B_2=\{u_{13},u_{14},\ldots,u_{24}\};$$
$$B_3=\{v_1,v_2,\ldots,v_{12}\};\ B_4=\{v_{13},.v_{14},\ldots,v_{24}\}.$$

If we take the bipartite graphs $K_{12,12}$ with parts

B_1 and B_3; B_1 and B_4; B_2 and B_3; B_2 and B_4,

it is clear that $K_{24,24}=4K_{12,12}$. Every vertex of a $K_{24,24}$ belongs to two of its subgraphs $K_{12,12}$.

Let us colour a bipartite graph $K_{12,12}$ in 12 colours; for example let it be $K_{12,12*}$ with parts B_1 and B_3. For i=1, 2,...,12 we give to the edge (u_i,v_{i+k}) the colour A_k, k= 1, 2,...,12. Obviously every vertex in $K_{12,12*}$ has degree 1 in every colour. Let u be an arbitrary vertex of K_{24n+1} except $24n+1$; u belongs to n-1 graphs $K_{24,24}$ and it belongs to $2(n\text{-}1)$ graphs $K_{12,12}$; so its degree in the whole colouring is: $2+2(n\text{-}1)=2n$. This way the required colouring of the edges of K_{24n+1} is done.

We take one multicoloured C_{12} from each K_{25}; so we have n multicoloured C_{12} which are subgraphs of K_{24n+1}. This completes the proof of the main result.

References

[1] Paul Erdos, Some of my favourite solved and unsolved problems in graph theory. Quaestiones Mathematicae 16(3) (1993), 333-350.

[2] Marina Martinova, A solution of an Erdos' problem, to appear in Annuaire de L'Institut d'Architecture et de Genie Civil - Sofia, Fascicule II, MATHEMATIQUES.

[3] Juraj Bosak, Decomposition of Graphs, Klumer Academic Publishers, Dodricht, 1990.